Matemática
Contemporânea
Vol. 53, 23–51
http://doi.org/10.21711/231766362023/rmc533

On classification and moduli spaces of holomorphic distributions

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Abstract. In this work we present a survey on classifications of holomorphic distributions and their moduli spaces on complex projective spaces.

Keywords: Holomorphic distributions, foliations, moduli spaces, reflexive sheaves.

2020 Mathematics Subject Classification: 58A17,14D20,14J60, 14F06

1 Introduction

The study of distributions and foliations has been developed from works of Pfaff, Jacobi, Grassmann, Frobenius, Darboux, Poincaré and Cartan, see [24]. The qualitative study of singular polynomial differential equations was investigated by Poincaré, Darboux and Painlevé. Nowadays, this corresponds to the study of singular holomorphic distributions and foliations on complex projective spaces.

Jouanolou in [32] gave the description of the space of codimension one foliations of degree 1 on $\mathbb{P}^n$. He has proved that a codimension one foliation of degree 1 on $\mathbb{P}^n$ either is given by a pencil of quadrics with a double hyperplane or is the linear pull-back of a degree one foliation by curves on the projective plane. In particular, the moduli space of foliation of codimension one and degree 1 has two irreducible components. Holomorphic foliations of arbitrary dimension and degrees 0 and 1 have been classified in [12, Théorème 3.8] and [34, Theorem 6.2] as follows: let $\mathcal{F}$ be a foliation of dimension k and degree $d \in \{0, 1\}$ on $\mathbb{P}^n$. Then,

1. (Cerveau–Déserti): $\mathcal{F}$ is tangent to a rational linear projection $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-k}$ if and only if $d = 0$.

2. (Loray– Pereira–Touzet):

   - either $\mathcal{F}$ is the linear pull-back of a degree 1 foliation by curves on $\mathbb{P}^{n-k+1}$,
   - or $\mathcal{F}$ is tangent to a dominant rational map $\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}(1^{(n-k+1)}, 2)$, defined by $n - k + 1$ linear forms and one quadratic form.

In particular, the moduli space of foliation of dimension $k \geq 2$ and degree $d \in \{0, 1\}$ has $d + 1$ irreducible components.

Cerveau and Lins Neto [10] described codimension one foliations of degree 2, They prove the following: let $\mathcal{F}$ be a foliation of codimension 1 and degree 2 on $\mathbb{P}^n$. Then one of the following holds.

1. $\mathcal{F}$ is tangent to a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1, 3)$, defined by one linear form and one cubic form;

2. $\mathcal{F}$ is tangent to a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^1$, defined by 2 quadratic forms.

3. $\mathcal{F}$ is the linear pull-back of a degree two foliation by curves on $\mathbb{P}^2$.

4. $T\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^n}(1)^{n-3} \oplus (\mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n})$, where $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ is an abelian Lie algebra of dimension 2.

5. $T\mathcal{F} \simeq \mathcal{O}_{\mathbb{P}^n}(1)^{n-3} \oplus (\mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n})$, where $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ is isomorphic to $\mathfrak{aff}(\mathbb{C})$.

6. $\mathcal{F}$ is the pull-back by a rational map $\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}(1,1,2)$ of a non-algebraic foliation by curves induced by a global vector field on $\mathbb{P}(1,1,2)$.

Moreover, the space of foliations of degree 2 and codimension 1 has six irreducible components. In [15] we show that a degree 2 holomorphic foliation of codimension q is either algebraically integrable, or is the linear pull-back of a degree two foliation by curves on $\mathbb{P}^{q+1}$, or is the linear pull-back of a degree two foliation by surfaces with trivial tangent sheaf or is a pull-back by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^{(q+1)}, 2)$ of a non-algebraic foliation by curves induced by a global vector field on $\mathbb{P}(1^{(q+1)}, 2)$.

Cerveau and Lins Neto in [11] prove that degree 3 codimension one foliations are either rational pull-backs of foliations on projective surfaces or are transversely affine. In [23] da Costa, Lizarbe, and Pereira have proved that there are exactly 18 distinct irreducible components parameterizing moduli spaces of foliations of degree 3 without rational first integrals, and at least 6 distinct irreducible components parameterizing foliations with rational first integrals. In [20] we have investigate foliations by curves on $\mathbb{P}^3$ with no isolated singularities and degree ≤ 3. We prove that the foliations of degree 1 or 2 are contained in a pencil of planes or is Legendrian, and are given by the complete intersection of two codimension one distributions, and that the conormal sheaf of a foliation by curves of degree 3 with reduced singular scheme either splits as a sum of line bundles or is an instanton bundle with second Chern class $\in \{1, 2, 3, 4\}$. As a consequence, we conclude that there are 6 different types of degree 3 foliations by curves with only reduced curves as singular schemes.

In [5] we have initiated a systematic study of codimension one holomorphic distributions on $\mathbb{P}^3$, analyzing the properties of their singular schemes and tangent sheaves and also we give a partial classification of distributions of degree ≤ 2. A complete classification of distributions of codimension 2 and degree 2 in $\mathbb{P}^3$ was given by Galeano, Jardim and Muniz in [29]. They show that there are 17 different types of such distributions. In the recent work [7], we classify dimension two distributions, of degree at most 2, on

$\mathbb{P}^4$ with either locally free tangent sheaf or locally free conormal sheaf and whose singular scheme has pure dimension one.

There is a correspondence between reflexive sheaves and holomorphic distributions due to a generalization of a Bertini–type theorem due to Ottaviani. In fact, holomorphic distributions appear implicitly in the theory of reflexive sheaves. Denote as usual by $\mathcal{M}^{st}(c_1, c_2, \ldots, c_n)$ the moduli space of stable reflexive sheaves on $\mathbb{P}^n$ with Chern classes c_i, with $i = 1, \ldots, n$. In [37] Okonek showed that

$$\mathcal{M}^{st}(-1, 1, 1, 2) = \left\{\mathrm{coker}(\phi); \phi : \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2} \to T\mathbb{P}^4\right\}$$

and that $\mathcal{M}^{st}(-1, 1, 1, 2)$ is isomorphic to $\mathbb{G}(1, 4)$ the grassmannian of lines in $\mathbb{P}^4$. From the distributions point of view the space $\{\mathrm{coker}(\phi); \phi : \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 2} \to T\mathbb{P}^4\}$ corresponds to the space of holomorphic foliations of degree zero in $\mathbb{P}^4$. Okonek proved in [38] that

$$\mathcal{M}^{st}(-1, 2, 2, 5) \simeq \left\{\mathrm{coker}(\phi); \phi : \mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 2} \to \Omega^1_{\mathbb{P}^4}\right\}.$$

Foliations by curves which are given by a global complete intersection of two codimension one distributions in $\mathbb{P}^3$ appear in [13], where Chang provides a characterization of certain class of arithmetically Buchsbaum scheme of curves on $\mathbb{P}^3$, see [16, Corollary 3].

A class example of stable vector bundle which occurs as tangent bundle of distributions is the so called null-Correlation bundle N which is given by an exact sequence of vector bundles

$$0 \to N(1) \to T\mathbb{P}^{2r+1} \to \mathcal{O}_{\mathbb{P}^{2r+1}}(2) \to 0.$$

This survey is organized as follows. In Section 2, we introduce basic concepts on distributions and we state a Bertini type which allows us to construct holomorphic distributions. In Section 3, we recall the definition of moduli spaces of foliations and distributions. In Section 4, we give a result on numerical invariants of the singular locus of distributions. In the next sections we survey on the recent progress of the classification of distributions of low degree and their moduli spaces.

2 Holomorphic distributions and foliations

Let X be a complex manifold. By a sheaf we mean a coherent analytic sheaf of $\mathcal{O}_X$-modules. Suppose that X has complex dimension $n = k + s$, $s \geq 1$. A saturated codimension k singular holomorphic distribution on X is given by a short exact sequence of analytic coherent sheaves

$$\mathcal{F} : 0 \to T\mathcal{F} \xrightarrow{\varphi} TX \xrightarrow{\pi} N\mathcal{F} \to 0, \tag{2.1}$$

The sheaf $N\mathcal{F}$ is the *normal sheaf* of $\mathcal{F}$, is a non trivial torsion free sheaf of generic rank k on X. The coherent sheaf $T\mathcal{F}$ is the *tangent sheaf* of $\mathcal{F}$ of rank $s = n - k \geq 1$.

The *singular scheme* of $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is defined as follows. Taking the maximal exterior power of the dual morphism $\varphi^\vee : \Omega^1_X \to T\mathcal{F}^*$ we obtain a morphism

$$\Omega^s_X \to \det(T\mathcal{F})^*$$

the image of such morphism is an ideal sheaf $\mathcal{I}_Z$ of a closed subscheme $Z_\mathcal{F} \subset X$ of codimension ≥ 2, which is called the singular scheme of $\mathcal{F}$, twisted by $\det(T\mathcal{F})^*$. As a subset of X, $\mathrm{Sing}(\mathcal{F})$ corresponds to the singular set of its normal sheaf.

Fix a line bundle $\mathcal{L}$ on a projective manifold X. The space of holomorphic distributions of codimension k in X in which the determinant of the normal sheaves are isomorphic to $\mathcal{L}$ may be identified by a class of sections

$$[\omega] \in \mathbb{P}H^0(X, \Omega^k_X \otimes \mathcal{L}).$$

In general, a twisted k-form $\omega \in H^0(X, \Omega^k_X \otimes \mathcal{L})$ is called by a *Pfaff system*, see [16, 28].

The distribution is *involutive* if $[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$ and by the Frobenius theorem, the distribution is *integrable*, i.e., tangent to a foliation.

We may consider a dual perspective: Consider the dual map $N\mathcal{F}^* \subset \Omega^1_X$, that we identify with the anhilator of the sheaf $T\mathcal{F}$. We get an exact sequence

$$\mathcal{F} : 0 \to N\mathcal{F}^* \to \Omega^1_X \to \mathcal{Q}_\mathcal{F} \to 0. \tag{2.2}$$

The sheaf $N\mathcal{F}^*$ has generic rank k is reflexive and $\mathcal{Q}_{\mathcal{F}}$ is torsion free of rank q. With this formulation, the distribution is involutive if the following holds:

Consider the ideal $\mathcal{J} \subset \Omega^*_X$ generated by $N\mathcal{F}^*$, then the distribution $\mathcal{F}^o$ is involutive if $d\mathcal{J} \subset \mathcal{J}$, and then, again Frobenius Theorem implies that is integrable, i.e., $N\mathcal{F}^*$ is the *conormal sheaf* of a foliation.

The integrability condition can also be defined as follows. Let $U \subset X$ an open set and $\omega \in \Omega^k_X(U)$. For any $p \in U \setminus \mathrm{Sing}(\omega)$ there exist a neighborhood V of p, $V \subset U$, and 1-forms $\eta_1, \ldots, \eta_k \in \Omega^1_X(V)$ such that:

$$\omega\,|_V = \eta_1 \wedge \cdots \wedge \eta_k.$$

We say that ω satisfies the integrability condition if and only if,

$$d\eta_j \wedge \eta_1 \wedge \cdots \wedge \eta_r = 0,$$

for all $j = 1, \ldots, k$.

Let $\mathcal{F}$ be a codimension one distribution on X. The distribution $\mathcal{F}$ corresponds to a unique (up to scaling) twisted 1-form

$$\omega \in H^0(X, \Omega^1_X \otimes \det(N\mathcal{F}))$$

non vanishing in codimension one. For every integer $i \geq 0$, there is a well defined twisted $(2i+1)$-form

$$\omega \wedge (d\omega)^i \ \in \ H^0\Big(X, \Omega^{2i+1}_X \otimes \det(N\mathcal{F})^{\otimes(i+1)}\Big).$$

The *class* of $\mathcal{F}$ is the unique non negative integer $\ell = \ell(\mathcal{F})$ such that

$$\omega \wedge (d\omega)^\ell \not\equiv 0 \quad \text{and} \quad \omega \wedge (d\omega)^{\ell+1} \equiv 0.$$

By Frobenius theorem, a codimension one distribution is a foliation if and only if $\ell(\mathcal{F}) = 0$.

Let $U \subset X_{\mathrm{reg}}$ be the maximal open set where $T\mathcal{F}_{|X_{\mathrm{reg}}}$ is a holomorphic subbundle of $T_{X_{\mathrm{reg}}}$. A *leaf* of $\mathcal{F}$ is a connected, locally closed holomorphic submanifold $L \subset U$ such that $TL = T\mathcal{F}_{|L}$. A leaf is called *algebraic* if it is open in its Zariski closure. The foliation $\mathcal{F}$ is said to be *algebraically integrable* if its leaves are algebraic, and in this case $\mathcal{F}$ is *tangent to a rational fibration* $X \dashrightarrow Y$, see [3, Section 3].

Let X be an irreducible projective manifold of dimension n; let G be a reflexive sheaf on X, and let TX be the tangent sheaf of X. Set $k := \mathrm{rk}(G)$; assume that $k < n$. The *degeneracy locus* of a morphism of sheaves $\phi : G \to TX$ is defined by

$$D(\phi) := \mathrm{Supp}(\mathrm{coker}(\phi^\vee)).$$

Theorem 2.1. *[5, Lemma A.1][Bertini type] If $G^\vee \otimes TX$ is globally generated, then, for a generic morphism $\phi \in \mathrm{Hom}(G, TX)$, the following holds:*

$$\mathrm{codim}_X(D(\phi)) \leq min\{n - k + 1, \mathrm{codim}_X \mathrm{Sing}(G^\vee)\}.$$

In particular, $\phi : G \to TX$ is a holomorphic distribution, of dimension k, on X.

3 Moduli spaces of distributions and foliations

We now recall the definition of the moduli space of distributions. Let X be a smooth projective variety and fix $H = \mathcal{O}_X(1)$ an ample line bundle. We will write $\mathcal{O}_X(k)$, $k \in \mathbb{Z}$, for its tensor powers.

Definition 3.1. [40, 5] Let X be a smooth projective variety and let TX_S denote the relative tangent sheaf of $X \times_{\mathbb{C}} S \to S$. A family of distributions on X parameterized by S is given by a subsheaf $\boldsymbol{\phi}\colon \mathbf{F} \hookrightarrow TX_S$ such that for each $s \in S$ the restriction $\boldsymbol{\phi}_s\colon \mathbf{F}|_s \to TX_s$ is injective and $\mathrm{coker}(\phi_s)$ is torsion free.

Let $\mathfrak{Sch}_{/\mathbb{C}}$ denote the category of schemes of finite type over $\mathbb{C}$, and $\mathfrak{Sets}$ be the category of sets. Fix a polynomial $P \in \mathbb{Q}[t]$, and consider the functor

$$\mathcal{D}ist_X^P\colon \mathfrak{Sch}_{/\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathfrak{Sets},$$

$$\mathcal{D}ist_X^P(S) := \left\{ (\mathbf{F},\boldsymbol{\phi}) \mid \begin{array}{c} \mathbf{F} \overset{\phi}{\hookrightarrow} T\mathcal{X}_S \text{ is a family of} \\ \text{distributions with} \\ \chi(F|_s(t)) = P(t),\ \forall s \in S \end{array} \right\} / \sim,$$

where a family is considered as in Definition 3.1, and we say that $(\mathbf{F},\boldsymbol{\phi}) \sim (\mathbf{F}',\boldsymbol{\phi}')$ if there exists an isomorphism $\boldsymbol{\beta}\colon \mathbf{F} \to \mathbf{F}'$ such that $\boldsymbol{\phi} = \boldsymbol{\phi}' \circ \boldsymbol{\beta}$. It follows from [5, Proposition 2.4] that the functor $\mathcal{D}ist_X^P$ is represented by a quasi-projective scheme $\mathcal{D}^P(X)$, that is, there exists an isomorphism of functors

$$\mathcal{D}ist_X^P \xrightarrow{\sim} \mathrm{Hom}(\bullet, \mathcal{D}^P(X)).$$

Furthermore, the scheme $\mathcal{D}^P(X)$ is an open subset of the quot scheme $\mathrm{Quot}^{P_{TX}-P}(TX)$. Consider the subfunctor $\mathcal{F}ol_X^P$ of $\mathcal{D}ist_X^P$ of integrable distributions. Quallbrunn proved in [40, Proposition 6.3] that there is a subscheme $\mathcal{F}^P(X) \subset \mathcal{D}^P(X)$ which represents $\mathcal{F}ol_X^P$ and this space is called the space of holomorphic foliations with Hilbert polynomial equal to P, see [21] for more details.

3.1 Distributions and semi-stability conditions

Let $\mathcal{L}$ be an ample line bundle on n-dimensional projective variety X. Let $\mathcal{E}$ be a torsion-free sheaf of generic rank r on X.

Definition 3.2. The slope of $\mathcal{E}$ with respect to $\mathcal{L}$ to be

$$\mu_{\mathcal{L}}(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot \mathcal{L}^{n-1}}{r}.$$

We say that $\mathcal{E}$ is $\mu_{\mathcal{L}}$-*(semi)stable* if for any subsheaf $\mathcal{G}$ of $\mathcal{E}$ we have $\mu_{\mathcal{L}}(\mathcal{G}) < \mu_{\mathcal{L}}(\mathcal{E})(\mu_{\mathcal{L}}(\mathcal{G}) \leq \mu_{\mathcal{L}}(\mathcal{E}))$.

We show in [5, Theorem 2.3] that the functor $\mathcal{D}ist^P$ is represented by the quasi-projective scheme $\mathcal{D}^P$, that is, there exists an isomorphism of functors $\mathcal{D}ist^P \xrightarrow{\sim} \mathrm{Hom}(\cdot, \mathcal{D}^P)$. Moreover, the study of the forgetful morphism

$$\varpi : \mathcal{D}^P \to M^P \quad , \quad \varpi([\mathcal{F}]) := [T\mathcal{F}],$$

where M^P denotes de moduli spaces of reflexive sheaves with Hilbert polynomial P and $T\mathcal{F}$ is the tangent sheaf of $\mathcal{F}$, allows us to give information about the geometry of the moduli space $\mathcal{D}ist^P$.

Other important map to consider is the natural morphism from the moduli space of the distributions to the irreducible component of the Hilbert scheme of the singular scheme of the distribution

$$\mathbf{S}^P : \mathcal{D}^P \to \mathrm{Hilb}^{R(P)} \quad , \quad \mathbf{S}^P([\mathcal{F}]) := [\mathrm{Sing}(\mathcal{F})],$$

where $R(P)$ is determined by P. This map is related with the following two important problems in the following way:

(a) Determination of the distributions by their singular scheme is the same as the injectivity of the map $\mathbf{S}^P$.

(b) Existence of distributions with prescribed singular schemes is the same as surjectivity of the map $\mathbf{S}^P$.

Let $\mathcal{D}ist^{P,ss}$ be the subfunctor of $\mathcal{D}ist^P$ given by

$$\mathcal{D}ist^{P,ss} : \mathfrak{Sch}^{\mathrm{op}}_{/\mathbb{C}} \to \mathfrak{Sets} \quad , \quad \mathcal{D}ist^{P,ss}(S) := \{(F, \phi)\}/\sim$$

where F_s is now assumed to be semistable for each $s \in S$. Clearly, $\mathcal{D}ist^{P,ss} \simeq \mathrm{Hom}(\cdot, \mathcal{D}^{P,ss})$, where

$$\mathcal{D}^{P,ss} := \{[F, \phi] \in \mathcal{D}^P \mid F \text{ is semistable } \};$$

note that $\mathcal{D}^{P,ss}$ is an open subset of $\mathcal{D}^P$. Similarly, we will also consider the following open subset of $\mathcal{D}^P$:

$$\mathcal{D}^{P,st} := \{[F, \phi] \in \mathcal{D}^P \mid F \text{ is stable } \}.$$

In order to study the geometry of moduli spaces of distributions we have the following result.

Lemma 3.3. [5, Lemma 2.5] There exists a forgetful morphism

$$\varpi : \mathcal{D}^{P,ss} \to M^P \quad , \quad \varpi([\mathcal{F}]) := [T\mathcal{F}].$$

In addition, if $T\mathcal{F}$ is stable and satisfies $\mathrm{Ext}^1(T\mathcal{F}, TX) = \mathrm{Ext}^2(T\mathcal{F}, T\mathcal{F}) = 0$, then $[\mathcal{F}]$ is a nonsingular point of $\mathcal{D}^{P,st}$, ϖ is a submersion at $[T\mathcal{F}] \in M^{P,st}$ and

$$\dim_{[\mathcal{F}]} \mathcal{D}^{P,st} = \dim \mathrm{Ext}^1(T\mathcal{F}, T\mathcal{F}) + \dim \mathrm{Hom}(T\mathcal{F}, TX) - 1.$$

4 Distributions and foliations on projective spaces

Let $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^{n-k}_{\mathbb{P}^n}(r))$ be the twisted $(n-k)$-form associated to a holomorphic distribution $\mathcal{F}$ of dimension k on $\mathbb{P}^n$. Now, take a generic non-invariant linearly embedded subspace $i : H \simeq \mathbb{P}^{n-k} \hookrightarrow \mathbb{P}^n$. We have an induced non-trivial section

$$i^*\omega \in \mathrm{H}^0(H, \Omega^{n-k}_H(r)) \simeq \mathrm{H}^0(\mathbb{P}^{n-k}, \mathcal{O}_{\mathbb{P}^{n-k}}(-n+k-1+r)),$$

since $\Omega^{n-k}_{\mathbb{P}^{n-k}} = \mathcal{O}_{\mathbb{P}^{n-k}}(-n+k-1)$. The tangency set between ω and H, denoted by $Z(i^*\omega)$, is defined as the hypersurface of zeros of $i^*\omega$ on H. The *degree* of $\mathcal{F}$, denoted by $\deg(\mathcal{F})$, is defined as the degree of $Z(i^*\omega)$ in H and, therefore, is given by

$$\deg(\mathcal{F}) = -n + k - 1 + r.$$

In particular, we obtain that $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+n-k+1))$, where $\deg(\mathcal{F}) = d$. Since $\det(N\mathcal{F}) \simeq \mathcal{O}_{\mathbb{P}^k}(r)$ and $\det(N\mathcal{F}) \simeq \det(T\mathbb{P}^n) \otimes \det(T\mathcal{F})^*$, then

$$c_1(\det(T\mathcal{F})) = \dim(\mathcal{F}) - \deg(\mathcal{F}).$$

4.1 Numerical invariants of the singular locus of distributions

Let $\mathcal{F}$ be a codimension one distribution on $\mathbb{P}^3$, with tangent sheaf $T\mathcal{F}$ and singular scheme Z. Let $\mathcal{U}$ denote the maximal 0-dimensional subsheaf of $\mathcal{O}_Z$, so that the quotient sheaf is the structure sheaf of a subscheme $C \subset Z \subset \mathbb{P}^3$ of pure dimension 1. We will say that C is the *1-dimensional component of Sing*$(\mathcal{F})$, while $\mathrm{Supp}(\mathcal{U})$ is the *0-dimensional component of Sing*$(\mathcal{F})$. We will denote $C := \mathrm{Sing}_1(\mathcal{F})$.

Theorem 4.1. *[5, Theorem 3.1] We have that*

$$\begin{aligned} c_2(T\mathcal{F}) &= d^2 + 2 - \deg(C) \\ c_3(T\mathcal{F}) = \mathrm{length}(\mathcal{U}) &= d^3 + 2d^2 + 2d - \deg(C)\cdot(3d-2) + 2p_a(C) - 2 \end{aligned} \tag{4.1}$$

where $p_a(C)$ denotes the arithmetic genus of C.

Similar formula for Fano 3-folds has been proved in [8]. We recall that $\mathrm{length}(\mathcal{U})$ is the number of isolated singularities of the distributions. We refer to [19, 41, 42, 35, 44] when the authors determine the number of isolated singularities of foliations by curves.

5 Moduli of distributions, singular schemes and classification of distributions of degree 1

We consider a morphism from the moduli space of the distributions $\mathcal{D}^P$ to a Hilbert scheme

$$\mathbf{S}^P : \mathcal{D}^P \to \mathrm{Hilb}^{R(P)} \quad , \quad \mathbf{S}^P([\mathcal{F}]) := [\mathrm{Sing}(\mathcal{F})], \tag{5.1}$$

where the polynomial Q is determined by P.

In [21] we prove the following result:

Theorem 5.1. *[21, Theorem A] Let X be a smooth projective scheme over an algebraically closed ground field κ of characteristic 0 with $\mathrm{Pic}(X) = \mathbb{Z}$. Let $\mathcal{D}^P$ be the moduli scheme parametrizing isomorphism classes of codimension one distributions on X with tangent sheaves having Hilbert polynomial equal to P. For any $\mathcal{F} \in \mathcal{D}^P(X)$, the fiber of the morphism Σ, parameterizing distributions with singular scheme equal to $Z \in Hilb^Q$ is a Zariski open subset of $\mathbb{P}\left(H^0(\Omega^1_X(c) \otimes \mathcal{I}_Z)\right)$ for some integer c.*

When $X = \mathbb{P}^n$ and Z is not contained in a hypersurface of degree less than or equal to d, then $H^0(\Omega^1_{\mathbb{P}^n}(d+2) \otimes \mathcal{I}_Z)$ is isomorphic to the space of linear first syzygies of the homogeneous ideal associated with Z.

This result allows us consider two important problems in the theory of holomorphic distributions related to properties of the morphism $\mathbf{S}^P$.

Gomez-Mont and Kempf [26] have considered the problem (a) for foliation by curves in projective spaces, with reduced, zero dimensional singular schemes. Campillo and Olivares showed in [9] that when the hypothesis that the singular scheme is reduced may be removed. In [1] we have provided sufficient conditions for distributions of arbitrary dimension to be uniquely determined by their singular schemes.

Theorem 5.1 has been used to give a full description of all the irreducible components of the moduli space of isomorphism classes of codimension one distributions of degree 1 on $\mathbb{P}^3$. By using Theorem 4.1 and investigating the stability of the tangent sheaf we can give a classification of such distributions in terms of the possible Chern classes. By [5] the classification of such distributions can be summarize in the following Table 5.1.

$c_2(T\mathcal{F})$	$c_3(T\mathcal{F})$	$T\mathcal{F}$	$\mathrm{Sing}(\mathcal{F})$
3	5	stable	5 points
2	2	stable	line and 2 points
1	1	stable	conic and 1 point
0	0	split	twisted cubic

Table 5.1: Classification of codimension one distributions of degree 1 on $\mathbb{P}^3$, according to [5, Section 8].

Let $\mathcal{D}(a,b)$ denote the moduli space of degree 1 distributions $\mathcal{F}$ such that $(c_2(T\mathcal{F}), c_3(T\mathcal{F})) = (a,b)$. We prove:

Theorem 5.2. *[21, Main Theorem 1] For each possible pair (a,b), the moduli space $\mathcal{D}(a,b)$ is an irreducible, smooth quasi-projective variety. In* *addition:*

- *the image of the morphism* $\Sigma\colon \mathcal{D}(0,0) \to \mathrm{Hilb}^{3t+1}(\mathbb{P}^3)$ *is an open subset of the aCM component of* $\mathrm{Hilb}^{3t+1}(\mathbb{P}^3)$*, and its fibres are open subsets of* $\mathbb{P}^1$*;* $\dim \mathcal{D}(0,0) = 13$.
- *the morphism* $\Sigma\colon \mathcal{D}(2,2) \to \mathrm{Hilb}^{t+3}(\mathbb{P}^3)$ *is dominant, and its fibres are open subsets of* $\mathbb{P}^5$*;* $\dim \mathcal{D}(2,2) = 15$.
- *the image of the morphism* $\Sigma\colon \mathcal{D}(1,1) \to \mathrm{Hilb}^{2t+2}(\mathbb{P}^3)$ *is an open subset of the non pure component of* $\mathrm{Hilb}^{2t+2}(\mathbb{P}^3)$*, and its fibres are open subsets of* $\mathbb{P}^3$*;* $\dim \mathcal{D}(1,1) = 14$.

In [5] we prove that the tangent sheaf of a codimension one distribution with only isolated singularities is always stable.

Theorem 5.3. *[5] Let* $\mathcal{F}$ *be a codimension one distribution on* $\mathbb{P}^3$ *of degree* $d > 0$*. If* $Sing(\mathcal{F})$ *is a 0-dimensional scheme, then* $T\mathcal{F}$ *is stable.*

In [6] we have proved that codimension one distribution with at most isolated singularities on certain smooth projective threefolds with Picard rank one have also stable tangent sheaves. As a consequence we provide a characterization of certain irreducible components of the moduli space of stable rank 2 reflexive sheaves on $\mathbb{P}^3$, and we are able to give the construction of stable rank 2 reflexive sheaves with prescribed Chern classes on general threefolds.

In [18] we classify non-integrable codimension one distributions, of degree one, and arbitrary class by applying a singular Darboux type theorem for homogeneous polynomial closed 2-forms of degree one. More precisely, we prove the following result.

Theorem 5.4. *Let* $\mathcal{F}$ *be a distribution on* $\mathbb{P}^n$ *of degree one and class* $\ell \geq 1$*. Then, there is a rational map* $\xi : \mathbb{P}^n \dashrightarrow \mathbb{P}(1^{\ell+1}, 2^{\ell+1})$ *determined by* $\ell+1$ *linear polynomials and* $\ell+1$ *quadratic polynomials, and a rational linear map* $\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^{2\ell+1}$ *such that* $\mathcal{F}$ *is induced by* $\rho^*\alpha + \xi^*\theta_0$*, where* $\alpha \in H^0(\mathbb{P}^{2k+1}, \Omega^1_{\mathbb{P}^{2\ell+1}}(3))$ *and* $\theta_0 = \sum_i (u_i dw_i - 2w_i du_i)$ *is the canonical contact distribution on* $\mathbb{P}(1^{\ell+1}, 2^{\ell+1})$.

6 Classification of degree zero rank 2 distributions

In [25] Glover, Homer and Stong classify non singular holomorphic distributions on complex projective spaces.

Theorem 6.1 (Glover–Homer–Stong)**.** *A holomorphic distribution $\mathcal{F}$ on $\mathbb{P}^n$ is non singular if and only if n is odd, $\mathcal{F}$ has codimension one and degree zero.*

A *null-correlation bundle* is a locally free sheaf E of rank 2 defined by the following exact sequence [36, Lemma 4.3.2]

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1) \to E \to 0;$$

Note that $E^{**} \simeq E$, since $\det(E) \simeq \mathcal{O}_{\mathbb{P}^3}$. Therefore, dualizing the above sequence, and twisting the result by $\mathcal{O}^3_{\mathbb{P}}(1)$ we obtain the following non singular dimension 2 distribution of degree 0 given by

$$0 \to E(1) \to T\mathbb{P}^3 \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0.$$

That is, the twisted null-correlation bundle $E(1)$ is a non-integrable sub-vector bundle of $T\mathbb{P}^3$, i.e., it is a *contact structure* on $\mathbb{P}^3$, see [5, Proposition 7.1].

Since any distribution $\mathcal{F}$ of dimension 2 and degree zero on $\mathbb{P}^n$, with $n > 3$, is given in homogeneous coordinate by a constant bi-vector σ of rank 2, then σ is decomposable and integrable, thus $\mathcal{F}$ is tangent to a linear rational projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}$. By Theorem 6.1, we have that $\mathcal{F}$ is non singular if and only if $n = 3$. Therefore, if $n = 3$ then $\mathcal{F}$ either is a pencil of planes or it is a non singular dimension 2 distribution, i.e., a holomorphic contact structure on $\mathbb{P}^3$. By this observation we have the following.

Proposition 6.2. *[7, Proposition 2] Let $\mathcal{F}$ be a distribution of dimension 2 and degree zero on $\mathbb{P}^n$. Then, $\mathcal{F}$ either is integrable and tangent to a linear rational projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}$ or is contact structure on $\mathbb{P}^3$ and its tangent sheaf is a twisted null-correlation bundle.*

6.1 Geometry of spaces of codimension one degree 0 distributions

In [2, Proposition 4.3] we have proved that a non-integrable distribution on $\mathbb{P}^n$ of degree zero and class $r \geq 1$ is a linear pull-back of a contact structure on $\mathbb{P}^{2r+1}$. Moreover, we prove the following: If $D_r \subseteq \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)))$ is the variety parametrizing codimension one distributions on $\mathbb{P}^n$ of class $\leq r$ and degree zero, then $D_r \simeq \mathrm{sec}_{r+1}(\mathbb{G}(1,n))$ and the stratification

$$D_0 \subseteq D_1 \subseteq ... \subseteq D_{r-1} \subseteq ... \subseteq \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)))$$

corresponds to the natural stratification

$$\mathbb{G}(1,n) \subseteq \mathrm{sec}_2(\mathbb{G}(1,n)) \subseteq ... \subseteq \mathbb{P}(\wedge^2\mathbb{C}^{n+1}) \simeq \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2))),$$

where $\mathrm{sec}_j(\mathbb{G}(1,n))$ is the *i-secant variety* of the Grassmannian $\mathbb{G}(1,n)$ of lines in $\mathbb{P}^n$.

7 Classification rank 2 distributions of degree 2 on $\mathbb{P}^3$

In [5] we have proposed a classification in terms of the Chern classes of tangent sheaf, and we provide a full classification in degrees 1 and 0. Moreover, we give a classification of codimension one distribution on $\mathbb{P}^3$ of degree 2 with locally free tangent sheaf and reduced singular scheme. We study codimension one holomorphic distributions on the projective three-space, analyzing the properties of their singular schemes and tangent sheaves. We describe the moduli space of distributions in terms of Grothendieck's Quot-scheme for the tangent bundle.

Recall that an *instanton bundle* on $\mathbb{P}^3$ is a stable rank 2 locally free sheaf E satisfying $h^1(E(-2)) = 0$. Moreover, $c_2(E)$ is called the *charge* of E. We prove the following result:

Theorem 7.1. *[5, Theorem pg 2] Let $\mathcal{F}$ be a codimension one distribution of degree* 2 *on* $\mathbb{P}^3$ *with locally free tangent sheaf* $T\mathcal{F}$*, and such that* $Sing(\mathcal{F})$ *is reduced, up to deformation. Then:*

1. *$T\mathcal{F}$ splits as a sum of line bundles and*
   - (a) *either $T\mathcal{F} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$, and $Sing(\mathcal{F})$ is a connected curve of degree* 7 *and arithmetic genus* 5*.*
   - (b) *or $T\mathcal{F} = \mathcal{O} \oplus \mathcal{O}$, and $Sing(\mathcal{F})$ is a connected curve of degree* 6 *and arithmetic genus* 3*.*
2. *$T\mathcal{F}$ is stable, and:*
   - (a) *either $T\mathcal{F}$ is a null-correlation bundle, and $Sing(\mathcal{F})$ is a connected curve of degree* 5 *and arithmetic genus* 1*; in addition, $\mathcal{F}$ possesses sub-foliations of degree* 2 *which are singular along two skew lines;*
   - (b) *or $T\mathcal{F}$ is an instanton bundle of charge* 2*, and $Sing(\mathcal{F})$ is the disjoint union of a line and a twisted cubic (or a degeneration of such curve); in addition, $\mathcal{F}$ possesses sub-foliations of degree* 2 *which are singular along three skew lines.*

In addition, this result is effective, in the sense that there exist injective morphisms $\phi : T\mathcal{F} \to T\mathbb{P}^3$ with torsion free cokernel for each of the possibilities listed above.

A complete classification of degree 2 codimension one distributions in $\mathbb{P}^3$ has been given by Galeano, Jardim and Muniz in [29]. They prove the following result.

Theorem 7.2. *[29, Main Theorem] Let $\mathcal{F}$ be a codimension one distribution of degree 2 on $\mathbb{P}^3$. Then $T\mathcal{F}$ is μ-semistable whenever it does not split as a sum of line bundles; it can be strictly μ-semistable only when* $(c_2(T\mathcal{F}), c_3(T\mathcal{F})) = (1, 2)$ *or* $(2, 4)$*. In addition, the second and third Chern classes of $T\mathcal{F}$ are listed in Table 7.1, where $Sing_1(\mathcal{F})$ is given.*

$c_2(T\mathcal{F})$	$c_3(T\mathcal{F})$	$Sing_1(\mathcal{F})$
6	20	empty
5	14	line
4	10	conic
	8	two skew lines
	6	double line of genus -2
3	8	plane cubic curve
	6	twisted cubic
	4	conic $\sqcup$ line
	2	three skew lines
	0	double line of genus $-2 \sqcup$ line
2	4	elliptic quartic curve
	2	rational quartic curve
	0	twisted cubic $\sqcup$ line
1	2	curve of degree 5, genus 2
	0	elliptic curve of degree 5
0	split	ACM curve of degree 6 genus 3
-1	split	ACM curve of degree 7 genus 5

Table 7.1: Classification of codimension one distributions of degree 2 on $\mathbb{P}^3$, according to [29, Section 8].

A *null-correlation distribution* $\mathcal{F}$ on $\mathbb{P}^3$ is a codimension one distribution whose tangent sheaf is isomorphic to a null-correlation bundle up to twist. To be more precise, assume that $T\mathcal{F} \simeq N(-a)$ for some null-correlation bundle N; one can check that $Hom(N(-a), T\mathbb{P}^3) = H^0(N \otimes T\mathbb{P}^3(a)) = 0$ for $a \leq -2$. The moduli space of null-correlation distributions, that is, those of the form

$$\mathcal{F} \ : \ 0 \to N(-a) \to T\mathbb{P}^3 \to I_Z(4+2a) \to 0,$$

denoted by $\mathcal{D}^{st}(2+2a, 1+a^2, 0)$.

The following result is a consequence of the Lemma 3.3.

Proposition 7.3. *[5, Proposition 11.1] $\mathcal{D}^{st}(2+2a, 1+a^2, 0)$ is an irreducible, nonsingular quasi-projective variety of dimension*

$$8\binom{a+4}{3} - 2\binom{a+3}{3} - 3a - 6.$$

Now, let $[\mathcal{F}] \in \mathcal{D}^{st}(2,2,0)$, so that $T\mathcal{F}$ is a stable rank 2 bundle with $c_2 = 2$; since every rank 2 bundle E on $\mathbb{P}^3$ with $(c_1(E), c_2(E)) = (0,2)$ is an instanton bundle of charge 2 [30, Section 9], it follows that $T\mathcal{F}$ is isomorphic to an instanton bundle of charge 2. Conversely, every rank 2 instanton bundle is the tangent sheaf of a codimension one distribution $[\mathcal{F}] \in \mathcal{D}^{st}(2,2,0)$. In other words, the forgetful morphism

$$\varpi : \mathcal{D}^{st}(2,2,0) \to \mathcal{M}(0,2,0)$$

is surjective. Therefore, the following result is also a consequence of Lemma 3.3.

Proposition 7.4. *[5, Proposition 11.3] $\mathcal{D}^{st}(2,2,0)$ is an irreducible, nonsingular quasi-projective variety of dimension 22.*

8 Classification of foliations by curves of degree ≤ 3 with curves as singular sets

The first step to the study foliations by curves on $\mathbb{P}^3$ with no isolated singularities and degree ≤ 3 was give in [20] . The study of such foliations is equivalent to assuming that the conormal sheaf is locally free and in this case we say that the foliations are of *local complete intersection type.* We provide a classification of such foliations by curves up to degree 3, also describing the possible singular schemes.

We prove the following result for foliations of degree $d \in \{1,2\}$:

Theorem 8.1. *[20, Main Theorem 1] Let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^3$ of degree $d \in \{1,2\}$. If $Sing(\mathcal{F})$ is a curve, then its conormal sheaf $N\mathcal{F}^*$ splits as a sum of line bundles. More precisely, we have that*

1. *if* $d = 1$*, then* $N\mathcal{F}^* \simeq \mathcal{O}(-2)^{\oplus 2}$ *and* $Sing(\mathcal{F})$ *consists of two disjoint lines;*

2. *if* $d = 2$*, then* $N\mathcal{F}^* \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-3)$ *and* $Sing(\mathcal{F})$ *is a connected curve of degree* 5 *and arithmetic genus* 1*.*

In particular, $\mathcal{F}$ *is contained in a pencil of planes or is Legendrian, and it is given by the complete intersection of two codimension one distributions.*

A foliation by curves is called *Legendrian* if it is a sub-distribution of a contact distribution on $\mathbb{P}^3$. In [22] the first author and I. Vainsencher provide formulas for the dimensions and degrees of the varieties of Legendrian foliations, and of the varieties of foliations tangent to a pencil of planes. For instance, the variety of Legendrian foliations of degree 2 has dimension **20** and degree **2224**.

We prove the following result for foliations by curves of degree 3.

Theorem 8.2. *[20, Main Theorem 2] Let* $\mathcal{F}$ *be a foliation by curves on* $\mathbb{P}^3$ *of degree* 3*. If* $\mathcal{F}$ *is of local complete intersection type, then one of the following possibilites hold:*

1. $N\mathcal{F}^* = \mathcal{O}(-2) \oplus \mathcal{O}(-4)$*, and* $Sing(\mathcal{F})$ *is a connected curve of degree* 10 *and arithmetic genus* 5*;*

2. $N\mathcal{F}^* = \mathcal{O}(-3)^{\oplus 2}$*, and* $Sing(\mathcal{F})$ *is a connected curve of degree* 9 *and arithmetic genus* 3*;*

3. $N\mathcal{F}^* = E(-3)$*, where* E *is stable rank 2 locally free sheaf with* $c_1(E) = 0$ *and* $1 \leq c_2(E) \leq 5$*; the singular scheme* $Sing(\mathcal{F})$ *is a curve of degree* $9 - c_2(E)$ *and arithmetic genus* $p_a(C) = 8 - 3c_2(E)$*.*

If, in addition, $Sing(\mathcal{F})$ *is reduced, then* $1 \leq c_2(E) \leq 4$*,* E *is an instanton bundle (though not a special 't Hooft instanton bundle of charge 3 or a 't Hooft instanton bundle of charge 4), and* $Sing(\mathcal{F})$ *is connected if and only if* $c_2(E) = 1, 2$*.*

Conversely, for each $n \in \{1, 2, 3, 4\}$*, there is foliation by curves* $\mathcal{F}$ *of degree 3 on* $\mathbb{P}^3$ *such that* $N\mathcal{F}^*(3)$ *is an instanton bundle of charge* n*.*

Recall that an instanton bundle E on $\mathbb{P}^3$ is said to be a *'t Hooft* instanton bundle if, $h^0(E(1)) \geq 1$, and a *special 't Hooft* instanton bundle if, in addition, $h^0(E(1)) = 2$. From Theorem 8.2 we conclude that there are 6 different types of degree 3 foliations by curves with only reduced curves as singular schemes.

9 Distributions of rank 2 on $\mathbb{P}^4$

In the recent work [7], we classify dimension two distributions, of degree at most 2, on $\mathbb{P}^4$ with either locally free tangent sheaf or locally free conormal sheaf and whose singular scheme has pure dimension one. We investigate the geometry of such distributions, studying from maximally non-integrable to integrable distributions.

Let $\mathcal{F}$ be a distribution of dimension 2 on a complex 4-fold. Roughly speaking, the non-integrability condition of $\mathcal{F}$ is measured by its associated *first derived distribution* defined as follows. Suppose that $\mathcal{F}$ is not integrable and define $\mathcal{F}^{[1]} := (T\mathcal{F} + [T\mathcal{F}, T\mathcal{F}])^{**} \subset TX$ the first derived distribution of $\mathcal{F}$. A maximally non-integrable distribution $\mathcal{F}$ of codimension 2 on a 4-fold X, called by *Engel structure*, is a distribution such that $\mathcal{F}^{[1]}$ is not integrable. Consider the O'Neill tensor

$$\begin{aligned} \mathcal{T}(\mathcal{F}^{[1]}) : \wedge^{[2]}\mathcal{F}^{[1]} &\longrightarrow TX/\mathcal{F}^{[1]} \\ u \wedge v &\longmapsto \pi([u, v]), \end{aligned}$$

where $\pi : \mathcal{F}^{[1]} \to TX/\mathcal{F}^{[1]}$ denotes the projection and $\wedge^{[2]}\mathcal{F}^{[1]} := (\wedge^2\mathcal{F}^{[1]})^{**}$. Then, there is an unique foliation by curves $\mathrm{Ker}(\mathcal{T}(\mathcal{F}^{[1]})^{**} := \mathcal{L}(\mathcal{F}) \subset \mathcal{F}^{[1]}$, called by *characteristic foliation* of $\mathcal{F}^{[1]}$, such that $[\mathcal{L}(\mathcal{F}^{[1]}), \mathcal{F}^{[1]}] \subset \mathcal{F}^{[1]}$.

Definition 9.1. (Cartan prolongation) Let $\mathcal{F}$ be a contact singular structure on a threefold X such that $T\mathcal{F}$ is locally free. Consider the 4-fold $\mathbb{P}(T\mathcal{F}^*)$ with the natural projection $\pi : Y := \mathbb{P}(T\mathcal{F}^*) \to X$ and the twisted relative Euler sequence

$$0 \to \mathcal{O}_X(-1) \to \pi^*T\mathcal{F} \to T_{X|Y}(-1) \to 0$$

The pull-back $\pi^*\mathcal{F}$ is codimension one non-integrable distribution, such that the $T_{X|Y} \subset T\pi^*\mathcal{F}$. The kernel $\mathcal{D}$ of the composition

$$T\pi^*\mathcal{F} \to T\mathcal{F} \to T_{X|Y}(-1)$$

is an Engel structure called by Cartan prolongation of $\mathcal{F}$.

Definition 9.2. (Lorentzian type structures) Let $\mathcal{F}$ be a Engel structure on a 4-fold X. Suppose that there is a fibration $f : X \dashrightarrow Y$ by rational curves, over a projective manifold Y. We say that $\mathcal{F}$ is of Lorentzian type if $T_{X/Y} \subset T\mathcal{F}$, i.e, fibers of f are tangent to $\mathcal{F}$, and the characteristic foliation $\mathcal{L}(\mathcal{F})$ is induced by a section of $f^*TY \otimes T_{X/Y}$. We say that the rational map $f : X \dashrightarrow Y$ is Lorentzian with respect to $\mathcal{F}$. See [17, 43] for more details on Engel distributions.

In [7] we prove the following result.

Theorem 9.3. *[7, Theorem 1] Let $\mathcal{F}$ be a dimension two distribution of degree $d \in \{1, 2\}$ with locally free tangent sheaf $T\mathcal{F}$ and whose singular scheme has pure dimension one. Then $T\mathcal{F}$ splits as a sum of line bundles, and one of the following cases hold:*

1. *$T\mathcal{F} = \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}$ and its singular scheme is a rational normal curve of degree 4.*
2. *$T\mathcal{F} = \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}$, and its singular scheme is an arithmetically Cohen–Macaulay curve of degree 10 and arithmetic genus 6.*
3. *$T\mathcal{F} = \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(-1)$, and its singular scheme is an arithmetically Cohen-Macaulay curve of degree 15 and arithmetic genus 17.*

If $\mathcal{F}$ is integrable, then it is either a linear pull-back of a degree d foliation by curve on $\mathbb{P}^3$ or $T\mathcal{F} \simeq \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^4}$, where either $\mathfrak{g}$ is an abelian Lie algebra of dimension 2 or $\mathfrak{g} \simeq \mathfrak{aff}(\mathbb{C})$.
If $\mathcal{F}$ is not integrable and $\mathcal{F}^{[1]}$ is integrable, denote $\mathfrak{f}^{[1]} := H^0(\mathcal{F}^{[1]})$ the Lie algebra of rational vector fields tangent to $\mathcal{F}^{[1]}$. Then one of the following cases hold:

a) $\mathcal{F}^{[1]}$ *is linear pull-back of a foliation, of degree* $\leq d$*, on* $\mathbb{P}^2$ *and* $\dim[\mathfrak{f}^{[1]}, \mathfrak{f}^{[1]}] = 1$.

b) $\mathcal{F}^{[1]}$ *has degree* $\in \{1, 2, 3\}$*, it is a linear pull-back of a foliation on* $\mathbb{P}^3$ *and* $\dim[\mathfrak{f}^{[1]}, \mathfrak{f}^{[1]}] \leq 2$.

c) $\mathcal{F}^{[1]}$ *has degree* 3 *and* $T\mathcal{F}^{[1]} \simeq \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^4}$*, where* $\mathfrak{g}$ *is a non-abelian Lie algebra of dimension* 3*. Moreover, one of the following holds:*

- *if* $\dim \mathfrak{f}^{[1]} = 1$*, then* $\mathfrak{f}^{[1]} \simeq \mathfrak{aff}(\mathbb{C}) \oplus \mathbb{C}$;
- *if* $\dim \mathfrak{f}^{[1]} = 2$*, then* $\mathfrak{f}^{[1]} \simeq \mathfrak{r}_{3,\lambda}(\mathbb{C}) := \{[v_1, v_2] = v_2;\ [v_1, v_3] = \lambda v_3;\ [v_2, v_3] = 0\}$*, where* $\lambda \in \mathbb{C}^*$ *with* $\lambda = -1$*, or* $0 < |\lambda| < 1$*, or* $|\lambda| = 1$ *and* $\mathfrak{Im}(\lambda) > 0$;
- *if* $\dim \mathfrak{f}^{[1]} = 3$ *, then* $\mathfrak{f}^{[1]} \simeq \mathfrak{sl}(2, \mathbb{C})$.

If $\mathcal{F}$ *is an Engel structure. Then:*

d) $\mathcal{F}$ *is the blow-down of the Cartan prolongation of a singular contact struture of degree* 1, 2 *or* 3 *on* $\mathbb{P}^3$.

e) $\mathcal{F}$ *has degree* 2 *and it is the blow-down of the Cartan prolongation of a singular contact structure on a weighted projective* 3*-space.*

f) $\mathcal{F}$ *has degree* 2 *and it is of Lorentzian type.*

In [14, Table I, pg 104] appear certain Pfaff systems in $\mathbb{P}^4$ whose singular schemes are very special algebraic varieties. We recall that variety $Z \subset \mathbb{P}^n$ is called arithmetically Buchsbaum (Buchsbaum for short) if the multiplication on the graded module $H^i(Z, \mathcal{I}_{Z\cap H}(m))$ is trivial for all linear space H and $m \in \mathbb{Z}$.

Theorem 9.4. *Let* Z *be a smooth non-general type Buchsbaum surface on* $\mathbb{P}^4$ *such that* $K_Z^2 \notin \{-2, -7\}$*. Then, we have the following possibilities:*

1. Z *is a singular locus of a Pfaff system of dimension* 2 *and degree* 2 *and it is a quintic elliptic scroll.*

2. *Z is a singular locus of a foliation by curves of degree* 1 *and it is a projected Veronese surface.*

3. *Z is a singular locus of a Pfaff system of dimension* 2 *and degree* 3 *and it is a* $K3$ *surface of genus* 8.

4. *Z is a singular locus of a foliation by curves of degree* 3 *and it is a* $K3$ *surface of genus* 7.

We also have investigate holomorphic distributions of dimension 2 in $\mathbb{P}^4$ whose tangent sheaves are not split. The only known non-decomposable vector bundles of rank 2 are the so-called Horrocks-Mumford bundles [31].

Theorem 9.5. *[7, Theorem 2] Let $\mathcal{F}_a$ be a dimension* 2 *Horrocks-Mumford holomorphic distribution $E(-a-7) \to \Omega^1_{\mathbb{P}^4}$. Then:*

1. *$\mathcal{F}_a$ is a degree $2a+5$ maximally non-integrable distribution, for all $a \geq 1$.*

2. *$Z_a = \mathrm{Sing}(\mathcal{F}_a)$ is a smooth and connected curve with the following numerical invariants*

$$\deg(Z_a) = 4a^3 + 33a^2 + 77a + 46,$$

$$p_a(Z_a) = 9a^4 + 89a^3 + \frac{553}{2}a^2 + \frac{573}{2}a + 45.$$

Moreover, Z_a is never contained in a hypersurface of degree $\leq 2a + 5 = \deg(\mathcal{F}_a)$.

3. *If $\mathcal{F}'$ is a dimension two distribution on $\mathbb{P}^4$, with degree $2a+5$, such that $\mathrm{Sing}(\mathcal{F}_a) \subset \mathrm{Sing}(\mathcal{F}')$, then $\mathcal{F}' = \mathcal{F}_a$.*

4. *$\mathcal{F}_a$ is invariant by a group $\Gamma_{1,5} \simeq H_5 \rtimes SL(2,\mathbb{Z}_5) \subset Sp(4,\mathbb{Q})$, where H_5 is the Heisenberg group of level* 5 *generated by*

$$\sigma : z_k \to z_{k-1} \text{ and } \tau : z_k \to \epsilon^{-k} z_k, \text{ with } k \in \mathbb{Z}_5 \text{ and } \epsilon = e^{\frac{2\pi i}{5}}.$$

Denote by $\mathrm{HM}\mathcal{D}ist(2a+5)$ the moduli spaces of Horrocks-Munford Holomorphic distributions of degree $2a+5$.

Theorem 9.6. *[7, Theorem 3] The moduli space* $\mathrm{HM}\mathcal{D}ist(2a+5)$ *of dimension* 2 *Horrocks-Mumford holomorphic distributions, of degree* $2a+5$*, is an irreducible quasi-projective variety of dimension*

$$\frac{1}{3}a^4 + 7a^3 + \frac{277}{6}a^2 + \frac{199}{2}a + 43$$

for all $a \geq 1$*.*

We also show that the space of codimension one distributions, of degree $d \geq 6$, on $\mathbb{P}^4$ have a family of degenerated flat holomorphic Riemannian metrics. Moreover, the degeneracy divisors of such metrics consist of codimension one distributions invariant by $H_5 \rtimes SL(2,\mathbb{Z}_5)$ and singular along a degenerate abelian surface with $(1,5)$-polarization and level-5-structure.

Theorem 9.7. *[7, Theorem 4] There exist a family* $\{g_\phi\}_{\phi\in\mathcal{A}}$ *of* $\Gamma_{1,5}$*-equivariant flat holomorphic conformal structure on the space of distributions of codimension one and degree* $d \geq 6$*, where* $\mathcal{A} \subset \mathbb{P}^M$ *is a Zariski open with*

$$M = \frac{1}{3}(d-5)^4 + \frac{23}{3}(d-5)^3 + \frac{343}{6}(d-5)^2 + \frac{899}{6}(d-5) + 74,$$

such that :

1. *there is a rational map* $\pi_\phi : \mathbb{P}H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(d+2)) \dashrightarrow \mathcal{H}_2/\Gamma_{1,5}$ *with rational fibers, a rational section* s_ϕ *whose image consists of codimension one distributions, of degree* d*, invariant by* $\Gamma_{1,5}$ *and singular along to an abelian surface with* $(1,5)$*-polarization and level-*5*-structure.*

2. $g_\phi = \pi_\phi^* g_0$*, where* g_0 *is the flat holomorphic conformal structure of* $\mathcal{H}_2/\Gamma_{1,5}$ *degenerating along a hypersurface* Δ_ϕ *of degree* 10 *which is a cone over a rational sextic curve in* $\mathbb{P}^3$*, and* Δ_ϕ *consists of codimension one distributions, of degree* d*, invariant by* $\Gamma_{1,5}$ *and singular along to either:*

- *a translation scroll associated to a normal elliptic quintic curve;*
- *or the tangent scroll of a normal elliptic quintic curve;*
- *or a quintic elliptic scroll carrying a multiplicity-2 structure;*
- *or a union of five smooth quadric surfaces;*
- *or a union of five planes with a multiplicity-2 structure.*

Acknowledgments

The author thanks the anonymous referees for giving many suggestions that helped improving the presentation of the survey. The author was partially supported by the Università degli Studi di Bari.

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