Matemática
Contemporânea

Vol. 52, 142–161
http://doi.org/10.21711/231766362022/rmc528

Homogenization of a Generalized Second-Order Differential Operator

Alexandre B. Simas[1] and **Fábio J. Valentim**[2]

[1]Universidade Federal da Paraíba, Cidade Universitária - Campus I, 58051-970, João Pessoa - PB, Brasil

[2]Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514, Goiabeiras, 29075-910, Vitória - ES, Brasil

Abstract. In this survey, we considere a generalization of Laplace operator and study the homogenization's problem, under litle assumptions regarding weak convergence and ellipticity conditions. We provide two examples: The first one consists on the case with litle regularity, the second one consists on the case with random environment associated with an ergodic group. Finally, we provide an application geared towards the hydrodynamical limit of certain gradient processes.

Keywords: Homogenization, Difference operators, Differential operators.

2020 Mathematics Subject Classification:46E35, 35J15, 35-02.

1 Introduction

In this survey we present some recent results concerning some aspects of homogenization theory for a class of second order elliptic difference operators. More precisely, we consider a function $W(x_1, \ldots, x_d) =$

fabio.valentim@ufes.br

$\sum_{k=1}^{d} W_k(x_k)$, where each $W_k : \mathbb{R} \to \mathbb{R}$ is a strictly increasing right continuous function with left limits. Given a matrix function $A = \text{diag}\{a_1, \dots, a_d\}$, let $\nabla A \nabla_W = \sum_{k=1}^{d} \partial_{x_k}(a_k \partial_{W_k})$ be a generalized second-order differential operator. We study the homogenization of generalized second-order difference operators, that is, we are interested in the convergence of the sequence of solutions of

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f^N$$

to the solution of

$$\lambda u - \nabla A \nabla_W u = f, \tag{1.1}$$

where the superscript N stands for some sort of discretization. Under litle assumptions regarding weak convergence and ellipticity of these matrices A^N, we show that every such sequence admits a homogenization. We provide two examples of matrix functions verifying these assumptions: the first one consists of fixing a matrix function A under litle regularity assumptions, and taking a convenient discretization A^N; the second one consists on the case where A^N represents a random environment associated to an ergodic group, a case in which we then show that the homogenized matrix A does not depend on the realization ω of the environment. For more details see [13, 14, 17].

The operator of the equation (1.1) represent a generalization of Feller operators introduced by Willian Feller in the '50s. For further details see [2, 3, 10]. Further, note that if $W_k(x) = x$, $k = 1, \dots, d$ and the matrix $A = Id$, then we recover the standard laplacian operator.

Recently attention has been raised by the the operator (1.1) and some non-linear versions. It were obtained as scaling limits of interacting particle systems in inhomogeneous media. They may model, for instance, diffusions with permeable membranes at the points of discontinuities of W, see [1, 4, 16, 17] and Section 4. On the other hand, beyond the interest in probability, in [13], the space of functions f having *weak generalized gradients* $\nabla_W f = (\partial_{W_1} f, \dots, \partial_{W_d} f)$, refered to as W-Sobolev spaces, have been studied and several properties, analogous to the classical Sobolev spaces, were proved. Moreover, in [15], the analysis has been further extended to W-Sobolev spaces of higher order where elliptic regularity results were obtained.

In the similar way, in last years significant progress has been achieved in the homogenization theory of random differential operators. We refer to the original works of Kozlov [7, 8, 9] and Papanicolaou and Varadhan [11], and to the book by Jikov, Kozlov, and Oleinik [5] wherein an additional bibliography can be found.

Our approach is inspired in [9, 12] where a number of homogenization results for difference schemes were obtained. To prove the main result we mainly use an estimate of energy and the discrete analogue of the compensated compactness technique originally introduced by Murat and Tartar for functions of continuous arguments.

This survey is organized as follows: in Section 2 we provide a brief review on W-Sobolev spaces and a discrete analogous to the continuous W-Sobolev spaces; In Section 3 we provide the main results of this article. Finally, in Section 4, as an application of this theory, we present a hydrodynamic limit result.

2 Notations and Basic Results

In this Section we present some results about the generalized second-order differential operator and the Sobolev spaces associeted. For more details, see [13, 14, 17].

2.1 W-Sobolev Space

Denote by $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0,1)^d$ the d-dimensional torus and fix a function $W : \mathbb{R}^d \to \mathbb{R}$ such that

$$W(x_1, \dots, x_d) = \sum_{k=1}^{d} W_k(x_k), \tag{2.1}$$

where each $W_k : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing* right continuous function with left limits (càdlàg), with periodic increments, in the sense that

$$W_k(u+1) - W_k(u) = W_k(1) - W_k(0), \qquad \text{for all } u \in \mathbb{R}.$$

The generalized partial derivative ∂_{W_k} of a function $f : \mathbb{T}^d \to \mathbb{R}$ is

$$\partial_{W_k} f(x_1,\ldots,x_k,\ldots,x_d) = \lim_{\epsilon\to 0} \frac{f(x_1,\ldots,x_k+\epsilon,\ldots,x_d) - f(x_1,\ldots,x_k,\ldots,x_d)}{W_k(x_k+\epsilon) - W_k(x_k)} \tag{2.2}$$

if the above limit exists. Denote the generalized gradient of f by $\nabla_W f = (\partial_{W_1} f, \ldots, \partial_{W_d} f)$, if the generalized partial derivatives ∂_{W_k} exist for all $k = 1, \ldots, d$. One may apply the operator $\partial_{x_k}\partial_{W_k}$ in the classical sense. Actually, we have $\nabla A \nabla_W f = \sum_{k=1}^d \partial_{x_k}(a_k \partial_{W_k} f)$ for $f \in \mathfrak{D}_W(\mathbb{T}^d)$, an appropriated domain to $\nabla A \nabla_W$. Also, $A = (a_1, \ldots, a_d)$ is a matrix function such that $\partial_{x_k} a_k$ exists. The set $\mathfrak{D}_W(\mathbb{T}^d)$ has good properties, can be used as a space of test functions(see [14]) and its definition is the following.

Denote by $D(f)$ the set of discontinuity points of a function $f : \mathbb{T} \to \mathbb{R}$. For $k = 1, \ldots, d$, let $C_{W_k}(\mathbb{T})$ be the set of càdlàg functions $f : \mathbb{T} \to \mathbb{R}$ such that $D(f) \subset D(W_k)$. We endow $C_{W_k}(\mathbb{T})$ with the sup norm $\|\cdot\|_\infty$.

Let $\mathfrak{D}_{W_k}(\mathbb{T})$ be the set of functions f in $C_{W_k}(\mathbb{T})$ such that $\frac{df}{dW_k}$ is well-defined and differentiable, and that $\frac{d}{dx}\big(\frac{df}{dW_k}\big)$ belongs to $C_{W_k}(\mathbb{T})$. In [4], it is proved that $\mathfrak{D}_{W_k}(\mathbb{T})$ is the set of functions f in $C_{W_k}(\mathbb{T})$ such that

$$f(x) \;=\; a \;+\; bW_k(x) \;+\; \int_{(0,x]} dW_k(y) \int_0^y g(z)\, dz \tag{2.3}$$

for some function g in $C_{W_k}(\mathbb{T})$, where a, b are real numbers satisfying

$$bW_k(1) \;+\; \int_{\mathbb{T}} dW_k(y) \int_0^y g(z)\, dz \;=\; 0 \quad \text{and} \quad \int_{\mathbb{T}} g(z)\, dz \;=\; 0\,. \tag{2.4}$$

Let us now define a d-dimensional counterpart to the sets $C_{W_k}(\mathbb{T})$ and $\mathfrak{D}_{W_k}$, respectively:

$$C_W(\mathbb{T}^d) = \operatorname{span}\{f; f = f_1 \otimes \cdots \otimes f_d, \quad f_k \in C_{W_k}(\mathbb{T}), k = 1, \ldots, d\} \quad \text{and} \tag{2.5}$$

$$\mathfrak{D}_W(\mathbb{T}^d) = \text{span}\,\{f; f = f_1 \otimes \cdots \otimes f_d, \quad f_k \in \mathfrak{D}_{W_k}, k = 1, \ldots, d\}\,. \quad (2.6)$$

In other words, $C_W(\mathbb{T}^d)$ and $\mathfrak{D}_W(\mathbb{T}^d)$ are the spaces generated by functions of the form $f_1 \otimes \cdots \otimes f_d$. Note that the closure of $\mathfrak{D}_W(\mathbb{T}^d)$ is the tensor product of the spaces $\mathfrak{D}_{W_1}(\mathbb{T}), \ldots, \mathfrak{D}_{W_d}(\mathbb{T})$.

Let us remember the definition of the space $H_{1,W}(\mathbb{T}^d)$, called *W-Sobolev space*. We denote by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space $L^2(\mathbb{T}^d)$ and by $\| \cdot \|$ its norm. Let $L^2_{x^k \otimes W_k}(\mathbb{T}^d)$ be the Hilbert space of measurable functions $f : \mathbb{T}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{T}^d} f^2 \, d(x^k \otimes W_k) \; < \; \infty,$$

where $d(x^k \otimes W_k)$ represents the product measure on $\mathbb{T}^d$ obtained from Lebesgue's measure in $\mathbb{T}^{d-1}$ and the measure induced by W_k on $\mathbb{T}$:

$$d(x^k \otimes W_k) \;=\; dx_1 \cdots dx_{k-1} \, dW_k \, dx_{k+1} \cdots dx_d.$$

Denote by $\langle \cdot, \cdot \rangle_{x^k \otimes W_k}$ the inner product of $L^2_{x^k \otimes W_k}(\mathbb{T}^d)$:

$$\langle f, g \rangle_{x^k \otimes W_k} \;=\; \int_{\mathbb{T}^d} f \, g \; d(x^k \otimes W_k)\,,$$

and by $\|\cdot\|_{x^k \otimes W_k}$ the norm induced by this inner product. Let $L^2_{x^k \otimes W_k, 0}(\mathbb{T}^d)$ be the closed subspace of $L^2_{x^k \otimes W_k}(\mathbb{T}^d)$ consisting of the functions f such that

$$\int_{\mathbb{T}^d} f \; d(x^k \otimes W_k) = 0.$$

A function $g \in L^2(\mathbb{T}^d)$ has *W-generalized weak derivative* if for each $k = 1, \ldots, d$ there exists $G_k \in L^2_{x^k \otimes W_k, 0}(\mathbb{T}^d)$ satisfying

$$\int_{\mathbb{T}^d} \left(\partial_{x_k} \partial_{W_k} f\right) \, g \; dx \;=\; - \int_{\mathbb{T}^d} (\partial_{W_k} f) \; G_k d(x^k \otimes W_k), \qquad (2.7)$$

for every function $f \in \mathfrak{D}_W(\mathbb{T}^d)$. The *W-Sobolev space* $H_{1,W}(\mathbb{T}^d)$ is the set of functions in $L^2(\mathbb{T}^d)$ having W-generalized weak derivative. G_k is unique almost everywhere and the set $H_{1,W}(\mathbb{T}^d)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{1,W} \;=\; \langle f, g\rangle + \sum_{k=1}^{d} \int_{\mathbb{T}^d} (\partial_{W_k} f)(\partial_{W_k} g)\, d(x^k \otimes W_k),$$

and denote by $\|\cdot\|_{1,W}$ its induced norm. Further, $\mathfrak{D}_W(\mathbb{T}^d) \subset H_{1,W}(\mathbb{T}^d)$ and if $g \in \mathfrak{D}_W(\mathbb{T}^d)$ then $G_k = \partial_{W_k} g$. For this reason, we denote G_k simply by $\partial_{W_k} g$, and we call it the k-th *generalized weak derivative* of the function g with respect to W (see [13]).

Denote by $H_W^{-1}(\mathbb{T}^d)$ the dual space to $H_{1,W}(\mathbb{T}^d)$. We conclude this subsection with the following characterization, whose proof can be found in [13].

Lemma 2.1. *$f \in H_W^{-1}(\mathbb{T}^d)$ if and only if there exist functions $f_0 \in L^2(\mathbb{T}^d)$ and $f_k \in L^2_{x^k \otimes W_k, 0}(\mathbb{T}^d)$, such that*

$$f = f_0 - \sum_{k=1}^{d} \partial_{x_k} f_k, \tag{2.8}$$

in the sense that for $v \in H_{1,W}(\mathbb{T}^d)$

$$f(v) := \int_{\mathbb{T}^d} f_0 v dx + \sum_{k=1}^{d} \int_{\mathbb{T}^d} f_k (\partial_{W_k} v) d(x^k \otimes W_k).$$

Furthermore,

$$\|f\|_{-1,W} = \inf\left\{ \left(\int_{\mathbb{T}^d} |f_0|^2 dx + \sum_{k=1}^{d} \int_{\mathbb{T}^d} |f_k|^2 d(x^k \otimes W_k) \right)^{1/2} ;\ f \text{ satisfies } (2.8) \right\}.$$

2.2 The discrete W-Sobolev space

In this subsection, we recall the notion of discrete W-Sobolev spaces. This can be seen as a counterpart to the continuous case considered in [13].

Denote by $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d = \{0, \ldots, N-1\}^d$ the d-dimensional discrete torus with N^d points. For $f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$, let the operators $\partial_{x_k}^N$, be the standard difference operator in the k-th canonical direction, and $\partial^N_{W_k}$, be the W_k-difference operator, be given, respectively, by:

$$\partial^N_{x_k} f\left(\frac{x}{N}\right) = N\left[f\left(\frac{x+e_k}{N}\right) - f\left(\frac{x}{N}\right)\right] \quad \text{and} \tag{2.9}$$

$$\partial^N_{W_k} f\left(\frac{x}{N}\right) = \frac{f\left(\frac{x+e_k}{N}\right) - f\left(\frac{x}{N}\right)}{W\left(\frac{x+e_k}{N}\right) - W\left(\frac{x}{N}\right)}, \quad \text{for} \quad x \in \mathbb{T}^d_N.$$

Denote by $L^2(\mathbb{T}^d_N)$, $L^2_{W_k}(\mathbb{T}^d_N)$ and $H_{1,W}(\mathbb{T}^d_N)$ the spaces of the functions defined on $\frac{1}{N}\mathbb{T}^d_N$ with respect to the following inner products and their induced norms, respectively:

$$\langle f, g\rangle_N := \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} f(x/N)g(x/N),$$

$$\langle f, g\rangle_{W_k,N} := \frac{1}{N^{d-1}}\sum_{x\in\mathbb{T}^d_N} f(x/N)g(x/N)\big(W((x+e_k)/N) - W(x/N)\big),$$

$$\langle f, g\rangle_{1,W,N} := \langle f, g\rangle_N + \sum_{k=1}^d \langle \partial^N_{W_k} f, \partial^N_{W_k} g\rangle_{W_k,N},$$

and

$$\|f\|^2_N = \langle f, f\rangle_N, \quad \|f\|^2_{W_k,N} = \langle f, f\rangle_{W_k,N} \text{ and } \|f\|^2_{1,W,N} = \langle f, f\rangle_{1,W,N},$$

These norms are natural discretizations of the norms considered in the continuous version.

We will now provide results on discrete elliptic equations. Let $\lambda \geq 0$ and $A = \operatorname{diag}\{a_1(x), \ldots, a_d(x)\}$, $x \in \mathbb{T}^d$, be a diagonal matrix function satisfying the ellipticity condition: for every $x \in \mathbb{T}^d$ and every $k = 1, \ldots, d$,

$$\text{There exists a constant } \theta \geq 1 \text{ such that } \theta^{-1} \leq a_k(x) \leq \theta, \tag{2.10}$$

Our interess is study the problem

$$T^N_\lambda u = f, \tag{2.11}$$

where $u, f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$, f is a known function, u is the unknown function and T_λ^N denotes the discrete generalized elliptic operator

$$T_\lambda^N u \;:=\; \lambda u - \nabla^N A \nabla_W^N u \qquad \text{with} \tag{2.12}$$

$$\nabla^N A \nabla_W^N u \;:=\; \sum_{k=1}^{d} \partial_{x_k}^N \Big(a_k(x/N) \partial_{W_k}^N u \Big).$$

For $u, v : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$, the bilinear form $B^N[\cdot,\cdot]$ associated with the elliptic operator T_λ^N is given by

$$B^N[u,v] \;=\; \lambda \langle u, v\rangle_N \, +$$
$$\frac{1}{N^{d-1}} \textstyle\sum_{k=1}^{d} \sum_{x \in \mathbb{T}_N^d} a_k(x/N)(\partial_{W_k}^N u)(\partial_{W_k}^N v)[W_k((x_k+1)/N) - W_k(x_k/N)]. \tag{2.13}$$

A function $u : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ is said to be a *weak solution* of (2.11) if

$$B^N[u,v] \;=\; \langle f, v\rangle_N \;\text{ for all }\; v : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}.$$

Denote by $H_{1,W}^{\perp}(\mathbb{T}_N^d)$ the subspace of $H_{1,W}(\mathbb{T}_N^d)$ formed by functions $f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ which are orthogonal to the constant functions with respect to the inner product $\langle \cdot, \cdot \rangle_{1,W,N}$. We conclude this subsection with two results relative the weak solution of (2.11). The proof can be found in [14].

Lemma 2.2. *Given a function $f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$, the equation*

$$\nabla^N A \nabla_W^N u = f,$$

has a weak solution $u : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ if and only if $f \in H_{1,W}^{\perp}(\mathbb{T}_N^d)$. In this case we have uniqueness of the solution disregarding addition by constants. Moreover, if $u \in H_{1,W}^{\perp}(\mathbb{T}_N^d)$ we have the bound

$$\|u\|_{1,W,N} \le C \|f\|_N,$$

where $C > 0$ does not depend on f nor on N.

Lemma 2.3. *Let $\lambda > 0$ and $f : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$. There exists a unique weak solution $u : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ of the equation*

$$\lambda u - \nabla^N A \nabla_W^N u = f. \tag{2.14}$$

Moreover,

$$\|u\|_{1,W,N} \leq C\|f\|_{-1,W,N}, \text{ and } \|u\|_N \leq \lambda^{-1}\|f\|_{-1,W,N},$$

where $C > 0$ does not depend neither on f nor on N.

3 Homogenization

In this Section we will present the main result. Two relevant particular cases are considered.

We focus on the analysis of the asymptotic behavior of the sequence (u_N) given by solutions of the

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f^N, \qquad \lambda \geq 0. \tag{3.1}$$

In [13] one can find the continuous counterpart of the theory results on existence, uniqueness and boundedness of weak solutions of the problem

$$\lambda u_0 - \nabla A \nabla_W u_0 = f, \qquad \lambda \geq 0. \tag{3.2}$$

We say that $A^N = \text{diag}\{a_1^N, \ldots, a_d^N\}$ H-converges, as $N \to \infty$, to the matrix $A = \text{diag}\{a_1, \ldots, a_d\}$, $A^N \xrightarrow{H} A$, if for every sequence f^N of functionals on $H_{1,W}(\mathbb{T}_N^d)$ and $f \in H_W^{-1}(\mathbb{T}^d)$, such that $f^N \longrightarrow f$ strongly in $H_W^{-1}(\mathbb{T}^d)$, we have

- $u_N \longrightarrow u_0$ weakly in $H_{1,W}(\mathbb{T}^d)$
- $a_k^N \partial_{W_k}^N u_N \longrightarrow a_k \partial_{W_k} u_0$ weakly in $L^2_{x^k \otimes W_k}(\mathbb{T}^d)$.

where $u_N : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ is the solution of the (3.1) and $u_0 \in H_{1,W}(\mathbb{T}^d)$ is the weak solution of the (3.2). A (resp. $\nabla A \nabla_W$) is said a *homogenization* of the sequence A^N (resp. $\nabla^N A^N \nabla_W^N$).

Theorem 3.1. *Let $A^N = \mathrm{diag}\{a_1^N, \ldots, a_d^N\}$ be a sequence of diagonal matrices and $\theta \geq 1$, such that $\theta^{-1} \leq a_k^N(x) \leq \theta$ for all $x \in \frac{1}{N}\mathbb{T}_N^d$. If*

$$1/a_k^N \longrightarrow b_k \quad \textit{weakly in} \quad L^2_{x^k \otimes W_k}(\mathbb{T}^d),$$

then, A^N admits a homogenization and the homogenized matrix is

$$A = \mathrm{diag}\{1/b_1, \ldots, 1/b_d\}.$$

In subsection 3.3 we shall give a sketch of the proof of Theorem 3.1.

3.1 Homogenization with little regularity

In this subsection we will provide an example of a very large class of matrix functions that admits homogenization.

Denote by $\mathbb{M}_W \subset \big(C_W(\mathbb{T}^d)\big)^d$ the space of functions $a = (a_1, \ldots, a_d)$ such that for every $k = 1, \ldots, d$, $\partial_{x_k} a_k$ exists. Let $Q_N(x) = \{y \in \mathbb{T}^d;\, 0 \leq y_i - x_i \leq 1/N\}$ and $\widetilde{u}_N : \mathbb{T}^d \to \mathbb{R}$, given by

$$\widetilde{u}_N(y) = \sum_{x \in \frac{1}{N}\mathbb{T}_N^d} u_N(x) 1_{Q_N(x)}(y),$$

be the *piecewise-constant interpolation* of a mesh function $u_N : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$.

Theorem 3.2. *Let $A = \mathrm{diag}\{a_1, \ldots, a_d\} \in \mathbb{M}_W$, such that there exists $\theta \geq 1$ with $a_k(x) \geq \theta^{-1}$, for all $x \in \mathbb{T}^d$. Then $A^N = \mathrm{diag}\{a_1^N, \ldots, a_d^N\}$,*

$$a_k^N(x) = \frac{N^{d-1}}{W_k(x_k + 1/N) - W_k(x_k)} \int_{Q_N(x)} a_k(y) d(y^k \otimes W_k) \qquad x \in \frac{1}{N}\mathbb{T}_N^d,$$

admits a homogenization, with homogenized matrix A.

Proof. From compatcness of $\mathbb{T}^d$ and definition of $C_W(\mathbb{T}^d)$ we obtaim that a_k is bounded, for $k = 1, \ldots, d$ ([14, Lemma 3.5]). So, without loss of generality, we may suppose that $\theta^{-1} \leq a_k(x) \leq \theta$ for all $x \in \mathbb{T}^d$. It is clear that $\theta^{-1} \leq a_k^N(x) \leq \theta$, $x \in \frac{1}{N}\mathbb{T}_N^d$, and the right-continuity implies the pointwise convergence of $\widetilde{a}_k^N$ to a_k. From the Dominated Convergence Theorem

$$a_k^N \longrightarrow a_k \text{ strongly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d).$$

On the other hand, we also have the pointwise convergence of $\widetilde{1/a_k^N}$ to $1/a_k$, and the bound $1/a_k^N \leq \theta$ implies that

$$1/a_k^N \longrightarrow 1/a_k \text{ strongly } L^2_{x^k\otimes W_k}(\mathbb{T}^d).$$

Therefore, the result follows from Theorem 3.1. □

3.2 Random environment homogenization

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\{T_x : \Omega \to \Omega; x \in \mathbb{Z}^d\}$ be a group of $\mathcal{F}$-measurable and ergodic transformations preserving the measure μ:

- $T_x : \Omega \to \Omega$ is $\mathcal{F}$-measurable for all $x \in \mathbb{Z}^d$,
- $\mu(T_x A) = \mu(A)$, for any $A \in \mathcal{F}$ and $x \in \mathbb{Z}^d$,
- $T_0 = I\,,\ \ T_x \circ T_y = T_{x+y}$,
- Any $f \in L^1(\Omega)$ such that $f(T_x\omega) = f(\omega)$ μ-a.e., for each $x \in \mathbb{Z}^d$, is equal to a constant μ-a.e..

$(\Omega, \mathcal{F}, \mu)$ is called *random environment*, and a point $\omega \in \Omega$ a *realization* of the random environment.

Let $\{b_k(\omega); k = 1, \ldots, d\}$ be $\mathcal{F}$-measurable functions such that $0 < \theta^{-1} \leq b_k(\omega) \leq \theta$, for all $\omega \in \Omega$. Define the *random* diagonal matrices $B^N = \text{diag}\{b_1^N, \ldots, b_d^N\}$ by

$$b_k^N(x) := b_k(T_{Nx}\omega)\,,\ \ x \in \frac{1}{N}\mathbb{T}_N^d.$$

By Birkhoff Ergodic Theorem, for $k = 1, \ldots, d$,

$$b_k^N \longrightarrow E[b_k] \qquad \text{and} \qquad 1/b_k^N \longrightarrow E[1/b_k] \qquad \text{weakly in } L^2(\mathbb{T}^d) \qquad \text{a.s.,}$$

We need an similar result for $L^2_{x^k\otimes W_k}(\mathbb{T}^d)$. Denote by μ_{W_k} the measure induced by function W_k. By Lebesgue Decomposition, there exist mensurable function g such that

$$\mu_{W_k} = g\lambda + \lambda^{\perp}$$

where $g\lambda$ and $\lambda^{\perp}$ are mutually singular measures and λ denotes the Lebesgue measure. Let $V_k \subset \mathbb{T}$ be the support of $\lambda^{\perp}$ and $\mathfrak{V}^k = \mathbb{T}\times\ldots\times\mathbb{T}\times V_k\times\mathbb{T}\times\ldots\times\mathbb{T}\subset\mathbb{T}^d$, V_k in the k-th component. Define $A^N = \operatorname{diag}\{a_1^N,\ldots,a_d^N\}$ where $a_k^N : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$, given by

$$a_k^N(x) \;=\; \begin{cases} b_k^N(x) & \text{if } \mathfrak{V}^k \cap Q_N(x) = \emptyset, \\ (E[1/b_k])^{-1} & \text{if } \mathfrak{V}^k \cap Q_N(x) \neq \emptyset \end{cases}$$

We have the following result.

Theorem 3.3. *$A^N(\omega)$ admits a homogenization a.s., where the homogenized matrix A does not depend on the realization ω.*

The proof follows from Theorem 3.1 and the Lemma below.

Lemma 3.4. *Let $a_k^N : \frac{1}{N}\mathbb{T}_N^d \to \mathbb{R}$ be as defined above. Then, a.s., weakly in $L^2_{x^k\otimes W_k}(\mathbb{T}^d)$*

$$1/a_k^N \longrightarrow E[1/b_k].$$

Proof. Let

$$\mathfrak{V}_N^k = \bigcup_{\substack{x\in\frac{1}{N}\mathbb{T}_N^d \\ \mathfrak{V}^k\cap Q_N(x)\neq\emptyset}} Q_N(x).$$

We have that

$$1_{\mathfrak{V}_N^k} \longrightarrow 1_{\overline{\mathfrak{V}}^k} \qquad \text{pointwise.}$$

Therefore,

$$1_{\mathfrak{V}_N^k} \longrightarrow 1_{\overline{\mathfrak{V}}^k} \qquad \text{strongly in } L^2(\mathbb{T}^d). \tag{3.3}$$

Let $\phi \in \mathfrak{D}_W(\mathbb{T}^d)$ be fixed. By Lebesgue decomposition we have,

$$\int_{\mathbb{T}^d} \widetilde{1/a_k^N}\phi d(x^k\otimes W_k) = \int_{\mathbb{T}^d} \widetilde{1/a_k^N}\phi g dx + \int_{\mathbb{T}^d} \widetilde{1/a_k^N}\phi d(x^k\otimes\lambda^{\perp}).$$

Note that the support of the measure $d(x^k \otimes \lambda^{\perp})$ is confined in the set $\mathfrak{V}^k$. Since $\mathfrak{V}^k \subset \mathfrak{V}_N^k$, we have that $\widetilde{1/a_k^N}$ is almost everywhere constant, namely $\widetilde{1/a_k^N} = E[1/b_k]$, with respect to the measure $d(x^k \otimes \lambda^{\perp})$. Thus, the second integral in the right-hand side in the previous expression is equal to

$$\int_{\mathbb{T}^d} E[1/b_k]\phi d(x^k \otimes \lambda^{\perp}) \, .$$

On the other hand, by (3.3) we have

$$1/a_k^N 1_{[\mathfrak{V}_N^k]^c} = 1/b_k^N 1_{[\mathfrak{V}_N^k]^c} \longrightarrow E[1/b_k] 1_{\left[\overline{\mathfrak{V}}^k\right]^c} \qquad \text{weakly in } L^2(\mathbb{T}^d) \qquad \text{a.s.,} \tag{3.4}$$

where X^c denotes the complementary set of X.

So, the first integral in the right-hand side in the above expression is equal to

$$\int_{\mathbb{T}^d} \widetilde{1/a_k^N} 1_{\mathfrak{V}_N^k} \phi g dx + \int_{\mathbb{T}^d} \widetilde{1/a_k^N} 1_{[\mathfrak{V}_N^k]^c} \phi g dx = \int_{\mathbb{T}^d} E[1/b_k] 1_{\mathfrak{V}_N^k} \phi g dx + \int_{\mathbb{T}^d} \widetilde{1/b_k^N} 1_{[\mathfrak{V}_N^k]^c} \phi g dx.$$

From the previous convergence, (3.3) and (3.4), the right hand-side in the previous expression converges to

$$\int_{\mathbb{T}^d} E[1/b_k] 1_{\overline{\mathfrak{V}}^k} \phi g dx + \int_{\mathbb{T}^d} E[1/b_k] 1_{\left[\overline{\mathfrak{V}}^k\right]^c} \phi g dx = \int_{\mathbb{T}^d} E[1/b_k] \phi g dx \, .$$

Then, we have shown that

$$\lim_{N\to\infty} \int_{\mathbb{T}^d} \widetilde{1/a_k^N} \phi d(x^k \otimes W_k) = \int_{\mathbb{T}^d} E[1/b_k]\phi d(x^k \otimes \lambda^{\perp}) + \int_{\mathbb{T}^d} E[1/b_k]\phi g dx = \int_{\mathbb{T}^d} E[1/b_k]\phi d(x^k \otimes W_k)$$

and this concludes the proof of the first statement.

□

3.3 Sketching of the proof of Theorem 3.1

In this subsection we present some step of the proof of Theorem 3.1. For more details see [14].

- Fix $f \in H_W^{-1}(\mathbb{T}^d)$, and consider the problem

$$\lambda u_N - \nabla^N A^N \nabla_W^N u_N = f_N, \tag{3.5}$$

  where $f_N \longrightarrow f$ strongly in $H_W^{-1}(\mathbb{T}^d)$.

- Results about existence, uniqueness and bounded implies: there exists a constant $C > 0$ such that $\|u_N\|_{1,W,N} \leq C$.

- So, there exists a convergent subsequence of u_N (which we will also denote by u_N) such that $u_N \longrightarrow u$, weakly in $H_{1,W}(\mathbb{T}^d)$. In particular,

$$\partial_{W_k}^N u_N \longrightarrow \partial_{W_k} u \quad \text{weakly in} \quad L^2_{x^k \otimes W_k}(\mathbb{T}^d). \tag{3.6}$$

- Applying (3.5) to u_N, we obtain

$$\lambda \|u_N\|_N^2 + \sum_{k=1}^{d} \|a_{kk}^N \partial_{W_k}^N u_N\|^2_{W_k,N} = f_N(u_N) \leq \|f_N\|_{-1,W,N} \|u_N\|_{1,W,N}.$$

- This implies the uniform bounded in N for $\|a_k^N \partial_{W_k}^N u_N\|_{W_k,N}$. Thus we can find a further subsequence such that

$$a_k^N \partial_{W_k}^N u_N \longrightarrow v_{0,k} \qquad \text{weakly in} \quad L^2_{x^k \otimes W_k}(\mathbb{T}^d), \tag{3.7}$$

  $v_{0,k} \in L^2_{x^k \otimes W_k}(\mathbb{T}^d)$.

- Since (u_N) is uniformly bounded in the Sobolev-norm and $L^2(\mathbb{T}^d)$ is precompact in this space, we obtaim $u_N \longrightarrow u$ strongly in $L^2(\mathbb{T}^d)$. In particular, $u_N \longrightarrow u$ strongly in $H_W^{-1}(\mathbb{T}^d)$.

- $(\lambda u_N - \nabla^N A^N \nabla_W^N u_N)$ converges strongly to f in $H_W^{-1}(\mathbb{T}^d)$. Therefore,

$$\nabla^N A^N \nabla_W^N u_N \longrightarrow v_0 \qquad \text{strongly in } H_W^{-1}(\mathbb{T}^d).$$

- From the very definition of the functional $\nabla^N A^N \nabla_W^N u_N$, means that for each k,

$$a_k^N \partial_{W_k}^N u_N \longrightarrow v_{0,k} \text{ strongly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d).$$

- From a Compensated Compactness Theorem:

$$\frac{1}{a_k^N} a_k^N \partial_{W_k}^N u_N \longrightarrow b_k v_{0,k}, \qquad \text{weakly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d).$$

On the other hand, by (3.6),

$$\frac{1}{a_k^N} a_k^N \partial_{W_k}^N u_N = \partial_{W_k}^N u_N \longrightarrow \partial_{W_k} u, \qquad \text{weakly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d)\,.$$

From uniqueness of the weak limit, we have that $\partial_{W_k} u = b_k v_{0,k}$. Since $b_k \neq 0$, we have that

$$v_{0,k} = \frac{1}{b_k}\partial_{W_k} u.$$

Thus, we can summarize our findings:

$$\begin{gathered} u_N \longrightarrow u \qquad \text{strongly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d)\,, \\ \partial_{W_k} u_N \longrightarrow \partial_{W_k} u \qquad \text{weakly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d), \qquad \text{and} \\ a_k^N \partial_{W_k} u_N \longrightarrow \frac{1}{b_k}\partial_{W_k} u \qquad \text{strongly in } L^2_{x^k\otimes W_k}(\mathbb{T}^d)\,. \end{gathered}$$

Therefore, u solves the problem

$$\lambda u - \nabla A \nabla_W u = f,$$

where $A = \operatorname{diag}\{1/b_1, \ldots, 1/b_d\}$. This concludes the proof.

4 Application: hydrodynamic limit

To conclude this paper we will provide an application of a result on probability theory: The hydrodynamic limit. Other application can be found in [16] where the equilibrium fluctuations problem is considered. We recommend the book of Kipnis and Landim [6] for a complete approach about scaling limits of interacting particle systems.

Let $\{e_1, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. For each $x \in \mathbb{T}_N^d$ and $k = 1, \ldots, d$, define the symmetric rate $\xi_{x,x+e_k} = \xi_{x+e_k,x}$ by

$$\xi_{x,x+e_k} = \frac{a_k^N(x/N)}{N[W((x+e_k)/N) - W(x/N)]} = \frac{a_k^N(x/N)}{N[W_k((x_k+1)/N) - W_k(x_k/N)]}. \tag{4.1}$$

Distribute particles on $\mathbb{T}_N^d$ in such a way that each site of $\mathbb{T}_N^d$ is occupied at most by one particle. Denote by η the configuration of the state space $\{0,1\}^{\mathbb{T}_N^d} = \{\eta : \mathbb{T}_N^d \to \{0,1\}\}$ so that $\eta(x) = 0$ if site x is vacant, and $\eta(x) = 1$ if site x is occupied.

The exclusion process with conductances is a continuous-time Markov process $\{\eta_t : t \geq 0\}$ with state space $\{0,1\}^{\mathbb{T}_N^d}$, whose generator L_N acts on functions $f : \{0,1\}^{\mathbb{T}_N^d} \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{k=1}^{d} \sum_{x \in \mathbb{T}_N^d} \xi_{x,x+e_k} \{f(\sigma^{x,x+e_k}\eta) - f(\eta)\}, \tag{4.2}$$

where $\sigma^{x,x+e_k}\eta$ is the configuration obtained from η by exchanging the variables $\eta(x)$ and $\eta(x+e_k)$:

$$(\sigma^{x,x+e_k}\eta)(y) = \begin{cases} \eta(x+e_k) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+e_k, \\ \eta(y) & \text{otherwise.} \end{cases} \tag{4.3}$$

We consider the Markov process $\{\eta_t : t \geq 0\}$ on the configurations $\{0,1\}^{\mathbb{T}_N^d}$ associated to the generator L_N in the diffusive scale, i.e., L_N is speeded up by N^2.

The descrition of the stochastic evolution of the process is the following: After a time given by an exponential distribution, at rate $\xi_{x,x+e_k} c_{x,x+e_k}(\eta)$ the occupation variables $\eta(x)$ and $\eta(x+e_k)$ are exchanged. Note that only nearest neighbor jumps are allowed. If W is differentiable at $x/N \in [0,1)^d$, the rate at which particles are exchanged is of order 1 for each direction, but if some W_k is discontinuous at x_k/N , it no longer holds. In fact, assume, to fix ideas, that W_k is discontinuous at x_k/N, and smooth on the segments $(x_k/N, x_k/N + \varepsilon e_k)$ and $(x_k/N - \varepsilon e_k, x_k/N)$. Assume, also, that W_j is differentiable in a neighborhood of x_j/N for $j \neq k$. In this case, the rate at which particles jump over the bonds $\{y - e_k, y\}$, with $y_k = x_k$, is of order $1/N$, whereas in a neighborhood of size N of these bonds, particles jump at rate 1. Thus, note that a particle at site $y - e_k$ jumps to y at rate $1/N$ and jumps at rate 1 to each one of the $2d - 1$ other options. Particles, therefore, tend to avoid the bonds $\{y - e_k, y\}$. However, since time will be scaled diffusively, and since on a time interval of length N^2 a particle spends a time of order N at each site y, particles will be able to cross the slower bond $\{y - e_k, y\}$. The scaling limits of this interacting particle systems in inhomogeneous media may, for instance, model diffusions in which permeable membranes, at the points of discontinuities of the conductances W, tend to reflect particles, creating space discontinuities in the density profiles. For more details see [17].

A sequence of probability measures $\{\mu_N : N \geq 1\}$ on $\{0,1\}^{\mathbb{T}_N^d}$ is said to be associated to a profile $\rho_0 : \mathbb{T}^d \to [0,1]$ if for every $\delta > 0$ and every function $H \in \mathfrak{D}_W(\mathbb{T}^d)$:

$$\lim_{N\to\infty} \mu_N \left\{ \eta; \ \Big| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N)\eta(x) - \int H(u)\rho_0(u)du \Big| > \delta \right\} = 0. \tag{4.4}$$

Let $\gamma : \mathbb{T}^d \to [l, r]$ be a bounded density profile and consider the parabolic differential equation

$$\begin{cases} \partial_t \rho = \nabla A \nabla_W \rho \\ \rho(0, \cdot) = \gamma(\cdot) \end{cases}, \tag{4.5}$$

A function $\rho : [0, T] \times \mathbb{T}^d \to [l, r]$ is said to be a weak solution of the parabolic differential equation (4.5) if the following conditions hold. $\Phi(\rho(\cdot,\cdot))$ and $\rho(\cdot,\cdot)$ belong to $L^2([0,T], H_{1,W}(\mathbb{T}^d))$, and we have the integral identity

$$\int_{\mathbb{T}^d} \rho(t,u)H(u)du - \int_{\mathbb{T}^d} \rho(0,u)H(u)du = \int_0^t \int_{\mathbb{T}^d} \rho(s,u)\nabla A \nabla_W H(u)du\, ds,$$

for every function $H \in \mathfrak{D}_W(\mathbb{T}^d)$ and all $t \in [0,T]$.

Existence of such weak solutions follows from the tightness of the process, which is proved in [14, subsection 6.2], and from the energy estimate given in [17, Lemma 6.2]. Uniqueness of weak solutions was proved in [13].

The main result of this Section, so-called *Hydrodinamic Limit*, is the following.

Theorem 4.1. *Fix a continuous initial profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and consider a sequence of probability measures μ_N on $\{0,1\}^{\mathbb{T}_N^d}$ associated to ρ_0, in the sense of (4.4). Then, for any $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N} \left\{ \Big| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} H(x/N)\eta_t(x) - \int H(u)\rho(t,u)\, du \Big| > \delta \right\} = 0$$

for every $\delta > 0$ and every function $H \in \mathfrak{D}_W(\mathbb{T}^d)$. Here, ρ is the unique weak solution of the non-linear equation (4.5) with $l = 0$, $r = 1$ and $\gamma = \rho_0$.

The strategy of the proof of Theorem 4.1 is the following. Fix $T > 0$ and let $D([0,T],\mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0,T] \to \mathcal{M}$ endowed with the *uniform* topology, where $\mathcal{M}$ be the space of positive measures on $\mathbb{T}^d$ with total mass bounded by one. Denote by $\mathbb{Q}_{\mu_N}^{W,N}$ the measure on the path space $D([0,T],\mathcal{M})$ induced by the measure μ_N and the empirical measure at time t

$$\pi_t^N = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N}\,.$$

Using estimates obtained in homogenization theory we show that:

- $\{\mathbb{Q}_{\mu_N}^{W,N} : N \geq 1\}$ is tight.

- As $N \uparrow \infty$, the sequence of probability measures $\mathbb{Q}_{\mu_N}^{W,N}$ converges in the uniform topology to $\mathbb{Q}_W$, where $\mathbb{Q}_W$ the probability measure on $D([0,T],\mathcal{M})$ concentrated on the deterministic path $\pi(t,du) = \rho(t,u)du$, where ρ is the unique weak solution of (4.5).

References

[1] A. Faggionato, Random walks and exclusion processs among random conductances on random infinite clusters: homogenization and hydrodynamic limit, *Electron. J. Probab.*. **13** (2008), 2217-2247.

[2] W. Feller, On Second Order Differential Operators, *Annals of Mathematics*, **61** (1955),No.1, 90-105.

[3] W. Feller, Generalized second order differential operators and their lateral conditions. *Illinois J. Math.* **1**, (1957), No. 4, 459-504.

[4] T. Franco, C. Landim, Hydrodynamic limit of gradient exclusion processes with conductances, *Archive for Rational Mechanics and Analysis* (Print), **195** (2009), 409-439.

[5] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, New York, 1994.

[6] C. Kipnis, C. Landim, *Scaling limits of interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 320. Springer-Verlag, Berlin, 1999.

[7] S.M. Kozlov, Averaging differential operators with almost periodic, rapidly oscillating coefficients, *Math. USSR-Sb.*, **35** (1979),481-498.

[8] S.M. Kozlov, The method of averaging and walks in inhomogeneous environments, *Russian Math. Surveys*, **40** (1985), pp. 73-145.

[9] S.M. Kozlov, Averaging of difference schemes, *Math. USSR Sbornik*, **57** (1987),No 2, 351-369.

[10] P. Mandl, *Analytical treatment of one-dimensional Markov processes*, Grundlehren der mathematischen Wissenschaften, 151. Springer-Verlag, Berlin, 1968.

[11] G. Papanicolaou, S.R.S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Seria Coll. Math. Soc. Janos Bolyai vol. 27, North-Holland (1979).

[12] A. Piatnitski, E. Remy, Homogenization of Elliptic Difference Operators, *SIAM J. Math. Anal.* **33**, (2001), 53-83.

[13] A.B. Simas, F.J. Valentim, W-Sobolev spaces, *Journal of Mathematical Analysis and Applications*, **382**,(2011), 1, 214-230.

[14] A.B. Simas, F.J. Valentim, Homogenization of Generalized Second-Order Elliptic Difference Operators, *Journal of Differential Equations*, **265**, (2018), 3709-3753.

[15] A.B. Simas, F.J. Valentim, W-Sobolev spaces: Higher order and regularity, *Communications on Pure and Applied Analysis.* **14** (2015),597-607.

[16] J. Farfan, A. B. Simas, F. J. Valentim, Equilibrium fluctuations for exclusion processes with conductances in random environments. *Stochastic Processes and their Applications*, **120**, (2010), 1535-1562.

[17] F.J. Valentim. Hydrodynamic limit of a d-dimensional exclusion process with conductances, *Ann. Inst. H. Poincaré Probab. Statist.* **48**(2012), 1, 188-211.