Matemática
Contemporânea

Vol. 52, 119–141
http://doi.org/10.21711/231766362022/rmc527

Stability and periodic solutions for Nicholson equations with mixed monotone terms

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Abstract. We give a brief overview of some recent results for nonautonomous Nicholson's equation with multiple pairs of time-varying delays and nonlinear terms given by mixed monotone functions. For a particular class of such equations, sufficient conditions for the global attractivity of its positive equilibrium K are established. A family of nonautonomous periodic Nicholson equations is also analysed, and criteria for the existence of at least one positive periodic solution given. In this survey, previous results in the literature are presented, together with some original improvements and achievements. Some open problems are posed.

Keywords: Nicholson equation, mixed monotonicity, equilibria, periodic solution, global attractivity, difference equations, Schwarzian derivative.

2020 Mathematics Subject Classification: 34K12, 34K13, 34K20, 34K25, 92D25.

The author is partially supported by Fundação para a Ciência e a Tecnologia (Portugal) under project UIDB/04561/2020, e-mail: teresa.faria@fc.ul.pt.

1 Introduction

In 1980 Gurney et al. [17] introduced the model nowadays known as Nicholson's blowflies equation,

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)} \quad (p, \delta, \tau > 0), \tag{1.1}$$

to model the life cycle of the Australian blowfly, responsible for a plague with severe consequences in sheep as well as other cattle industries. Since its introduction, (1.1) has been extensively studied, since the model fits well the data collected by the Australian entomologist A. Nicholson in a series of experiments in the 1950's, see e.g. [32]. In (1.1), $x(t)$ stands for the size of the adult blowfly population at time t, τ is the time of the life-cycle of the fly, δ the adult mortality rate, p the maximal egg production rate, $1/a$ the size at which the population produces eggs at maximal rate, and xe^{-ax} the birth function. Innumerous scalar and multi-dimensional variants of (1.1) have been proposed and analysed, to account for different mathematical biology frameworks. Their potential to further developments of the theory of delay differential equations (DDEs) has had an important role. In view of all generalizations and applications of Nicholson equations and systems, their importance to motivate significant theoretical advances, and the extensive literature covering many relevant topics, it is not possible to mention all the major contributions on the subject. For some selected works on autonomous scalar models, see [3, 19, 27, 28, 40] as well as additional references in Section 2. Among many other real world applications, we stress that generalized and modified versions of (1.1) include a neoclassic growth model used in economics [20], periodic equations [3, 5, 6, 13, 22] and systems [12] to account for periodic environments (as treated in Section 4), a diffusive Nicholson's equation derived as a suitable model for an age structured single species population [16, 38, 39], n-dimensional Nicholson's systems with patch structured used in population dynamics [10, 15, 23, 24, 41] or as compartmental leukemia and marine models [4], etc.

In this note, we give a brief survey of classical results and some recent achievements for nonautonomous scalar Nicholson's equation with multiple pairs of delays and nonlinear mixed monotonicities. First, we concentrate our attention on Nicholson's equations with multiple pairs of time-varying delays of the form

$$x'(t) = \beta(t)\left(\sum_{j=1}^{m} p_j x(t-\tau_j(t))e^{-a_j x(t-\sigma_j(t))} - \delta x(t)\right), \tag{1.2}$$

where $p_j, a_j, \delta \in (0,\infty)$ and $\beta(t), \sigma_j(t), \tau_j(t)$ are continuous, non-negative and bounded, with $\beta(t)$ bounded from below by a positive constant, and give criteria established in [14] for the global attractivity of the positive equilibrium. Secondly, we shall consider more general nonautonomous equations

$$x'(t) = \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-a_j(t)x(t-\sigma_j(t))} - \delta(t)x(t), \tag{1.3}$$

where all coefficients $\beta_j(t), a_j(t), \delta(t)$ and delays $\tau_j(t), \sigma_j(t)$ are ω-periodic, continuous and nonnegative functions, for some $\omega > 0$. Sufficient conditions for the existence (and stability, if $\tau_j(t) = \sigma_j(t)$ for all j) of positive periodic solutions for the periodic equation (1.3) are derived from material in [12, 13]. In both Sections 2 and 3, some original further consequences are exploited, examples given and open problems stated.

We recall some notation and standard definitions used in the next sections. A general scalar DDE with finite delay $\tau > 0$ is written in abstract form as $x'(t) = f(t, x_t)$, where $f : D \subset \mathbb{R} \times C \to \mathbb{R}$. Here, $C := C([-\tau, 0]; \mathbb{R})$ equipped with the supremum norm $\|\phi\| = \max_{\theta\in[-\tau,0]} |\phi(\theta)|$ is taken as the phase space, and as usual $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$ [37]. Due to the biological motivation for the models under consideration, only nonnegative solutions are of interest, and therefore admissible. The set $C_0^+ := \{\phi \in C : \phi(\theta) \geq 0 \text{ on } [-\tau, 0), \phi(0) > 0\}$ is taken as the set of admissible initial conditions. In this way, under uniqueness conditions, we write $x_t(t_0, \phi) \in C$, or $x(t; t_0, \phi) \in \mathbb{R}$, to denote the solution of $x'(t) = f(t, x_t)$ with the initial condition

$$x_{t_0} = \phi \in C_0^+, \tag{1.4}$$

for $(t_0, \phi) \in D$, defined on a maximal interval $[t_0, \eta(\phi))$. For $f(t, \phi)$, we recall Smith's *quasi-monotone condition* (Q) (see [34, p. 78]):

(Q) for $(t, \phi), (t, \psi) \in D, \phi \le \psi$ and $\phi(0) = \psi(0)$, then $f(t, \phi) \le f(t, \psi)$, $t \ge t_0$.

The property (Q) guarantees monotonocity of solutions to $x'(t) = f(t, x_t)$ relative to initial data and allows comparison between solutions of two DDEs, $x'(t) = f(t, x_t), x'(t) = g(t, x_t)$ with $f \le g$: if at least one of them satisfies (Q), then $x(t, \sigma, \phi, f) \le x(t, \sigma, \psi, g)$ for all $t \le \sigma$ and $\phi \le \psi$ ([34, Theorem 5.2.1]). For (1.2), the solutions with initial conditions (1.4) satisfy $-\delta\beta(t)x(t) \le x'(t) \le \beta(t)\Big(\sum_{j=1}^m p_j x(t-\tau_j(t)) - \delta x(t)\Big) =: L(t, x_t)$; moreover, the solutions of the ODE $x'(t) = -\delta\beta(t)x(t)$ with initial conditions $(t_0, x_0) \in \mathbb{R} \times (0, \infty)$ are positive, and $L(t, \cdot)$ is a linear bounded operator for all t, with $L(t, \phi)$ satisfying (Q). Thus, the above result of comparison of solutions leads to the conclusion that solutions of the initial value problems (1.2)-(1.4) are defined and positive on $[t_0, \infty)$. A similar deduction holds for (1.3) and other models presented here.

Definition 1.1. Consider a scalar DDE $x'(t) = f(t, x_t)$ with a nonnegative equilibrium x^*. x^* is said to be **stable** (on an interval $[\alpha, \infty)$) if for any $\varepsilon > 0$ and $t_0 \ge \alpha$ there is $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x(t; t_0, \phi) - x^*| < \varepsilon$ for all $t \ge t_0, \phi \in C_0^+$, whenever $\|\phi - x^*\| < \delta$; x^* is called a **global attractor** if $\lim_{t\to\infty} x(t) = x^*$ for all solutions $x(t) = x(t; t_0, \phi)$ with initial conditions (1.4); x^* is **globally asymptotically stable** (GAS) if it is stable and globally attractive. The equation $x'(t) = f(t, x_t)$ is (uniformly) **persistent** (in the set C_0^+) if there exists $m > 0$ such that

$$\liminf_{t\to\infty} x(t; t_0, \phi) \ge m, \quad \phi \in C_0^+,$$

and it is **permanent** (in the set C_0^+) if there exist $m, M > 0$ such that

$$m \le \liminf_{t\to\infty} x(t; t_0, \phi) \le \limsup_{t\to\infty} x(t; t_0, \phi) \le M, \quad \phi \in C_0^+.$$

The remainder of the paper is organized as follows: in Section 2, we give some preliminary results for (1.1) and autonomous generalizations of (1.1), criteria from [1, 9], and some insights on the procedure linking difference equations to DDEs, namely in [8]. For the latter, the relevance of the use of Schwarzian derivatives will be apparent. With $p > \delta$, in Section 3 we give a main result from [14] on the global attractivity of K, as well as some improvements under slightly weaker conditions. In Section 4, a periodic version of (1.2) is considered. Firstly, we consider the periodic case without mixed monotonicities, i.e., (1.3) with $\tau_j \equiv \sigma_j$ for all j. By imposing conditions for the permanence, a criterion for the existence and global attractivity of a positive periodic solution is derived from broader results proven in [12] for Nicholson systems. Then, for the mixed monotone situation and without assuming the permanence, a fixed point theorem on cones is used to establish the existence of at least one positive periodic solution for (1.3), along the ideas in [5, 13]. Only proofs of new results are presented. References for the omitted proofs are given.

2 Preliminary results

The global and local behaviour of solutions to (1.1) and generalized versions of (1.1) have been deeply studied, yet many problems remain unsolved [3]. The literature is very extensive: here we only refer to a few selected works, also for further references therein.

We start with an overview of some results for (1.1), where, after a scaling, we take $a = 1$. Besides the trivial equilibrium 0, if $p/\delta > 1$ there exists a positive equilibrium (often called the carrying capacity) $K = \log(p/\delta)$. A sharp criterion for extinction versus the (absolute) global attractivity of K is well known [21, 34, 40]: if $p/\delta \leq 1$, then 0 is GAS – which corresponds to the extinction of the fly population; if $1 < p/\delta \leq e^2$ (or equivalently, when $0 < K \leq 2$), then K is globally attractive for all delays $\tau > 0$. However, when $p/\delta > e^2$, K attracts all positive solutions only if the delay τ is sufficiently small, say $\tau < \tau^*$; moreover, a Hopf bifurcation occurs at some values $\tau_n\,(n\in\mathbb{N}_0)$, the first bifurcation point being $\tau_0=\frac{\pi-\arccos(1/(K-1))}{\delta\sqrt{(K-1)^2-1}}$ [37], leading to the appearance of stable non-trivial periodic solutions. Several estimates for the above mentioned upper bound value τ^* of the delay have been given in the literature [3, 4, 19, 28, 34, 40]. Nicholson's equations with multiple discrete delays

$$x'(t)=-\delta x(t)+\sum_{j=1}^{m}p_jx(t-\tau_j)e^{-ax(t-\tau_j)}. \tag{2.1}$$

have also been studied by several authors. Similarly to what happens for the original equation (1.1), for (2.1) the equilibrium zero is GAS if $p:=\sum_{j=1}^m p_j\leq\delta$, whereas $p>\delta$ implies the existence of a positive equilibrium given by $K=\frac{1}{a}\log(\frac{p}{\delta})$; from Liz et al. [28], with $1<p/\delta\leq e^2$, K is still GAS for all values of delays τ_j.

The equation with several delays and different coefficients a_j's, as in

$$x'(t)=-\delta x(t)+\sum_{j=1}^{m}p_jx(t-\tau_j)e^{-a_jx(t-\tau_j)}, \tag{2.2}$$

has not been often addressed. One difficulty is that, when $p:=\sum_{j=1}^m p_j>\delta$, the positive equilibrium K still exists but it is known only implicitly by the equation

$$\sum_{j=1}^{m}p_je^{-a_jK}=\delta, \tag{2.3}$$

in contrast with the explicit value

$$K=\frac{1}{a}\log(\frac{p}{\delta}) \tag{2.4}$$

for (2.1). Nevertheless, a more general criterion for the absolute global attractivity of K can be achieved. A simple proof is given below.

Proposition 2.1. If $1<p/\delta\leq e^{2a^-/a^+}$, where $a^-=\min_j a_j, a^+=\max_j a_j$, then the positive equilibrium K is a global attractor of all positive solutions of (2.2).

Proof. From $1 < p/\delta \le e^{2a^-/a^+}$ and (2.3), we deduce that $a_j K \le \frac{a_j}{a^-}\log(\frac{p}{\delta}) \le 2$ for all $j = 1, \dots, m$. Denote $h(x) = xe^{-x}$ for $x \ge 0$. When $x \in (0, 2]$, the properties of h lead to (see [15])

$$|h(y) - h(x)| < e^{-x}|y - x| \quad \text{for} \quad y > 0, y \ne x. \tag{2.5}$$

Effecting the change $y(t) = x(t) - K$ and using (2.3), (2.2) becomes

$$\begin{aligned} y'(t) &= -\delta y(t) + \sum_{j=1}^{m} p_j \Big[(K + y(t-\tau_j))e^{-a_j(K+y(t-\tau_j))} - Ke^{-a_j K}\Big] \\ &= -\delta y(t) + \sum_{j=1}^{m} \frac{p_j}{a_j}\Big[h\Big(a_j(K + y(t-\tau_j))\Big) - h(a_j K)\Big] \end{aligned} \tag{2.6}$$

Since condition $p > \delta$ implies the permanence of (2.2), for any fixed solution $y(t)$ of (2.6) the limits

$$-v = \liminf_{t\to\infty} y(t), \qquad u = \limsup_{t\to\infty} y(t)$$

are well defined and satisfy $-K < -v \le u < \infty$. To conclude that $y(t) \to 0$ as $t \to \infty$, it suffices to show that $\max(v, u) = 0$.

For the sake of contradiction, assume that $\max(v, u) > 0$. Suppose that $\max(v, u) = u$. Let $t_k \to \infty$, with $y(t_k) \to u$ and $y'(t_k) \to 0$. By taking a subsequence if necessary, assume also that $y(t_k - \tau_j) \to u_j$, $1 \le j \le m$. From the definition of u, we have $|u_j| \le u$ for all j. Next, from (2.6) and letting $t_k \to \infty$, we obtain

$$\delta u = \sum_{j=1}^{m} \frac{p_j}{a_j}\Big[h\Big(a_j(K + u_j)\Big) - h(a_j K)\Big].$$

If $u_j = 0$ for all j, then $u = 0$, a contradiction. Otherwise, $u_j > 0$ for some j, and (2.5) and the fact that $a_j K \le 2$ yield

$$\delta u < \sum_{j=1}^{m} \frac{p_j}{a_j} e^{-a_j K} a_j |u_j| \le \delta u,$$

again a contradiction. The case $\max(v, u) = v$ is treated in a similar way. Thus $\max(v, u) = 0$. □

Recently, the autonomous equation with two different delays

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\sigma)}, \tag{2.7}$$

where $p > \delta > 0$ and $\tau \geq \sigma \geq 0$, was studied by El-Morshedy and Ruiz-Herrera [9], who gave conditions for the global attractivity of its positive equilibrium $K = \frac{1}{a}\log(p/\delta)$. In (2.7), there is a *mixed monotonicity* nonlinear term, expressed by $g(x(t-\tau), x(t-\sigma))$, with $g(x, y) = pxe^{-ay}$ monotone increasing in the first variable and monotone decreasing in the second one.

For the last two decades, authors have considered DDEs with a so-called mixed monotonicity of the form

$$x'(t) = Lx_t + g(x(t-\tau), x(t-\sigma)), \tag{2.8}$$

where $L : C([-r, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ (for $r = \max(\sigma, \tau)$ and $C([-r, 0]; \mathbb{R}^n)$ equipped with the supremum norm) is a linear bounded operator and $g(x, y)$ is monotone increasing in x and monotone decreasing in y. Models with distributed delays or nonautonomous versions of (2.8) have also been studied. The literature goes back to the works Chen [6] and Smith [35, 36]. Lately the topic has received much attention, see the works of Berezansky and Braverman [1, 2], El-Moshedy and Ruiz-Herrera [9, 10], Győri et al. [18], Huang et al. [20], among others.

For DDEs, in general "small delays are harmless" (see e.g. [37]), yet this might be valid only if there is a single delay in each term. As shown in [2] for a Mackey-Glass-type equation, the consideration of a pair of delays in each nonlinearity as in (2.8) can lead to unbounded solutions or chaotic oscillations. In fact, Berezansky and Braveman [2] set as an open problem to provide stability criteria for the equilibria of a Nicholson model with two time-varying delays as follows:

$$x'(t) = \beta(t)\big(px(t-\tau(t))e^{-ax(t-\sigma(t))} - \delta x(t)\big), \tag{2.9}$$

where $p, a, \delta \in (0, \infty)$, $\beta(t) > 0$, and $\tau(t), \sigma(t)$ represent respectively the incubation and the maturation delays for the blowfly. For the particular case (2.7), El-Morshedy and Ruiz-Herrera [9] established the criterion below.

Theorem 2.1. [9] Assume $\tau \geq \sigma \geq 0$ and $p > \delta$. Then (2.7) is permanent. Moreover, if in addition

$$(e^{\delta\sigma} - 1)\log(p/\delta) \leq 1, \tag{2.10}$$

the equilibrium $K = \frac{1}{a}\log(p/\delta)$ of (2.7) is globally attractive.

Clearly, this result on the global attractivity of K depends on the size of both delays: on the one hand τ is required to be larger than σ, and on the other hand (2.10) imposes an upper bound for the size of σ. We also stress that, contrary to what happens with the Nicholson's equations (1.1) or (2.2), the carrying capacity K is never *absolutely* GAS, (i.e. GAS for all values of delays σ, τ). This is shown by the simple example below.

Example 2.1. Consider (2.7) with $a = 1$, and denote $r = \tau/\sigma$. After the scaling $t \mapsto t/\sigma$, the linearized equation about $K = \log(\frac{p}{\delta})$ is

$$y'(t) = \delta\sigma[-y(t) - Ky(t-1) + y(t-r)],$$

with characteristic equation

$$\Delta(\lambda) := \lambda + \delta\sigma[1 + Ke^{-\lambda} - e^{-\lambda r}] = 0.$$

With $r = r_{n,p} := \frac{2p\pi}{\delta K \sigma_n}$, $\sigma = \sigma_n := \frac{1}{\delta K}(\frac{\pi}{2} + 2n\pi)$ $(n \in \mathbb{N}_0, p \in \mathbb{N})$, we have $\Delta(i\delta K\sigma_n) = 0$. Moreover, choosing σ as the bifurcating parameter and fixing e.g. $r = 2\pi$, for $\lambda = \lambda(\sigma) = \alpha(\sigma) + i\omega(\sigma)$ with σ close to σ_n and $\omega(\sigma_n) = i\delta K\sigma_n$, we have $\Delta(\lambda(\sigma)) = 0$ and $\alpha'(\sigma_n) \neq 0$. As a consequence, Hopf bifurcations occur at each bifurcating point $\sigma = \sigma_n$, leading to the appearance of periodic solutions. In particular, K is not a global attractor for (2.7) for some delays sufficiently large, in spite of how close to 1 the ratio $\frac{p}{\delta}$ is.

Going back to the Nicholson's equation with multiple pairs of time-varying delays (1.2), in what follows we impose the general assumptions below:

$p_j, a_j, \delta \in (0,\infty)$, $\beta, \sigma_j, \tau_j : [t_0,\infty) \to [0,\infty)$ are continuous and bounded ($1 \le j \le m$), with

$$0 < \beta^- := \inf_{t\ge t_0} \beta(t) \le \beta(t) \le \sup_{t\ge t_0} \beta(t) =: \beta^+. \tag{2.11}$$

Throughout this note, denote

$$p = \sum_{j=1}^m p_j, \quad a^+ = \max_j a_j, \quad a^- = \min_j a_j$$

and

$$\tau = \max\{\sup_{t\ge t_0} \tau_j(t), \sup_{t\ge t_0} \sigma_j(t) : j = 1,\dots,m\}.$$

As mentioned, $C := C([-\tau,0];\mathbb{R})$ with the sup norm is the phase space and the solutions of (1.2) with the initial conditions (1.4) are defined and positive on $[t_0,\infty)$.

In [29], Long and Gong showed that the equilibrium 0 of (1.2) is a global attractor of all positive solutions if $p \le \delta$, with 0 globally exponentially stable if $p < \delta$.

The study of the global attractivity of a positive equilibrium, when it exists, is more interesting in terms of applications. For (1.2) with $p > \delta$, again the positive equilibrium K is implicitly given by (2.3), whereas for (2.9) K has the value in (2.4). To address the global attractivity of K, Faria and Prates [14] extended to (1.2) the method developed in [9] for (2.7): the main idea is to associate to the DDE a suitable difference equation $x_{n+1} = h(x_n)$, where h has negative Schwarzian derivative and for which we prove that K is a global attractor. Then, we show that this fact implies that K is a global attractor for (1.2) as well. The main novelty in [14] is that this technique was used to handle a situation with more than one pair of *nonautonomous* delays. It is our belief that the method in [14] can be applied to other equations or systems.

As anticipated in the Introduction, links between DDEs and difference equation (DEs) of the form

$$x_{n+1} = h(x_n), \qquad n \in \mathbb{N}_0, \tag{2.12}$$

where $h : I \to I$ is continuous on a real interval I, have been established by many authors, see Mallet-Paret and Nussbaum [30, 31], Győri and Trofimchuk [19], Liz et al. [28], Liz and Ruiz-Herrera [26]. We also refer to the early works of Coppel [7] and Singer [33], on the attractivity of equilibria for (2.12), and more recently to El-Morshedy and López in [8]. In [8, 19, 25, 33], Schwarzian derivatives were used. If h is a C^3 function on an interval I whose derivative does not vanish, the Schwarzian derivative of h is defined by

$$Sh(x) := \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2.$$

The lemma below is extracted from results in [8] and was crucial for our purposes.

Lemma 2.1. [8]. Let $h : I \to I$ be continuous on a real interval $I = [a, b]$ ($a < b$ with $b \in \mathbb{R}$ or $b = \infty$) with a unique fixed point $x^* \in (a, b)$ and such that

$$(h(x) - x)(x - x^*) < 0 \qquad \text{for all } x \in (a, b), x \neq x^*.$$

(a) If x^* is a global attractor for (2.12), then there are no points $c, d \in I$ with $c < x^* < d$ and $h([c, d]) \supset [c, d]$.

(b) If h is a C^3 function, decreasing on I and such that $Sh(x) < 0$ for $x \in I$ and $-1 \leq h'(x^*) < 0$, then x^* is a global attractor for (2.12).

3 Global attractivity of the positive equilibrium

In this section, sufficient conditions for the global attractivity of the positive equilibrium K of (1.2) (in the set of all positive solutions) are given. As an essential precondition, we need the permanence of (1.2), which follows easily from [1, Theorem 5.6]:

Theorem 3.1. [14] If $p > \delta$, then (1.2) is permanent.

In fact, explicit positive uniform lower and upper bounds for solutions of (1.2) are easily derived from [1, Theorem 5.6], and we obtain

$$Ke^{-2\delta\beta^+\tau} \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le Ke^{2(\delta+p)\beta^+\tau} \tag{3.1}$$

for any solution $x(t)$ of (1.2) with initial condition in C_0^+. Here, the permanence of (1.2) is relevant simply as a preliminary result to derive conditions for the global attractivity of the carrying capacity. Nevertheless, it is clear that Theorem 3.1 largely extends the permanence result obtained in [9] for the Nicholson's equation (2.7).

The global attractivity of the positive equilibrium K of (1.2) is given in the main result below.

Theorem 3.2. [14] Define

$$\begin{aligned} f(x) &= \frac{1}{\delta}\Big(\sum_{j=1}^m p_j e^{-a_j x}\Big),\ \ x \ge 0,\\ \zeta_M &= \max_{1\le j\le m} \limsup_{t\to\infty} \int_{t-\sigma_j(t)}^t \beta(s)\,ds. \end{aligned} \tag{3.2}$$

Assume $p > \delta$ and that:

(H1) $Sf(x) < 0$;

(H2) $a^+K(e^{\delta\zeta_M} - 1) \le 1$.

Then the equilibrium K of (1.2) is globally attractive.

Here, we just highlight the main argument for a proof, which can be found in [14]. To start with, note that $f(K) = 1$ and $f(x)$ is decreasing on $[0,\infty)$. Take a solution $x(t)$ with initial condition in C_0^+ and define

$$l := \liminf_{t\to\infty} x(t) \quad \text{and} \quad \limsup_{t\to\infty} x(t) =: L. \tag{3.3}$$

By the permanence in Theorem 3.1, we have $0 < l \le L < \infty$. If $l = L$, from (1.2), we get $x'(t) = \beta(t)g(x_t)$, where $\lim_{t\to\infty} g(x_t) = \delta L\big(f(L) - 1\big)$. If $L \ne K$, then $f(L) \ne 1$, leading to $x(t) \to \pm\infty$ as $t \to +\infty$, which is not possible. Hence $L = K$.

Next, for the sake of contradiction, suppose that $l < L$. By using equation (1.2), the fluctuation lemma and several other techniques, we deduce that $e^{-\delta\zeta_M}K =: \theta < l < K < L$ and that

$$L \leq h(l), \quad l \geq h(L), \tag{3.4}$$

where

$$h(x) = \frac{e^{-\delta\zeta_M}K}{1-(1-e^{-\delta\zeta_M})f(x)} \quad \text{for} \quad x > \theta,$$

provided that $f(x)(1-e^{-\delta\zeta_M}) < 1$ for $x > \theta$, or, in other words, that $\theta \geq \theta_M$ where $\theta_M > 0$ is such that $f(\theta_M) = (1-e^{-\delta\zeta_M})^{-1}$.

From this point onward, Lemma 2.1 is applied. The function h is decreasing on (θ_M, ∞) with $K = 1$ the unique fixed point. From (3.4), $[l, L] \subset h([l, L])$, therefore Lemma 2.1(a) implies that K is not a global attractor of the DE $x_{n+1} = h(x_n)$ in the interval $I = [l, \infty) \subset (\theta, \infty)$. On the other hand, to apply Lemma 2.1(b) and reach the opposite conclusion, the following conditions should hold: (i) $Sh(x) < 0$ for $x > 0$; (ii) $|h'(K)| \leq 1$; (iii) $\theta \geq \theta_M$. Since $Sh(x) = Sf(x)$ [33], and $Sf(x) < 0$ by (H1), condition (i) is satisfied. Next, one can prove that the remaining conditions (ii) and (iii) follow from (H2). Thus a contradiction is reached, and consequently $l = L = K$.

For the Nicholson's equation with a single pair of time-dependent delays, the function f reads as $f(x) = \frac{p}{\delta}e^{-ax}$ with Schwarzian derivative $Sf(x) = -a^2/2 < 0$. The corollary below strongly improves the criterion in Theorem 2.1.

Corollary 3.1. Assume $p > \delta$, denote $\zeta_M := \limsup_{t\to\infty} \int_{t-\sigma(t)}^{t} \beta(s)\, ds$ and further assume

$$(e^{\delta\zeta_M} - 1)\log\frac{p}{\delta} \leq 1.$$

Then the equilibrium K of (2.9) is globally attractive (in C_0^+).

A criterion where a constraint implying $Sf(x) < 0$ replaces (H1) and where (H2) does not depend on K (since in the case of multiple pairs of delays K is not given explicitly) is useful.

Corollary 3.2. With the previous notations, assume $p > \delta$, and

(H1*) $\dfrac{a^+}{a^-} < \dfrac{3}{2}$,

(H2*) $\dfrac{a^+}{a^-}(e^{\delta\zeta_M} - 1)\log\dfrac{p}{\delta} \le 1$.

Then the equilibrium K of (1.2) is globally attractive.

In fact, we have $f(K) = 1$, $a^-K \le \log(p/\delta) \le a^+K$ and consequently (H2*) implies (H2). Moreover, for $f(x)$ as in (3.2),

$$\begin{aligned} Sf(x) &= \frac{\sum_j p_j a_j^3 e^{-a_j x}}{\sum_j p_j a_j e^{-a_j x}} - \frac{3}{2}\left(\frac{\sum_j p_j a_j^2 e^{-a_j x}}{\sum_j p_j a_j e^{-a_j x}}\right)^2 \\ &= \frac{1}{2(\sum_j p_j a_j e^{-a_j x})^2} \sum_{j,i} p_j p_i e^{-(a_j + a_i)x} a_j^2 a_i (2a_j - 3a_i). \end{aligned}$$

Thus, $Sf(x) < 0$ if assumption (H1*) holds.

With $m = 2$, an estimate better than (H1*) is given below.

Lemma 3.1. Consider $f(x) = p_1 e^{-a_1 x} + p_2 e^{-a_2 x}$, $x > 0$, where $p_j, a_j > 0$ for $j = 1, 2$. Then, $Sf(x) < 0$ for all $x > 0$ if

$$\frac{a^+}{a^-} < 2 + \sqrt{3}. \tag{3.5}$$

Proof. With x fixed and $c_j = p_j e^{-a_j x} \in (0, p_j)$, we need to show that

$$A := 2(c_1 a_1^3 + c_2 a_2^3)(c_1 a_1 + c_2 a_2) - 3(c_1 a_1^2 + c_2 a_2^2)^2 < 0.$$

Take e.g. $a_2 \ge a_1$. With $X = a_2/a_1$,

$$\begin{aligned} A &= -(c_1^2 a_1^4 + c_2^2 a_2^4) + 2c_1 c_2 (a_1^3 a_2 + a_1 a_2^3 - 3a_1^2 a_2^2) \\ &= a_1^4 \left\{ -(c_1 - c_2 X^2)^2 + 2c_1 c_2 X[(X-2)^2 - 3] \right\} \end{aligned}$$

If $1 \le X \le 2 + \sqrt{3}$, we obtain $A < 0$. □

Example 3.1. Consider (1.2) with $m = 2$, $p_1 = \frac{3}{5}, p_2 = \frac{1}{2}, a_1 = \frac{5}{2}, a_2 = 1, \delta = 1$, and $\beta(t), \tau_j(t), \sigma_j(t)\,(j = 1, 2)$ satisfing the general conditions set above. We have $p = 1.1 > \delta$ and $a^+/a^- = \frac{5}{2} < 2 + \sqrt{3}$. If $\zeta_M \le$

$\log\left(1+\frac{2}{5\log(1.1)}\right) \approx 1.6480$, (H2*) is satisfied and we conclude that the positive equilibrium K is globally attractive. Keeping the same values with the exception of a_1, which is increasead to $a_1 = \frac{7}{2}$, the same conclusion is drawn if $\zeta_M \le \log\left(1+\frac{2}{7\log(1.1)}\right) \approx 1.3857$.

4 Periodic Nicholson equations with mixed monotone terms

We start by considering a periodic Nicholson version of (2.2) given by

$$x'(t) = \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-a_j(t)x(t-\tau_j(t))} - \delta(t)x(t), \qquad (4.1)$$

with all coefficients $\beta_j(t), a_j(t), \delta(t)$ and delays $\tau_j(t)$ in the set $C_\omega^+(\mathbb{R})$ of the ω-periodic continuous and nonnegative functions, for some $\omega > 0$. The permanence of (4.1), the existence and attractivity of a positive periodic solution were studied in [3, 22] and some references therein.

Recall that a solution $x^*(t)$ of (4.1) is said to be **globally exponentially stable** if there exist $K, \beta > 0$ such that $|x(t; t_0, \phi) - x^*(t)| \le Ke^{-\beta(t-t_0)}\|\phi - x^*_{t_0}\|$ for all $t \ge t_0$ and $\phi \in C_0^+$. Applying to the scalar case the results recently established in [12] for periodic Nicholson-type systems, we derive the following:

Theorem 4.1. Define $\beta(t) = \sum_{j=1}^m \beta_j(t)$ and assume that $\min_{t\in[0,\omega]} \frac{\beta(t)}{\delta(t)} > 1$. Then, (4.1) is permanent and there exists a positive ω-periodic solution $x^*(t)$. If in addition

$$\max_{t\in[0,\omega]} \frac{\beta(t)}{\delta(t)} < 2e\frac{a^-}{a^+},$$

where $a^- = \min_j \min_{[0,\omega]} a_j(t)$, $a^+ = \max_j \max_{[0,\omega]} a_j(t)$, then the periodic solution $x^*(t)$ is globally exponentially stable.

An interesting open problem stated in [12] is whether the global attractivity of $x^*(t)$ remains true under the weaker condition $\max_{t\in[0,\omega]} \frac{\beta(t)}{\delta(t)} < e^{2a^-/a^+}$ (compare with Proposition 2.1). In fact, this is the case if all the delays $\tau_j(t)$ are constant and multiples of the period, $\tau_j(t) = n_i\omega$ for some $n_i \in \mathbb{N}, 1 \le j \le m$ [11], but the result is unknown for general time-varying periodic delays.

Next, we consider the mixed monotone periodic version of (1.2) given by (1.3), with all coefficients and delays in $C_\omega^+(\mathbb{R})$ and $D(\omega) := \int_0^\omega d(t)\,dt > 0$. $C_\omega^+(\mathbb{R})$ is a cone of the usual Banach space $C_\omega(\mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{R} | \, \phi$ is continuous and ω-periodic$\}$ with the supremum norm $\|\phi\| = \max_{t\in[0,\omega]} |\phi(t)|$.

In this section, we give conditions for the existence of a positive ω-periodic solution of (1.3), which were established in [5] by using the Krasnoselskii fixed point theorem on cones (see also [13]). As a suitable cone, we define

$$\mathcal{K} = \mathcal{K}(\sigma) = \{x \in C_\omega^+ : x(t) \ge \sigma\|x\| \text{ for } t \in [0, \omega]\},$$

where $\sigma = e^{-D(\omega)}$.

Theorem 4.2. [5] Consider a scalar equation

$$x'(t) = -d(t)x(t) + g(t, x_t), \quad t \ge 0, \tag{4.2}$$

where $d, g(\cdot, \phi) \in C_\omega^+(\mathbb{R})$, $D(\omega) := \int_0^\omega d(t)\,dt > 0$, $g(t, x_t)$ is bounded on bounded sets of $\mathbb{R} \times C_\omega^+$, and $t \mapsto g(t, x_t)$ is uniformly equicontinuous for $t \in [0, \omega]$ on bounded sets of C_ω^+. Assume:

(A1) There are constants r_0, R_0 with $0 < r_0 < R_0$ and functions $b_1, b_2 \in C_\omega^+(\mathbb{R})$ with $\int_0^\omega b_i(t)\,dt > 0\,(i = 1, 2)$, such that for $x \in \mathcal{K}$ and $t \in [0, \omega]$ it holds:

$$\begin{aligned} g(t, x_t) &\ge b_1(t)u \quad \text{if } 0 < u \le x \le r_0, \\ g(t, x_t) &\le b_2(t)u \quad \text{if } R_0 \le x \le u. \end{aligned}$$

(A2) for b_1, b_2 as in (A1),

$$\begin{aligned} \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d(r)\,dr} b_1(s)\,ds &> e^{D(\omega)} - 1, \\ \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d(r)\,dr} b_2(s)\,ds &< e^{D(\omega)} - 1. \end{aligned}$$

Then there exists at least one positive ω-periodic solution of system (4.2).

As a consequence, some further useful criteria are easily deduced:

Corollary 4.1. [5] Assume (A1) and in addition, for b_1, b_2 as in (A1), one of the following conditions:

(i) $b_2(t) \le d(t) \le b_1(t),\ t \in [0,\omega]$ and $b_1(t_1) \ne d_1(t_1), b_2(t_2) \ne d(t_2)$ for some $t_1, t_2 \in [0,\omega]$;

(ii) $\displaystyle\int_0^\omega b_1(t)\,dt \ge e^{\int_0^\omega d(s)\,ds} - 1,\ \int_0^\omega b_2(t)\,dt \le 1 - e^{-\int_0^\omega d(s)\,ds}$.

Then there exists at least one positive ω-periodic solution of (4.2).

Applying Corollary 4.1 to (1.3) leads to:

Theorem 4.3. There is at least one positive ω-periodic solution of (1.3) if one of the following conditions holds:

(i) $\sum_{j=1}^m \beta_j(t) \geq_{\not\equiv} \delta(t) e^{\int_0^\omega \delta(s)\,ds},\ t \in [0,\omega]$;

(ii) $\int_0^\omega \sum_{j=1}^m \beta_j(t)\,ds \ge e^{\int_0^\omega \delta(s)\,ds}\big(e^{\int_0^\omega \delta(s)\,ds} - 1\big)$.

Proof. (1.3) has the form (4.2) with

$$g(t,\phi) = \sum_{j=1}^m \beta_j(t)\phi(-\tau_j(t))e^{-a_j(t)\phi(-\sigma_j(t))}.$$

For the cone $\mathcal{K} = \mathcal{K}(\sigma)$ as above, $x \in \mathcal{K}$ if and only if $x \in C_\omega^+$ and $x(t) \ge \sigma\|x\|$ where $\sigma = e^{-\int_0^\omega \delta(s)\,ds}$. Write $a^- = \min\limits_j \min\limits_{t\in[0,\omega]} a_j(t), a^+ = \max\limits_j \max\limits_{t\in[0,\omega]} a_j(t)$. For $x \in \mathcal{K}, t \in \mathbb{R}$,

$$\begin{cases} g(t,x_t) \le \sum_{j=1}^m \beta_j(t)e^{-\sigma a^- u}u & \text{if} \quad x \le u \quad \text{and } u \ge R, \\ g(t,x_t) \ge \sum_{j=1}^m \beta_j(t)e^{-a^+ u}\sigma u & \text{if} \quad x \ge u > 0 \quad \text{and } u \le r, \end{cases}$$

for $R > 0$ sufficiently large and $r > 0$ sufficiently small. On the other hand, $e^{-\sigma a^- u} \to 0^+$ as $u \to \infty$ and $e^{-a^+ u} \to 1^-$ as $u \to 0^+$. This implies that, for any $\varepsilon > 0$, (A1) holds with

$$b_1(t) = (1-\varepsilon)\sigma \sum_{j=1}^m \beta_j(t), \quad b_2(t) = \varepsilon.$$

Corollary 4.1 gives the conclusion. □

We remark that in an early work [6], Chen considered a periodic model with $m = 1$,

$$x'(t) = \beta(t)x(t - \tau(t))e^{-a(t)x(t-\sigma(t))}) - \delta(t)x(t),$$

and showed that there exists a positive ω-periodic solution if $\int_0^\omega \beta(s)\,ds > e^{2\int_0^\omega \delta(s)\,ds} \int_0^\omega \delta(s)\,ds$. Clearly, the criterion in Theorem 4.3.(ii) improves Chen's result.

Example 4.1. Consider a 1-periodic Nicholson's equation with two pairs of "mixed delays":

$$y'(t) = -\delta(t)y(t)+\beta_1(t)y(t-\tau_1(t))e^{-y(t-\sigma_1(t))}+\beta_2(t)y(t-\tau_2(t))e^{-y(t-\sigma_2(t))}, \tag{4.3}$$

where $\eta_1, \eta_2 > 0$, $\delta(t) = \frac{1}{2}(1 + \cos(2\pi t)), \beta_1(t) = \eta_1 \left(1 + \frac{1}{2}\cos(2\pi t)\right)$, $\beta_2(t) = \eta_2 \left(1 + \frac{1}{2}\sin(2\pi t)\right)$ and the delay functions $\tau_1(t), \sigma_1(t), \sigma_2(t)$ are 1-periodic.

Here, $e^{\int_0^1 \delta(s)\,ds} = \sqrt{e}, \int_0^1 (\beta_1(s) + \beta_2(s))ds = \eta_1 + \eta_2$. For (4.3), condition (i) in Theorem 4.3 is satisfied if $\eta_1 + \frac{1}{2}\eta_2 \geq \frac{\sqrt{e}}{2} \approx 0.83$, whereas condition (ii) reads as $\eta_1 + \eta_2 \geq e - \sqrt{e} \approx 1.07$. In both cases, equation (4.3) possesses at least one 1-periodic positive solution $x^*(t)$.

We end this note with a couple of other open problems. For the periodic Nicholson eq. (1.3) under the assumptions (i) or (ii) in Theorem 4.3, a pertinent question is whether the positive periodic solution $x^*(t)$ is globally attractive or not. In fact, Theorem 4.3 does not even guarantee the uniqueness of a positive solution. Further investigations should be carried out, to find a criterion for its global attractivity.

In view of the techniques in Section 3, it is also natural to inquire whether it is possible to relate the global attractivity of a positive periodic solution $x^*(t)$ for (1.3), with the global attractivity of a positive equilibrium for a suitable auxiliary difference equation of type (2.12). More generally, an open question is whether the method in Section 3 can be applied to derive the global attractivity of any positive solution for a general nonautonomous equation of the form (1.3), not necessarily periodic.

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