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Summary of "Gelfand–type problems involving the 1–Laplacian operator"

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Abstract. In this paper, the theory of Gelfand problems is adapted to the 1−Laplacian setting. Concretely, we deal with the following problem

$$\begin{cases} -\Delta_1 u = \lambda e^u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain and $\lambda \geq 0$. We prove the existence of a threshold $\lambda^* > 0$ such that Gelfand problem has no solution when $\lambda > \lambda^*$ and there exists a solution when $\lambda \leq \lambda^*$. The 1 dimensional and the radial cases are analyzed in more detail, showing the existence of multiple radial solutions. We also study the behavior of solutions to problems involving the p–Laplacian as p tends to 1, which allows us to identify proper solutions through an extra condition.

Keywords: nonlinear elliptic equations, 1–Laplacian operator, Gelfand problem.

2020 Mathematics Subject Classification: 35J75, 35J20, 35J92.

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1 Introduction

The classical Gelfand problem studies existence and boundedness of positive solutions to

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is an open bounded set in $\mathbb{R}^N$ and $\lambda > 0$. In the threee dimensional setting it has been considered by R. Emden in 1896, A. Fowler in 1926 and I. Gelfand in 1963. Emden and Fowler use it as a model in the study of stellar structure, while Gelfand studies this problem in a combustion reaction setting. It corresponds to a stationary situation where the exponential means an Arrhenius reaction term. For a good introduction to this problem, we refer to [6].

We summarize the main results of this theory:

1. The 1D case has an explicit solution. In the interval $(-1, 1)$ it is

   $$u(x) = \mu - \log\left(1 + \tan^2\left(\sqrt{\frac{\lambda e^\mu}{2}}x\right)\right)$$

   where μ is choosen so that $u(1) = 0 = u(-1)$.

2. There exists $\lambda^* \leq \lambda_1$ such that a solution exists only for $0 < \lambda \leq \lambda^*$.

3. There exist minimal solutions for all $0 < \lambda < \lambda^*$.

4. Bifurcation diagrams in the radial case are known (and they depends on the dimension). For $N \leq 2$, there exist two solution for all $0 < \lambda < \lambda^*$. For $3 \leq N \leq 9$, there exists $0 < \overline{\lambda} < \lambda^*$ such that problem (1.1) exhibits infinitely many solutions when $\lambda = \overline{\lambda}$ and the others values give rise to a finite number of solutions; its bifurcation diagram exhibits oscillations around $\overline{\lambda}$. For $N \geq 10$, every $0 < \lambda < \lambda^*$ has associated an unique solution.

Bifurcation diagrams also indicates what happens when $\lambda = \lambda^*$. The extremal solution is bounded if $N \le 9$, while unbounded if $N \ge 10$.

Moreover, in the case $3 \le N \le 9$ there exists an unbounded solution such that $e^u \in L^1(\Omega)\backslash L^\infty(\Omega)$ which is not extremal.

Similar results are obtained when diffusion is not governed by the Laplacian operator, for instance, is driven by the $p-$Laplacian (with $p > 1$). This subject began to study in the 90s and had been analyzed by many authors (for more information on this setting, we recommend [5] and the recent work [4]). Even though results are analogous, dimensions depend on the specific operator. The three cases become $N \le p$, $p < N < \frac{3p+p^2}{p-1}$ and $N \ge \frac{3p+p^2}{p-1}$.

Our aim is to analyze if similar results hold when 1-Laplacian operator is considered; that is, our objective is to study problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) = \lambda e^u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

Moreover, we want to check if these solutions come from limits of the solutions to Gelfand problems driven by the p–Laplacian as $p \to 1$. Observe that, fixed N, there is a $p > 1$ close to 1 such the bifurcation diagrams shows two different features:

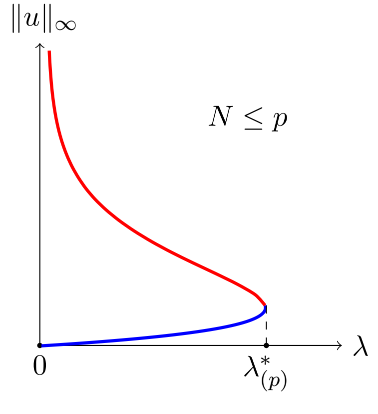

Figure 1.1:

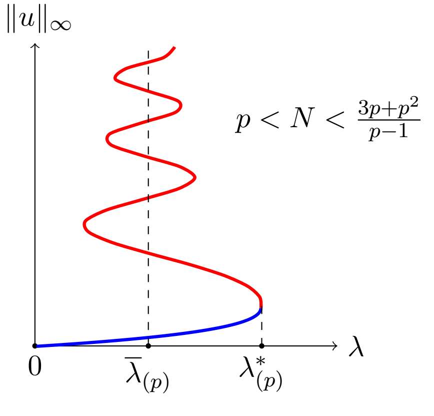

Figure 1.2:

In Section 4, we will show that minimal solutions (represented in blue) tend to trivial solutions as p goes to 1.

This paper is a summarised version of [8], where proofs and more references can be found. It was presented in the Special Session on Elliptic Equations of "ICMC Summer Meeting on Differential Equations – 2022 Chapter" held in São Carlos–SP, Brazil.

2 Preliminaries

The natural space to study problems involving the 1–Laplacian is the space of functions of bounded variation, defined as

$$BV(\Omega) = \left\{ u \in L^1(\Omega) \, : \, Du \text{ is a bounded Radon measure } \right\}$$

where $Du : \Omega \to \mathbb{R}^N$ denotes the distributional gradient of u. We recall that the space $BV(\Omega)$ with norm

$$\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \int_\Omega |u|$$

is a Banach space which is non reflexive and non separable. For more information on functions of bounded variation, we refer to [1].

Definition 2.1. A solution to (1.2) is $u \in BV(\Omega)$ such that $e^u \in L^{N,\infty}(\Omega)$ and there exists $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ satisfying:

1. $\|\mathbf{z}\|_\infty \leq 1$,
2. $-\operatorname{div} \mathbf{z} = \lambda e^u$ in the sense of distributions,
3. $(\mathbf{z}, Du) = |Du|$ as measures,
4. $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$ on $\partial\Omega$.

This concept of solution lies on that introduced by Andreu, Ballester, Caselles and Mazón (see [2]).

The space $L^{N,\infty}(\Omega)$ is called Marcinkiewicz or weak Lebesgue space.

The vector field $\mathbf{z}$ plays the role of the quotient $\dfrac{Du}{|Du|}$ since it satisfies $\|\mathbf{z}\|_\infty \le 1$ and $(\mathbf{z}, Du) = |Du|$. This "dot product" $(\mathbf{z}, Du)$ was defined by Anzellotti in [3] for vectors fields $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ and functions u of bounded variations that can be paired in a certain sense. For instance,

1. $\operatorname{div} \mathbf{z} \in L^N(\Omega)$ and $u \in BV(\Omega)$.

2. $\operatorname{div} \mathbf{z} \in L^1(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$.

Using truncations this concept can be extended to $\operatorname{div} \mathbf{z} \in L^{N,\infty}(\Omega)$ and $u \in BV(\Omega)$.

The trace on $\partial\Omega$ of the normal component of $\mathbf{z}$ is denoted by $[\mathbf{z}, \nu]$, this concept also goes back to Anzellotti's theory. The most important result of this theory collects the previous elements in a Green's formula. The requirement $[\mathbf{z}, \nu] \in \operatorname{sign}(-u)$ is a weak form of the Dirichlet boundary condition. (We note that the boundary condition is not satisfied in the sense of traces.)

3 Gelfand problem governed by the 1-Laplacian

A threshold that, in our setting, plays the role of the first eigenvalue of the Laplacian is the Cheeger constant. Recall that the first eigenvalue of the Laplacian in Ω is given as

$$\lambda_1 = \inf\left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \;:\; u \in H_0^1(\Omega)\backslash\{0\} \right\}$$

On the other hand, Cheeger's constant of Ω is defined by $h = \inf \dfrac{P(D)}{|D|}$ where the infimum is taken among all subsets $D \subset \Omega$ that have finite perimeter. Kawohl and Fridman proved in [7] that it can be written as

$$h = \inf\left\{ \frac{\int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}}{\int_\Omega |u|} \;:\; u \in BV(\Omega)\backslash\{0\} \right\}$$

Notice that if Ω is a ball of radius R in $\mathbb{R}^N$, then its Cheeger's constant is $h = \dfrac{N}{R}$.

3.1 1–dimensional case

Assume that the domain is an interval (a,b), in this case the equation is

$$-\left(\frac{u'}{|u'|}\right)' = \lambda e^u$$

In this setting, we prove the following result.

Theorem 3.1. *Set* $\lambda^* = \frac{2}{b-a}$.

1. *If* $\lambda > \lambda^*$, *problem has no solution*
2. *If* $0 < \lambda \le \lambda^*$, *the minimal solution is the trivial solution*
3. *If* $0 < \lambda < \lambda^*$, *there exist non trivial solutions*
4. *If* $\lambda = \lambda^*$, *there exists only trivial solution*

Remark 3.2.

1. We point out that λ^* is the Cheeger constant of interval (a,b).
2. Trivial solution $u(x) = 0$ has as associated function
$$\mathbf{z}(x) = \lambda \frac{b+a-2x}{2}$$
3. Non trivial solutions are given by
$$u(x) = \log\left(\frac{2}{(b-a)\lambda}\right)$$
and its associated function is
$$\mathbf{z}(x) = \frac{a+b-2x}{b-a}$$

3.2 Minimal solutions

Fix $N \ge 2$ and a bounded open $\Omega \subset \mathbb{R}^N$ having Lipschitz continuous boundary.

Theorem 3.3. *Denote by* λ^* *the Cheeger constant of* Ω*:*

1. *If* $0 < \lambda \leq \lambda^*$, *then* $u \equiv 0$ *is a solution.*

2. *If* $\lambda > \lambda^*$, *then problem has no solution.*

For $\lambda = \lambda^*$, the minimal solution vanishes. Hence, the extremal solution is always bounded, independent of the dimension. This fact is coherent with what occurs for $p > 1$ as p goes to 1.

3.3 Radial solutions

Now, we consider the same problem, but in the unit ball (recall that its Cheeger's constant is N). We start by analyzing how the associated vector field should be.

Lemma 3.4. *Let u be a solution with associated vector field $\mathbf{z}$. If u is constant in one radial zone and nonconstant in another zone, then there exists $0 < \rho < 1$ such that u is constant in $B_\rho(0)$ and nonconstant in $B_1(0) \backslash B_\rho(0)$. Moreover,*

$$\mathbf{z}(x) = \begin{cases} -\dfrac{x}{\rho}\,, & \text{if } |x| < \rho\,; \\ -\dfrac{x}{|x|}\,, & \text{if } \rho < |x| < 1\,. \end{cases}$$

Obviously, there is the possibility of being $\rho = 0$ and that of being $\rho = 1$.

Once we have identified the vector fields, it is easy to get the solutions.

$$\operatorname{div} \mathbf{z}(x) = \begin{cases} -\dfrac{N}{\rho}\,, & \text{if } |x| < \rho\,; \\ -\dfrac{N-1}{|x|}\,, & \text{if } \rho < |x| < 1\,. \end{cases}$$

$$u(x) = \begin{cases} \log\left(\dfrac{N}{\lambda\rho}\right), & \text{if } |x| < \rho\,; \\ \\ \log\left(\dfrac{N-1}{\lambda|x|}\right), & \text{if } \rho < |x| < 1\,. \end{cases}$$

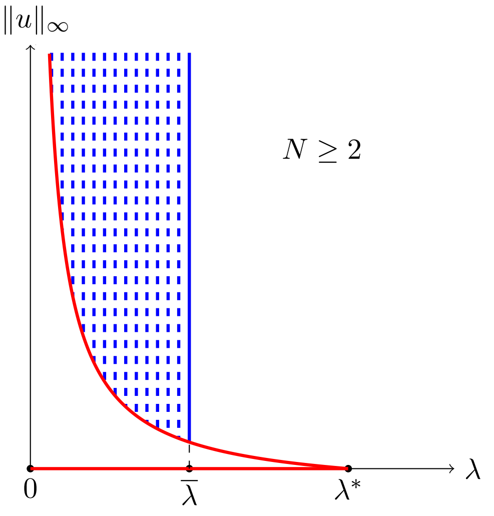

Figure 3.1:

These computations lead to the following result.

Theorem 3.5. *Let $N \geq 2$ and set $\lambda^* = N$ and $\overline{\lambda} = N - 1$. Then,*

1. *For every $0 < \lambda < \lambda^*$ there exists a constant nontrivial solution*

2. *For every $0 < \lambda \leq \overline{\lambda}$ there exists a singular solution*

3. *For every $0 < \lambda \leq \overline{\lambda}$ there exist infinitely many bounded solutions. More precisely, for each $\alpha \in]\log(\frac{N}{\lambda}), +\infty[$, we can find a solution satisfying $\|u\|_\infty = \alpha$.*

4. *If $\lambda > \overline{\lambda}$, then every solution is constant.*

Remark 3.6. The bifurcation diagram can be seen in Figure 3.1.

Let us illustrate in Figure 3.2 different possibilities for solutions having the same infinity norm:

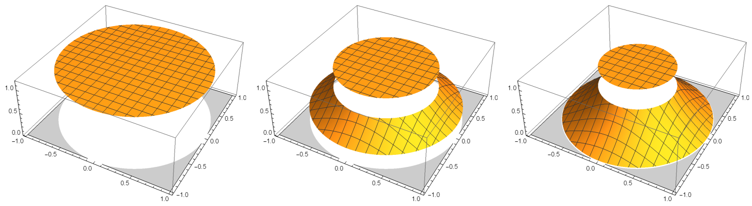

Figure 3.2:

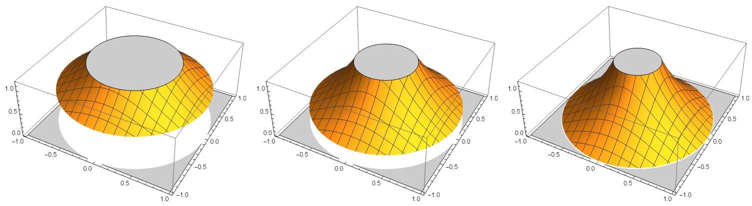

Figure 3.3:

1. Solutions associated to λ smaller are constant; these are the only continuous solutions.

2. Solutions corresponding to $\lambda = \overline{\lambda}$ satisfy the Dirichlet boundary condition in the sense of traces.

3. When $\rho = 0$ singular solutions are obtained.

On the other hand, we show in Figure 3.3 unbounded solutions that have the same values λ as the previous ones. As above, the solution corresponding to $\lambda = \overline{\lambda}$ satisfies the Dirichlet boundary condition in the sense of traces.

4 Limit of Gelfand problems involving the p-Laplacian operator

Since there are too many radial solutions, it is natural to ask which of these solutions come from solutions to problems driven by the p–Laplacian

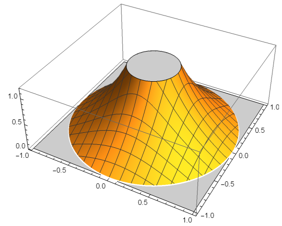

Figure 4.1:

as p goes to 1. In other words, we wonder how these snake–shape p–diagrams are transformed. To obtain these limit solutions, we follow the same procedure used in [9] and so we get a similar additional condition.

The limit for singular solutions is straightforward. When $1 < p < N$, singular solutions are given by :

$$u_p(x) = \log\left(\frac{1}{|x|^p}\right) \qquad \overline{\lambda}_p = p^{p-1}(N-p)$$

In this case, the limit as p tends to 1 can easily be computed:

$$\lim_{p\to 1} u_p(x) = \log\left(\frac{1}{|x|}\right) \qquad \lim_{p\to 1} \overline{\lambda}_p = N-1 = \overline{\lambda}$$

Its graphic can be seen in Figure 4.1. This is the only singular solution which is limit of singular solutions.

To go further, we need precise what we understand as the limit of problems driven by the p–Laplacian as p goes to 1.

Gelfand problem involving the p-Laplacian is

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda e^u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

We will write $(\lambda, u) \in \mathcal{G}_p$ if $\lambda > 0$ and $u \in W_0^{1,p}(\Omega)$ is a solution to problem (Q_λ).

We will write $(\lambda, u) \in \mathcal{G}$ if $\lambda > 0$, $u \in BV(\Omega)$ and there exist sequences $(p_n)_n$, $(\lambda_n)_n$ and $(u_n)_n$ satisfying

1. $p_n \to 1$

2. $\lambda_n \to \lambda$

3. $u_n(x) \to u(x)$ a.e.in Ω

4. $(\lambda_n, u_n) \in \mathcal{G}_{p_n}$ for all $n \in \mathbb{N}$

The next step is to find a condition that satisfies the limit solution in the ball. To this end, the equations that satisfy the profiles of the radial solutions are considered. Calling $r = |x|$ and $v(r) = u(x)$, it has to hold

$$\begin{cases} r^{1-N}\left(r^{N-1}|v'|^{p-2}v'\right)' + \lambda e^v = 0, & r \in (0,1), \\ v > 0, & r \in (0,1), \\ v(1) = 0\,. \end{cases} \tag{4.1}$$

This is the problem that provides the profile of the p–solutions To solve the problem that gives us the profiles, we analyze the system

$$\begin{cases} |v'|^{p-2}v' = w, \\ w' = -\dfrac{N-1}{r}w - \lambda e^v, \\ v(0) = \alpha, \quad w(0) = 0, \end{cases} \tag{4.2}$$

where $\alpha > 0$ is chosen to get $v(1) = 0$. Associated to this system is an energy functional:

$$E_p(v,w) = \frac{1}{p'}|w|^{p'} + \lambda(e^v - 1), \tag{4.3}$$

whose derivative along the trajectories is given by

$$\frac{d}{dr}E_p(v,w) = -\frac{N-1}{r}|w|^{p'} = -\frac{N-1}{r}|v'|^p.$$

Furthermore, we find the proper spaces where the limits belong and the features they satisfy.

Proposition 4.1. *1.* $v_1 \in BV(\sigma, 1)$ *for all* $\sigma > 0$.

2. *w_1 is Lipschitz–continuous in $(\sigma, 1)$ for all $\sigma > 0$.*

3. $-w_1' - \frac{N-1}{t} w_1 = \lambda_1 e^{v_1}$ *in the sense of distributions.*

4. $|v_1'| = (w_1, v_1')$ *as measures.*

5. *The identity*
$$\lambda_1 \frac{de^{v_1}}{dr} = -\frac{N-1}{r} \left| \frac{dv_1}{dr} \right| \tag{4.4}$$
holds in the sense of distributions.

This last identity is the translation of the derivative of energy functional E_p to the limit as p tends to 1.

We deduce that the limit $(v, w, \lambda) = (v_1, w_1, \lambda_1)$ is a solution to the limit system
$$\begin{cases} \frac{v'}{|v'|} = w, \\ w' = -\dfrac{N-1}{r} w - \lambda e^v \\ v(0) = \alpha, \quad w(0) = 0, \end{cases}$$
where $\alpha = \lim_{p \to 1} \alpha_p$. Moreover, the following extra condition holds
$$\lambda_1 \frac{de^{v_1}}{dr} = -\frac{N-1}{r} \left| \frac{dv_1}{dr} \right|$$
Then defining $u(x) = v(|x|)$ and $\mathcal{G}(x) = w(|x|) \frac{x}{|x|}$, we infer that u is a radial solution to the limit system which satisfies this extra energy condition. Thus, if $(\lambda, u) \in \mathcal{G}$, then this energy condition holds.

In the following result we check which are the solutions that satisfy the energy condition.

Theorem 4.2. *Assume that $N \geq 1$.*

Radial solutions to the Gelfand problem that satisfy the energy condition are continuous (so that they are constant).

As a consequence, for all $0 < \lambda < \lambda^*$ there exist exactly two bounded solutions:

1. *The trivial solution* $u(x) = 0$.
2. *The constant solution* $u(x) = \log\left(\frac{N}{\lambda}\right)$

Even more, if we assume that $N \geq 2$, *the unbounded solution* $u(x) = \log\left(\frac{N-1}{\lambda|x|}\right)$, *which exists for every* $0 < \lambda \leq \overline{\lambda}$, *also satisfies the energy condition.*

There is a remaining question on convergence, namely: the extremal solutions of the p-equations converge to 0? To see it, fix $N \geq 1$ and let $p > 1$ be small enough. We denote by λ_p^* the critical threshold, so that there are no bounded solutions for $\lambda > \lambda_p^*$ and we write as u_p^* its associated solution.

We prove that the following estimate holds

$$N\left(\frac{p}{e}\right)^{p-1} \leq \lambda_p^* \leq N\left(\frac{p}{e}\right)^{p-1} \frac{\Gamma\left(p+1+\frac{N(p-1)}{p}\right)}{\Gamma(p+1)\Gamma\left(2+\frac{N(p-1)}{p}\right)}$$

Theorem 4.3.

1. $\lambda_p^* \to N$, *as* $p \to 1$.
2. $\|u_p^*\|_\infty \to 0$, *as* $p \to 1$.

5 Extensions

The classical Gelfand problem has λe^u as reaction term. The exponential function $f(s) = e^s$ can be replaced by other functions that have similar characteristics like

$$f(s) = (1+s)^m \quad m > 1$$

$$f(s) = 1 + s^2$$

A general condition to deal with Gelfand problems driven by the p–Laplacian is given by: $f : [0,\infty[\to [0,\infty[$ *is an increasing and* C^1 *function such that* $\lim_{s\to\infty} f(s)/s^{p-1} = \infty$ *and* $f(0) > 0$

We study our problem for functions satisfying a wider condition: $f : [0,\infty[\to [0,\infty[$ *is an increasing and continuous function such that* $\lim\limits_{s\to\infty} f(s) = \infty$ *and* $f(0) > 0$. Adjusting the statements, the following results are true:

1. $\lambda^* = N/f(0)$
2. Non trivial solutions are given by $f^{-1}\left(\frac{N}{\lambda\rho}\right)$ or $f^{-1}\left(\frac{N-1}{\lambda|x|}\right)$
3. The energy condition becomes
$$\lambda\frac{dF(v)}{dr} = -\frac{N-1}{r}\left|\frac{dv}{dr}\right|$$
where $F(s) = \int_0^s f(t)\,dt$

Nevertheless, we cannot extend all the results. In particular we are not able to prove in general that the extremal solutions converge to the trivial solution.

The result we obtain is:

Theorem 5.1. *Fix $N \geq 1$ and let $p > 1$ be small enough. We denote by $\{w_{\lambda(p)}\}_{\lambda\in[0,\lambda_p^*]}$ the increasing branch of minimal solutions of the problem involving the p–Laplacian. Let λ_p^* denote the critical value such that there is not bounded solutions for $\lambda > \lambda_p^*$. Then,*

1. $\lambda_p^* \to \dfrac{N}{f(0)}$, *as* $p \to 1$.

2. $\|w_{\tilde\lambda(p)}\|_\infty \to 0$, *as* $p \to 1$ *for all* $\tilde\lambda \in [0, \frac{N}{f(0)}[$.

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