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Topology of first integrals via Milnor fibrations

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Abstract: In this survey, we consider a system of partial differential equations from the viewpoint of singularity theory. A topological classification is given to the first integrals of such systems using a powerful tool known as Milnor fibrations.
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1 Introduction

Differential equations remain a fundamental part of mathematics and continue to play a crucial role in various fields with a multitude of applications. These equations provide insight into the behavior of systems modeled by one or multiple dependent variables and as such play a crucial

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role in comprehending the behavior of many physical, biological, financial, and engineering systems. The study of differential equations continues to be an important and ever-evolving area of research, with ongoing advancements that deepen our understanding of these complicated systems.

The integrability of differential equations is one of the most important problems in the area and has a rich history, starting from the inception of Newtonian mechanics [21]. Over the centuries, this field has evolved into a complex and diverse area of study, incorporating a multitude of approaches and theories, one of the key contributions being Liouville's pioneering examination of Riccati equations in 1841 [15]. This work marked an essential moment in the development of integrability theory, as it demonstrated the limitations of traditional methods, such as the inability to integrate simple Riccati equations by quadratures. This realization prompted researchers to explore new theories and methods to determine the integrability of dynamical systems. The main one is the existence of *first integrals*, which are (non-constant) functions that remain constant along the solution curves of the equation.

In physics, the presence of the first integrals is often referred to as the constant of motions, which are crucial in determining the stability of a physical system. In the field of classical mechanics, for instance, the well-known Hamiltonian formalism [9] provides a framework for understanding the integrability of systems through the existence of first integrals. Additionally, the Liouville-Arnold theorem [3, 16] states that under certain conditions, a system is integrable if and only if it has n-independent first integrals.

The 19th and 20th centuries saw a significant expansion of the study of first integrals, particularly with the advent of foliation theory (for instance, see [4, 5]). Roughly speaking, a foliation can be visualized as the pages of a book, with each page representing a separate leaf. A first integral for a foliation is a function f on the manifold M such that the level sets of f coincide with the leaves of the foliation, meaning that f is constant on each leaf of the foliation.

Foliation theory is closely related to differential equations. For instance, the integral curves of a vector field and its associated flow dynamics can be viewed as a one-dimensional foliation. The first integral of foliations provides a tool for understanding the dynamics of systems by dividing the phase space into simpler connected components, referred to as fibers (or level sets), as they coincide with the leaves of the foliation. Often, the existence (or non-existence) of first integrals for foliations is referred to as the integrability problem (see [26] and the references therein).

Singularities arise in mathematics in a variety of contexts. In the field of differential equations, the examination of singularities has its roots in the works of Poincaré [22, 23, 24], who studied the stability and bifurcations of periodic orbits in the early 20th century. Since then, there has been significant progress in the field. The examination of singularities in dynamical systems and foliations has resulted in substantial advances, particularly in the classification of systems and the characterization of chaotic behavior, as well as solving problems in the theory of differential equations (see for instance [10]).

In this paper, we examine partial differential systems through the lens of singularity theory. Our comprehensive analysis of the topological classification of the fibers of first integrals highlights the importance of considering the singularity theory for improving our understanding of the integrability problem for differential equations. The results presented in this work serve as a demonstration of the value that singularity theory can bring to the field of differential equations.

Although the concept of integrability is well-defined for ordinary differential equations (see for instance [2, 3]), a universal definition for other structures, such as partial differential equations, has yet to be accepted. Nevertheless, there is a special type of partial differential system, that has a precise definition of integrability and it can be determined by analyzing the existence of first integrals.

Consider a system of first-order quasi-linear partial differential equations, given in local coordinates by:

$$\sum_{i=1}^{n} a_i^{\lambda}(x,u)\frac{\partial u}{\partial x_i} = b^{\lambda}(x,u), \tag{1.1}$$

where u is one unknown function of n independent variables $x = (x_1, \dots, x_n)$, $\lambda = 1, \dots, q$, with $1 \leq q \leq n$, and a_i^{λ} and b^{λ} are analytic functions defined on an open set $\Omega \subset \mathbb{R}^{n+1} = (x_1, \dots, x_n, u)$. The system (1.1) is said to be a *completely integrable system* or *integrable system* if it possesses p functionally independent first integrals[1] $f_1, f_2, \dots, f_p : \Omega \to \mathbb{R}$, where $p = n + 1 - q$.

Note that the domain of each f_j should be suitably adjusted, as the domain of definition of the solutions of the system (1.1) cannot be the entire space $\mathbb{R}^n$. However, for the purposes of the current discussion, we are only interested in the local situation, i.e., the topological classification around a singular point. In this context, we are concerned with the germs of functions, maps, vector fields, etc.

Our main strategy will be to consider the map germ

$$F_{(1.1)} := (f_1, \dots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0),$$

with $1 \leq p \leq n$ which we call the *first integrals map germ of system* (1.1), for short, *first integrals map* (see Definition 2.4). In fact, we aim to describe the topology of the fibers of the first integrals map. For this reason, we will make use of one of the most powerful tools of singularity theory for the study of spaces and maps: the fibered structures are known as *Milnor fibrations*. The existence of local Milnor fibrations for real and complex singularities is fundamental in the study of the topology of singularities.

The paper is structured as follows. Section 2 introduces the concept of a first integrals map germ and other relevant definitions. The definition of the singular set of a map germ is also reviewed, along with some pertinent remarks. In Section 3, we demonstrate the presence of obstructions for

[1] The precise notion of first integrals of a system of first-order quasi-linear partial differential equations is provided in Definition 2.1 in Section 2.

a system of first-order quasi-linear partial differential equations to admit first integrals map germs, along with several informative examples. Finally, in the last section, the topology of Milnor's fibers is leveraged to provide a topological characterization of $F_{(1.1)}$. As a result, the relationship between the topology of the Milnor's fibers of the first integrals map and the topology of the phase space of system (1.1).

2 The first integrals map germ for quasi-linear system

Consider a system of quasi-linear partial differential equations of first-order given by

$$\sum_{i=1}^{n} a_i^\lambda(x,u)\frac{\partial u}{\partial x_i} = b^\lambda(x,u), \tag{$*$}$$

in n independent variables $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, where the dependent variable $u(x_1, x_2, \dots, x_n)$ is a function which is a solution of ($*$), and a_i^λ, b^λ are not all zero analytic coefficients.

In the case of studying the relationship between a partial differential equation and its first integrals, it is often convenient to consider u as a coordinate along with the independent variables $x_1, x_2, \dots, x_n$ in $\mathbb{R}^{n+1}$. In this context, u is treated as a coordinate in the $(n+1)$-dimensional space. (see for instance Chapter 1 in [18] or [11, 12]).

Having said that, consider a_i^λ and b^λ analytic functions defined on an open set $\Omega \subset \mathbb{R}^{n+1} = \{(x_1, \dots, x_n, u)\}$ for $i = 1\dots, n$ and $\lambda = 1, \dots, q$, with $1 \leq q \leq n$.

Now, consider the *characteristic* vector fields

$$X^\lambda := \sum_{i=1}^{n} a_i^\lambda(x,u)\frac{\partial}{\partial x_i} + b^\lambda(x,u)\frac{\partial}{\partial u},$$

for each $\lambda = 1, \dots, q$.

Definition 2.1. A function $f : \Omega \to \mathbb{R}$ is a *first integral* of system ($*$) if f is constant along the integral curves of each X^λ, for all λ. That is,

f is a first integral of $(*)$ if $df(X^\lambda) \equiv 0, \forall \lambda$, where d denote the exterior derivative.

It follows from the classical method of characteristics due to Monge [20], that the graph of a solution u of $(*)$ is the union of the integral curves of the characteristic vector fields as detailed in [11, 12]. By Definition 2.1, a first integral f is a function that is constant along the integral curves of each X^λ. As a result, the graph of u is contained in the fibers $f^{-1}(\delta)$, where δ is a real constant. Therefore, if we have a sufficient number of first integrals $f_1, f_2 \dots, f_p$, we can completely describe the graph of the solutions of the system.

Let us remind that a set of functions $f_1, \dots, f_k$ are *functionally independent* if $df_1 \wedge \dots \wedge df_k \not\equiv 0$.

Definition 2.2. We say that $f_1, \dots, f_p$ is a *complete system of first integrals* for the system $(*)$, if $f_1, \dots, f_p$ are functionally independent first integrals and $p = n + 1 - q$. In this case, we say that $(*)$ is a *completely integrable system* or a *integrable system.*

Example 2.3. Consider the system of first-order quasi-linear pde's

$$\begin{cases} 2x_3 u \frac{\partial u}{\partial x_2} - 3u(x_1^2 + x_2^2)\frac{\partial u}{\partial x_3} = -3x_3(x_1^2 + x_2^2) \\ u\frac{\partial u}{\partial x_2} + u^3 \frac{\partial u}{\partial x_3} = -\frac{3}{2}(x_1^2 + x_2^2) - ux_3. \end{cases} \tag{2.1}$$

This quasi-linear system has two first integrals given by $f_1(x_1, x_2, x_3, u) = x_1$ and $f_2(x_1, x_2, x_3, u) = 3x_1^2 x_2 + x_2^3 + x_3^2 + u^2$, which are functionally independent. Indeed, consider the characteristic vector fields

$$X^1 = 2x_3 u \frac{\partial}{\partial x_2} - 3u(x_1^2 + x_2^2)\frac{\partial}{\partial x_3} - 3x_3(x_1^2 + x_2^2)\frac{\partial}{\partial u}$$

and

$$X^2 = u\frac{\partial}{\partial x_2} + u^3 \frac{\partial}{\partial x_3} - \left(\frac{3}{2}(x_1^2 + x_2^2) + ux_3\right)\frac{\partial}{\partial u}.$$

Through a straightforward computation we can obtain $df_i(X^\lambda) = 0$ for every $i, \lambda \in \{1, 2\}$ and $df_1 \wedge df_2 \neq 0$. Since $p = 2, q = 2, n = 3$ then $p = n + 1 - q$ and (2.1) is a completely integrable system.

Definition 2.4. Let $f_1, \ldots, f_p$ be first integrals functionally independent[2] for the system $(*)$ for some $p \geq 1$. The map germ

$$F_{(*)} := (f_1, f_2 \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0),$$

is called the *first integrals map germ of system* $(*)$, for short, *first integrals map*.

In this work, we will only consider the first integrals map $F_{(*)}$ which are analytic and their representatives carry the origin of the coordinate system into the origin.

Definition 2.5. Consider a first integrals map of system $(*)$, $F_{(*)} := (f_1, f_2 \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$. The singular set of $F_{(*)}$, is given by

$$\operatorname{Sing} F_{(*)} := \{x \in U : \operatorname{rank}(dF_{(*)}(x)) < p\},$$

where $dF_{(*)}(x)$ denotes the Jacobian matrix of $F_{(*)}$ at x, and $U \subset \mathbb{R}^m$ is an open set with $0 \in U$.

Example 2.6. The quasilinear equation

$$(3x_2u^2 + u)\frac{\partial u}{\partial x_1} + ux_1(3u-2)\frac{\partial u}{\partial x_2} = -(2x_2+1)x_1 \qquad (2.2)$$

admits a first integrals map $F_{(2.2)} = (f_1, f_2)$ where $f_1 = x_1^2 - x_2^2 + u^2$ and $f_2 = x_1^2 + x_2 + u^3$. Moreover, $\dim \operatorname{Sing} F_{(2.2)} > 0$.

When the singular set $\operatorname{Sing} F_{(*)} = \{0\}$ as a set germ, we say that $F_{(*)}$ has an *isolated singularity at the origin.* This condition means that there exists a neighborhood U of the origin in $\mathbb{R}^{n+1}$ such that the jacobian matrix $dF_{(*)}(x)$ has rank p at all points $x \in U$ other than the point $x = 0$.

Example 2.7. In Example 2.3 we have that $F_{(2.2)} = (f_1, f_2)$ is a first integrals map with $\operatorname{Sing} F = \{0\}$.

[2] Not necessarily a complete system of first integrals for the system $(*)$.

3 Singular set and obstructions of a first integrals map germ

Definition 3.1. A map germ $F := (f_1, \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^n, 0)$ is (x_α^j)*-separable* if f_j only depends on the variable x_α.

Example 3.2. The equation

$$-2x_1 u \frac{\partial u}{\partial x_2} = x_1^2 + 3x_2^2 \tag{3.1}$$

admits $F_{(3.1)} = (x_1, x_2(x_1^2 + x_2^2) + x_1 u^2)$ as a first integrals map wich is (x_1^1)-separable and has singular set with positive dimension.

Lemma 3.3. *A first integrals map* $F_{(*)} := (f_1, f_2 \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ *is* (x_α^j)*-separable then* $a_\alpha^\lambda \equiv 0$ *for any* $\lambda = 1, \ldots, q$ *and some* $\alpha \in \{1, \ldots, n+1\}$ *fixed.*

Proof. Remember that we denote $x_{n+1} := u$ and $a_{n+1}^\lambda := b^\lambda$ for the sake of simplicity. It follows from hypothesis that $f_j = f_j(x_\alpha)$ for some $j \in \{1, \ldots, p\}$ and $\alpha \in \{1, \ldots, n+1\}$. Then $\frac{\partial}{\partial x_i} f_j = 0$ for every $i \neq \alpha$. Since f_j is a first integral of $(*)$, it is non-constant and

$$0 = \sum_{i=1}^{n} a_j^\lambda(x, u) \frac{\partial f_j(x_\alpha)}{\partial x_i} + b^\lambda(x, u) \frac{\partial f_j(x_\alpha)}{\partial u} = a_\alpha^\lambda(x, u) \frac{\partial f_j(x_\alpha)}{\partial x_\alpha}.$$

Consequently, $a_\alpha^\lambda \equiv 0$. □

Remark 3.4. It follows immediately from Lemma 3.3 that if a first integral f_j of the system $(*)$ depends only on variable u, i.e., the first integrals map $F_{(*)}$ is $\left(x_{(n+1)}^j\right)$-separable, then $(*)$, becomes the homogeneous system

$$\sum_{i=1}^{n} a_i^\lambda(x, u) \frac{\partial u}{\partial x_i} = 0. \tag{$*$-homogeneous}$$

Proposition 3.5. *Let* $F := (f_1, f_2 \dots, f_n) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^n, 0)$*, be an analytic map germ such that* $f_1, f_2 \dots, f_n$ *are functionally independent. If for each* j *there exists* $\alpha(j)$ *such that* F *is* $(x^j_{\alpha(i)})$*-separable, then* F *cannot be a first integrals map of a system of first-order quasi-linear partial differential equations.*

Proof. Since $f_1, f_2 \dots, f_n$ are functionally independent, $\alpha(l) \neq \alpha(k)$ for any $l \neq k$. Therefore, without lost of generality we can assume that F is (x^i_i)-separable, for any $i = 1, \dots, n$. It follows from Lemma 3.3 that $a^\lambda_i \equiv 0$ for any $i = 1, \dots, n$ and $\lambda = 1, \dots, q$. Consequently, $b^\lambda \equiv 0$. □

4 The Milnor fibration for first integrals maps: isolated case

Let us consider a first integrals map $F_{(*)} := (f_1, f_2 \dots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ which has an isolated singularity at the origin. Let $V_{F_{(*)}} := F^{-1}_{(*)}(0)$. Since $V_{F_{(*)}} = f_1^{-1}(0) \cap f_2^{-1}(0) \cap \cdots \cap f_p^{-1}(0)$, one has that it is an analytic set, which is a smooth manifold of dimension n everywhere in $\left(V_{F_{(*)}} \setminus \{0\}\right) \cap U$.

In the famous book [19, §11], J. Milnor, proved the existence of a smooth fiber bundle in a neighborhood of the isolated singular point $x = 0$. In this section, we will use Milnor's result to study the topology of manifolds $M_j := f_1^{-1}(y_1) \cap f_2^{-1}(y_2) \cap \cdots \cap f_j^{-1}(y_j)$, where $1 \leq j \leq p$ and $y_1, \dots, y_j$ are arbitrary and small enough real numbers, such that $y_1^2 + \cdots + y_j^2 \neq 0$.

The result about the existence of a Milnor fibration can be state for a first integrals map of system $(*)$ as follows:

Theorem 4.1. *[19, Theorem 11.2] Let* $F_{(*)} = (f_1, f_2 \dots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ *be a first integrals map, with* $p \geq 2$*, which has an isolated singularity at the origin. There exist* $\epsilon_0 > 0$*, such that for all* $0 < \epsilon \leq \epsilon_0$ *the restricted map.*

$$F_{(*)_|} : B^{n+1}_\epsilon \cap F^{-1}_{(*)}(S^{p-1}_\eta) \to S^{p-1}_\eta \tag{4.1}$$

is a projection of a locally trivial smooth fibration, for all $0 < \eta \ll \epsilon$*. Moreover, the diffeomorphism type does not depend on the choice of* ϵ *and* η*.*

The projection (4.1) is known by *Milnor tube fibration* and its general fiber, $M_{F_{(*)}} := {F_{(*)}}^{-1}(y)$, where $y = (y_1, \dots, y_p) \in S_\eta^{p-1}$, is known as *Milnor fiber*. An immediate consequence of the proof of Theorem 4.1 is that the Milnor fiber is diffeomorphic to the manifold $B_\epsilon^{n+1} \cap M_p$, where $M_p := \bigcap_{k=1}^{p} f_k^{-1}(y_k)$.

Let us point out that if $F_{(*)}$ has isolated singularity at the origin then the first integrals map $F_{(*)}^j := (f_1, f_2 \dots, f_j) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^j, 0)$, with $2 \leq j < p$ has isolated singularity at the origin as well since $\operatorname{Sing} F_{(*)}^j \subset \operatorname{Sing} F_{(*)}$. Consequently, $F_{(*)}^j$ has a Milnor tube fibration, and again, the Milnor fiber $M_{F_{(*)}}^j$ is diffeomorphic to the manifold $B_\epsilon^{n+1} \cap M_j$.

4.1 Topology of integral manifolds

The existence of local fibration structures in the neighborhood of isolated singularities of the first integral maps $F_{(*)}$ and $F_{(*)}^j$ allows the comparison of the constructed manifolds M_p and M_j, which are intersections of integral manifolds. This is the purpose of this section, where we will apply the results of [8] and [13] to first integrals maps.

In what follows, whenever we use the notation M_j, we are referring to the representative $B_\epsilon^{n+1} \cap M_j$ of the germ $(M_j, 0)$, for $\epsilon > 0$ small enough and $j = 1, \dots, p$.

Let us remind the next definition.

Definition 4.2. Consider a first integral $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ of the system $(*)$. A manifold N in a domain B_ϵ^{n+1} for small enough $\epsilon > 0$ is a *integral manifold* of $(*)$ if N is a level set of f, in other words, a fiber of f.

As we have seen in Section 2, if we have a sufficient number of first integrals $f_1, f_2 \dots, f_p$ and we know the integral manifolds $N_j := f_j^{-1}(y_j)$, for any $j = 1, \ldots, p$, and y_j arbitrary small enough real numbers, not all equal to zero, one can describe the graph of the solutions of the system, provided we are able to describe locally, the topology of the intersection $\bigcap_{k=1}^{j} N_k$, for any $j = 1, \ldots, p$, that means, the topology of manifolds $M_{j's}$. Therefore, from the above discussion, it is enough to consider the Milnor fibers associated with a first integrals map with an isolated singularity at the origin.

Let $F_{(*)} = (f_1, f_2 \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ be a first integrals map, which has an isolated singularity at the origin. Then, each first integral f_j, for $j = 1, \ldots, p$, also has an isolated critical point at the origin. Let ∇f_j the gradient vector field of f_j and $\deg_0 \nabla f_j$ the topological degree of the mapping

$$\varepsilon \frac{\nabla f_j}{\|\nabla f_j\|} : S^n_\varepsilon \to S^n_\varepsilon,$$

for $j = 1, \ldots, p$.

Using these notations, in [8] the authors extended the PoincarÃ©-Hopf formula for a real analytic function given in [14] by Khimshiashvili, which for first integrals map, reads:

Theorem 4.3. *[8, Corollary 3.4] Let* $F_{(*)} = (f_1, f_2 \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ *be a first integrals map, which has an isolated singularity at the origin. The following holds true:*

(i) If $n+1$ *is even, then* $\chi(M_p) = 1 - \deg_0 \nabla f_1$*. Moreover, one has* $\deg_0 \nabla f_1 = \deg_0 \nabla f_2 = \ldots = \deg_0 \nabla f_p$.

(ii) If $n+1$ *is odd, then* $\chi(M_p) = 1$*. Moreover, one has* $\deg_0 \nabla f_i = 0$ *for* $i = 1, 2, \ldots, p$.

Example 4.4. Consider the first integrals map $F_{(2.1)} = (f_1, f_2)$ given in Example 2.3, where $f_1(x_1, x_2, x_3, u) = x_1$ and $f_2(x_1, x_2, x_3, u) = 3x_1^2 x_2 + x_2^3 + x_3^2 + u^2$. Since $F_{(2.1)}$ has isolated singularity at the origin, it follows from item (i) of the Theorem 4.3 that $\deg_0 \nabla f_2 = \deg_0 \nabla f_1 = 0$ and $\chi(M_2) = 1$. In other words, the graph of a solution u of the system (2.1) are contained in a manifold with Euler characteristic one.

Next, we present the main result of [13], which in our setting, helps to compare the intersections of integral manifolds.

Theorem 4.5. *[13, Corollary 7] The manifold M_j is homeomorphic to $M_p \times B^{p-j}$, for any $j = 1, \ldots, p$.*

The next result shows that under the hypothesis of singularity isolated at the origin, the Euler characteristic of each integral manifold is equal to the Euler characteristic of the Milnor fiber associated with the first integrals map.

Corollary 4.6. *[7, Corollary 6.3] Let $F_{(*)} = (f_1, f_2 \ldots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ be a first integrals map, with $p \geq 2$, which has an isolated singularity at the origin. Then $\chi(M_p) = \chi(M_j)$, for any $j = 1, \ldots, p$. In particular, $\chi(N_j) = \chi(M_p)$, for any $j = 1, \ldots, p$.*

Remark 4.7. Let us point out that for any $j = 1, \ldots, p$, the first integral $f_j : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ of the system $(*)$ admits two Milnor fibers, namely: $N_j = f^{-1}(\delta) \cap B_\epsilon^{n+1}$ and $N_j^- := f^{-1}(-\delta) \cap B_\epsilon^{n+1}$, where $0 < \delta \ll \epsilon$. Therefore, one has $\chi(N_j^-) = \chi(N_j) = \chi(M_p)$. For more details, see [7, Corollary 6.4]

Example 4.8. Consider the quasi-linear equation

$$2x_1 u \frac{\partial u}{\partial x_1} + 3x_1^2 + x_2^2 + u^2 = 0, \tag{4.2}$$

and the first integrals map $F_{(4.2)} := (f_1, f_2) : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by $f_1(x_1, x_2, u) = x_2$ and $f_2(x_1, x_2, u) = x_2^2 + x_1(x_1^2 + x_2^2 + u^2)$. It follows from item (ii) of Theorem 4.3 that $\chi(M_2) = 1$. Consequently, the Euler characteristic of the integral manifold N_2 is equal to one.

Let $F : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ with an isolated singularity at the origin. In [19, §11], Milnor proposed to call F *trivial* if the closure of the Milnor fiber M_F is diffeomorphic to an $(n - p + 1)$-dimensional closed disk. Therefore, we have the immediate result:

Lemma 4.9. *Let $F_{(*)} = (f_1, f_2 \dots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ be a first integrals map, which has an isolated singularity at the origin. If $F_{(*)}$ is trivial, then for any $j = 1, \dots, p$, the integral manifold N_j is homeomorphic to $B^{n-p+1} \times B^{p-1}$. Moreover, the manifold M_j is homeomorphic to $B^{n-p+1} \times B^{p-j}$ for any $j = 1, \dots, p$.*

In [6], P. Church and K. Lamotke used a construction of E. Looijenga (for more details, see [17]) and classified the pairs $(n+1, p)$ for which non-trivial examples exist. Their main result states that for $0 \leq n-p+1 \leq 2$ non-trivial examples occur precisely for the pairs $(2, 2), (4, 3)$ and $(4, 2)$. For $n - p + 1 = 3$, all examples are trivial except for the pairs $(5, 2), (8, 5)$ and the case $(6, 3)$ was proved in [1]. Moreover, since the PoincarÃ© Conjecture was shown to be true, for $n - p + 1 \geq 4$, non-trivial examples occur for all pairs $(n + 1, p)$.

Theorem 4.10. *Let $F_{(*)} = (f_1, f_2 \dots, f_p) : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^p, 0)$ be a first integrals map, which has an isolated singularity at the origin. If any of the following conditions bellow occur, then the integral manifold N_j is homeomorphic to $B^{n-p+1} \times B^{p-1}$, for any fixed $j = 1, \dots, p$. Moreover, the manifold M_j is homeomorphic to $B^{n-p+1} \times B^{p-j}$ for any $j = 1, \dots, p$:*

1. $(n + 1, p) = (4, 2)$ *and* $\deg_0 \nabla f_1 = 0$*;*
2. $(n + 1, p) = (5, 2)$ *and* $\pi_1(M_p) = 0$*, i.e.,* M_p *is simply connected;*
3. $(n+1, p) = (6, 3)$ *and the manifold* $V_{F(*)} \cap S^n_\varepsilon$ *is connected for small enough* $\varepsilon > 0$ *or* $\deg_0 \nabla f_1 = 0$*;*
4. $(n+1, p) = (8, 5)$ *and the manifold* $V_{F(*)} \cap S^n_\varepsilon$ *is not empty for small enough* $\varepsilon > 0$ *or* $\deg_0 \nabla f_1 = 0$*;*
5. $p = n$ *and* $(n + 1, p) \neq (4, 3)$*;*
6. $n - p = 1$ *and* $(n + 1, p) \neq (4, 2)$*;*
7. $n - p + 1 = 3$ *and* $(n + 1, p) \neq (5, 2)$, $(n + 1, p) \neq (6, 3)$ $(n + 1, p) \neq (8, 5)$.

Proof. It follows immediately from Lemma 4.9 combined with the following arguments, in each case:

Case $(4,2)$: Let $CM_{F_{(*)}}$ be the closed surface obtained from the Milnor fiber $M_{F_{(*)}}$ by gluing open disks to the boundary and $k \geq 1$ the number of closed curves components of the boundaries. If we assume that $\deg_0 \nabla f_1 = 0$, it follows from [8, Lemma 4.4] that $\chi(CM_{F_{(*)}}) = k+1$, with $k=1$. Consequently, $M_{F_{(*)}}$ is a closed disk and $F_{(*)}$ is trivial. See [8, Corollary 4.5].

Case $(5,2)$: If we assume that M_p is simply connected, then the Milnor fiber $M_{F_{(*)}}$ is simply connected and the [25, Lemma 2.3] combined with the Theorem 4.3 (b) ensure that $\partial M_{F_{(*)}}$ is a 2-sphere, then by [25, Lemma 2.4], we have that $F_{(*)}$ is trivial. See [25, Theorem 1.6 (3)].

Case $(6,3)$ Since $p=3$, the Milnor fiber $M_{F_{(*)}}$ is simply connected and the [25, Lemma 2.3] ensure that $\partial M_{F_{(*)}}$ is a disjoint union of 2-spheres. If we assume that $\deg_0 \nabla f_1 = 0$, it follows from Theorem 4.3 (a) that $\chi\left(M_{F_{(*)}}\right) = 1$. Consequently,

$$\chi\left(\partial M_{F_{(*)}}\right) = 2\chi\left(M_{F_{(*)}}\right) = 2.$$

Therefore, $\partial M_{F_{(*)}}$ is a 2-sphere, then by [25, Lemma 2.4], we have that $F_{(*)}$ is trivial.

Now, since that $\partial M_{F_{(*)}}$ is diffeomorphic to manifold $V_{F(*)} \cap S^n_\varepsilon$, if we assume that it is connected, then $\partial M_{F_{(*)}}$ must be diffeomorphic to one single copy of a 2-sphere and again by [25, Lemma 2.4], we have that $M_{F_{(*)}}$ is diffeomorphic to a 3-disk. See [25, Theorem 1.6 (1) and Proposition 3.1].

Case $(8,5)$ Assume that the manifold $V_{F(*)} \cap S^n_\varepsilon$ is not empty for small enough $\varepsilon > 0$. Since $p=5$, that Milnor Fiber $M_{F_{(*)}}$ is simply connected, compact 3-manifold with boundary. Now, it follows from Hurewicz Theorem combined with [25, Lemma 2.4 (ii)] that $M_{F_{(*)}}$ is diffeomorphic to a 3-disk. If we assume that $\deg_0 \nabla f_1 = 0$, we can use the same arguments as in Case $(6,3)$. See [25, Theorem 1.6 (2) and Corollary 3.4].

The remaining cases follow from the above discussion. For more details, see for example [6, p.149-150].

□

The next examples are based on the geometrical/topological description given in section 4.2 of [8].

Example 4.11. Consider the equation

$$2x_3u\frac{\partial u}{\partial x_2} - 3u(x_1^2+x_2^2)\frac{\partial u}{\partial x_3} = -3x_3(x_1^2+x_2^2). \tag{4.3}$$

The graph of the solutions u of (4.3) is included in 2-disk. In fact, the system (4.3) admits the first integrals map $F_{(4.3)}(x_1,x_2,x_3,u) = (x_1, 3x_1^2x_2 + x_2^3 + x_3^2 + u^2)$ which has an isolated singularity at the origin. Since $f_1 : (\mathbb{R}^4, 0) \to (\mathbb{R}, 0)$ given by $f_1(x_1,x_2,x_3,u) = x_1$ satisfies $\deg_0 \nabla f_1 = 0$, it follows from the item (1) of Theorem 4.10 that M_2 is homeomorphic to B^2.

Example 4.12. Consider the system

$$\begin{cases} x_3\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} - x_1\frac{\partial u}{\partial x_3} = -x_2 \\ u\frac{\partial u}{\partial x_1} - x_3\frac{\partial u}{\partial x_2} - x_2\frac{\partial u}{\partial x_3} = x_1. \end{cases} \tag{4.4}$$

The graph of the solutions u of (4.4) is included in a cylinder. Indeed, the first integrals map $F_{(4.4)} = (f_1, f_2) : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ with $f_1 = x_1^2 - x_2^2 + x_3^2 - u^2$ and $f_2 = 2(x_1x_2 + x_3u)$ has an isolated singularity at the origin. It follows from the geometrical/topological description of the Milnor fibers $M_{F_{(4.4)}}$, (see [8, Section §4.2]), that M_2 is topologically equivalent to the two-sphere minus two open disks removed.

Example 4.13. Consider the system

$$\begin{cases} -\frac{3}{2}\left(x_3^2 - u^2\right)\frac{\partial u}{\partial x_1} - 3x_3u\frac{\partial u}{\partial x_2} + x_1\frac{\partial u}{\partial x_3} - x_2 = 0 \\ -3x_3u\frac{\partial u}{\partial x_1} + \frac{3}{2}\left(x_3^2 - u^2\right)\frac{\partial u}{\partial x_2} + x_2\frac{\partial u}{\partial x_3} + x_1 = 0. \end{cases} \tag{4.5}$$

This system admits a first integrals map $F_{(4.5)} = (f_1, f_2) : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ with $f_1 = x_1^2 - x_2^2 + x_3^3 - 3x_3u^2$ and $f_2 = 2x_1x_2 + 3x_3^2u - u^3$. The graph of the solutions u of (4.5) is included in a manifold topologically equivalent to a torus minus a disk removed, (see [8, Section §4.2]).

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