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Almost elliptic structures

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Abstract. Given a manifold $\mathcal{M}$ and vector bundles $E^q \to \mathcal{M}$, a first order sequence is a sequence

$$\cdots \xrightarrow{\mathbb{P}_{q-1}} C^\infty(\mathcal{M}; E^q) \xrightarrow{\mathbb{P}_q} C^\infty(\mathcal{M}; E^{q+1}) \xrightarrow{\mathbb{P}_{q+1}} \cdots$$

of first order differential operators in which the composition of successive operators is at most of first (rather than second) order; the sequence need not be a complex.

It is shown, among other things, that under certain hypotheses on the principal symbols, sequences of operators that are somewhat loosely associated to a (noninvolutive) subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ are essentially completely determined, and that under additional conditions, the hypothesis that some composition $\mathbb{P}_{q_0+1} \circ \mathbb{P}_{q_0}$, $q_0 < m-1$, is of order zero, implies that $\mathcal{V}$ is involutive.

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1 Introduction

Suppose $\mathcal{M}$ is a smooth paracompact connected oriented manifold. As motivation for what will follow, recall that any involutive vector subbundle

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$\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ of the complexification of the tangent bundle of $\mathcal{M}$ induces a complex of first order differential operators on sections of the exterior powers of the dual bundle $\overline{\mathcal{V}}^*$ of the conjugate bundle $\iota : \overline{\mathcal{V}} \hookrightarrow \mathbb{C}T\mathcal{M}$,

$$\cdots \xrightarrow{\mathbb{D}_{q-1}} C^\infty(\mathcal{M}; {\textstyle\bigwedge}^q \overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_q} C^\infty(\mathcal{M}; {\textstyle\bigwedge}^{q+1} \overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_{q+1}} \cdots, \tag{1.1}$$

by way of the Cartan formula for the differential of a form. Namely, if $u \in C^\infty(\mathcal{M}; \bigwedge^q \overline{\mathcal{V}}^*)$ then $\mathbb{D}_q u$ is defined as follows: given local smooth sections $V_0, ..., V_q$ of $\overline{\mathcal{V}}$,

$$\begin{aligned}(q+1)\mathbb{D}_q u(V_0, ..., V_q) &= \sum_{i=0}^{q} (-1)^i V_i(u(V_0, ..., \widehat{V}_i, ..., V_q)) \\ &+ \sum_{0 \le i < j \le q} (-1)^{i+j} u([V_i, V_j], V_0, ..., \widehat{V}_i, ..., \widehat{V}_j, ..., V_q).\end{aligned} \tag{1.2}$$

Thus for example, if f is a function, then $\mathbb{D}_0 f = \iota^*(df)$, the restriction of $df : \mathbb{C}T\mathcal{M} \to \mathbb{C}$ to $\overline{\mathcal{V}}$. Automatically $\mathbb{D}_{q+1} \circ \mathbb{D}_q = 0$. The principal symbol of $\mathbb{D}_q$ at $\boldsymbol{\xi} \in T^*\mathcal{M}$ is $\alpha \mapsto i\iota^*(\boldsymbol{\xi}) \wedge \alpha$, $\alpha \in \bigwedge^q \overline{\mathcal{V}}^*_{\pi(\boldsymbol{\xi})}$. Here $\pi : T^*\mathcal{M} \to \mathcal{M}$ is the projection map, $\iota^* : \mathbb{C}T^*\mathcal{M} \to \overline{\mathcal{V}}^*$ is the dual of the inclusion map $\iota : \overline{\mathcal{V}} \to \mathbb{C}T\mathcal{M}$, and $i = \sqrt{-1}$. It follows from this that the sequence of principal symbols

$$\cdots \xrightarrow{\boldsymbol{\sigma}(\mathbb{D}_{q-1})(\boldsymbol{\xi})} {\textstyle\bigwedge}^q \overline{\mathcal{V}}^*_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})} {\textstyle\bigwedge}^{q+1} \overline{\mathcal{V}}^*_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{D}_{q+1})(\boldsymbol{\xi})} \cdots$$

is exact if and only if $\iota^*(\boldsymbol{\xi}) \neq 0$. Thus the characteristic set is the set of real covectors in the annihilator $\ker \iota^* = \overline{\mathcal{V}}^\perp$ of $\overline{\mathcal{V}}$: $\operatorname{Char} \mathcal{V} = \overline{\mathcal{V}}^\perp \cap T^*\mathcal{M}$. Since $\mathbb{C} \otimes \operatorname{Char} \mathcal{V} = \mathcal{V}^\perp \cap \overline{\mathcal{V}}^\perp$, the complex is elliptic if and only if $\mathcal{V}^\perp \cap \overline{\mathcal{V}}^\perp = 0$, equivalently, if and only if

$$\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}T\mathcal{M}. \tag{1.3}$$

This leads to the concept of elliptic structure: an involutive subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ such that (1.3) holds, see [7]. Extreme examples of elliptic structures are $\mathcal{V} = \mathbb{C}T\mathcal{M}$, for which (1.1) is the de Rham complex, and $\mathcal{V} = T^{1,0}\mathcal{M}$ if $\mathcal{M}$ is a complex manifold, in which case (1.1) is the Dolbeault complex. A theorem of Nirenberg [4] states that the general local example is that of an open set in $\mathbb{C}^m \times \mathbb{R}^\kappa$ with coordinates (z, t), with $\mathcal{V}$ the bundle spanned by the vector fields ∂_{z^i}, ∂_{t^μ}. Keeping involutivity and replacing (1.3) by $\mathcal{V} \cap \overline{\mathcal{V}} = 0$ leads to CR structures. For these reasons the study of complexes associated to involutive structures is of utmost importance. The literature pertaining complexes associated to involutive subbundles as above is very extensive. The excellent monographs [7] by Treves and [1] by Berhanu, Cordaro, and Hounie will give the interested reader a a good overview of the subject.

A subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ will be called an almost elliptic structure on $\mathcal{M}$ if it satisfies (1.3). Almost elliptic structures are not required to be involutive.

If $\mathcal{V}$ is not involutive, then there is no evident natural sequence (1.1). However, one can be defined with the aid of a connection on $\overline{\mathcal{V}}^*$, as shown in Section 2. The resulting sequence need not be a complex, but the principal symbol sequence is exact (if $\mathcal{V}$ is elliptic). This leads to the notion of a (first order) elliptic sequence, defined as follows. Let $E^0, \dots, E^m$ be vector bundles over $\mathcal{M}$, let $\mathbb{P}_q : C^\infty(\mathcal{M}; E^q) \to C^\infty(\mathcal{M}; E^{q+1})$ be first order differential operators. We say that

$$\cdots \xrightarrow{\mathbb{P}_{q-1}} C^\infty(\mathcal{M}; E^q) \xrightarrow{\mathbb{P}_q} C^\infty(\mathcal{M}; E^{q+1}) \xrightarrow{\mathbb{P}_{q+1}} \cdots \tag{1.4}$$

is an elliptic sequence if for each $\boldsymbol{\xi} \in T^*\mathcal{M}\backslash 0$, the sequence

$$\cdots \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_{q-1})(\boldsymbol{\xi})} E^q_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_q)(\boldsymbol{\xi})} E^{q+1}_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_{q+1})(\boldsymbol{\xi})} \cdots$$

is exact. In particular, for each q, $\mathbb{P}_{q+1} \circ \mathbb{P}_q$ is at most of first order. Dropping the requirement that the principal symbol sequence be exact we get a first order sequence.

We should perhaps make explicit that in the case of an elliptic sequence, the symbol sequence begins and ends with 0, that is, $\boldsymbol{\sigma}(\mathbb{P}_0)(\boldsymbol{\xi})$ is injective and $\boldsymbol{\sigma}(\mathbb{P}_{m-1})(\boldsymbol{\xi})$ is surjective.

The relevancy of the condition of exactness of the symbol sequence lies in the fact that it is equivalent to the condition that the associated Hodge

Laplacians be elliptic. These are defined in the same way as for a complex. Suppose the E^q are Hermitian, let $\mathfrak{m}$ be a fixed smooth positive density on $\mathcal{M}$. With these data we get the L^2 spaces of sections of each E^q. Let

$$\mathbb{P}_q^\star : C^\infty(\mathcal{M}; E^{q+1}) \to C^\infty(\mathcal{M}; E^q)$$

be the formal adjoint. Then

$$\square_q = \mathbb{P}_q^\star \mathbb{P}_q + \mathbb{P}_{q-1} \mathbb{P}_{q-1}^\star.$$

Theorem 1.1. *Let E^0, $E^1, \dots, E^m$ be Hermitian complex vector bundles over $\mathcal{M}$, with $\operatorname{rk} E^0 = \operatorname{rk} E^m$. Suppose*

$$\cdots \xrightarrow{\mathbb{P}_{q-1}} C^\infty(\mathcal{M}; E^q) \xrightarrow{\mathbb{P}_q} C^\infty(\mathcal{M}; E^{q+1}) \xrightarrow{\mathbb{P}_{q+1}} \cdots \tag{1.5}$$

is an elliptic sequence of first order differential operators. Thus $\boldsymbol{\sigma}(\mathbb{P}_q)$ can be regarded as a homomorphism $\mathbb{C}T^*\mathcal{M} \to \operatorname{Hom}(E^q, E^{q+1})$, and automatically

$$\boldsymbol{\sigma}(\mathbb{P}_{q+1})(\zeta) \circ \boldsymbol{\sigma}(\mathbb{P}_q)(\zeta) = 0 \quad \text{for each } \zeta \in \mathbb{C}T^*\mathcal{M}.$$

Suppose

(i) There is a subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ of rank m such that $\overline{\mathcal{V}}^\perp = \ker \boldsymbol{\sigma}(\mathbb{P}_q)$ for all q.

(ii) The maps $\mathbb{C}T^*\mathcal{M} \otimes E^q \to E^{q+1}$ induced by the bilinear map

$$(\zeta, \alpha) \mapsto \boldsymbol{\sigma}(\mathbb{P}_q)(\zeta)(\alpha)$$

are surjective.

Then

(a) there are unique isomorphisms $\Phi_q : E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^q$, $\Phi_0 = \mathrm{Id}$, such that

$$\Phi_{q+1}^{-1} \circ \boldsymbol{\sigma}(\mathbb{P}_q)(\zeta) \circ \Phi_q = d_q(\zeta) \quad \zeta \in \mathbb{C}T^*\mathcal{M}$$

with $d_q(\zeta)(\alpha \otimes \eta) = i\alpha \otimes \iota^*\zeta \wedge \eta$.

(b) If the sequence is elliptic, then $\mathcal{V}$ is an almost elliptic structure.

(c) If the operators $\mathbb{D}_q = \Phi_{q+1}^{-1} \circ \mathbb{P}_q \circ \Phi_q$ happen to be those defined by connections as explained in the next section, and $\mathbb{D}_{q_0+1} \circ \mathbb{D}_{q_0}$ is of order zero form some $q_0 < m-1$, then $\mathcal{V}$ is involutive.

In the next section we give some basic examples of sequences of first order differential operators constructed using connections. They involve an arbitrary vector bundle and a given subbundle $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$. The principal symbols of such operators are independent of the connections used to define them, so they (the symbols) can be used for stating generic hypotheses about sequences. In Section 3 we give a condition on a sequence with such standard symbol that implies involutivity of $\mathcal{V}$. In Section 4 we show that the hypotheses of Theorem 1.1 (without assuming ellipticity) already imply that the sequence is equivalent to one with standard principal symbols. The ellipticity of the sequence in the statement of Theorem 1.1 then implies that $\mathcal{V}$ is almost elliptic.

Elliptic sequences of first order differential operators were studied from with an analysis perspective by Krupchyk, Tarkhanov, and Tuomela in [3] and Tarkhanov in [6]. This last author also has a very interesting monograph [5] dealing in general with complexes.

2 Examples

Let $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be a subbundle, not necessarily involutive nor almost elliptic. As before, let $\iota : \overline{\mathcal{V}} \hookrightarrow \mathbb{C}T\mathcal{M}$ be the inclusion homomorphism and $\iota^* : \mathbb{C}T^*\mathcal{M} \to \overline{\mathcal{V}}^*$ the dual map. Pick a connection on its dual bundle,

$$\nabla_1 : C^\infty(\mathcal{M}; \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; \mathbb{C}T^*\mathcal{M} \otimes \overline{\mathcal{V}}^*).$$

The following examples are adapted from the standard extension of the connection to a differential operator

$$C^\infty(\mathcal{M}; \textstyle\bigwedge^q \mathcal{M} \otimes \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; \textstyle\bigwedge^{q+1} \mathcal{M} \otimes \overline{\mathcal{V}}^*).$$

Define $\nabla_q : C^\infty(\mathcal{M}; \bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; \mathbb{C}T^*\mathcal{M} \otimes \bigwedge^q \overline{\mathcal{V}}^*)$ by requiring

$$\nabla_q(\theta^1 \wedge \cdots \wedge \theta^q) = \sum_{j=1}^{q} (-1)^{j+1} (\nabla_1 \theta^j) \wedge \theta^1 \wedge \cdots \wedge \hat{\theta}^j \wedge \cdots \wedge \theta^q$$

when $\theta^1, \ldots, \theta^q$ are sections of $\overline{\mathcal{V}}^*$, not necessarily part of a frame. To check that ∇_q is well defined we note first that if $f^1, \ldots, f^q$ are functions with $f = f^1 \cdots f^q = 1$, then

$$\begin{aligned}\nabla_q(f^1\theta^1 \wedge \cdots \wedge f^q\theta^q) &= f\nabla_q(\theta^1 \wedge \cdots \wedge \theta^q) + f(d\log f) \otimes \theta^1 \wedge \cdots \wedge \theta^q \\ &= \nabla_q(\theta^1 \wedge \cdots \wedge \theta^q).\end{aligned}$$

Next, if $\theta^j = \sum_k a^j_k \eta^k$, then $\nabla_q(\theta^1 \wedge \cdots \wedge \theta^q)$ computed as

$$\nabla_q(\theta^1 \wedge \cdots \wedge \theta^q) =$$
$$\sum_j (-1)^{j+1} \nabla_1 (\sum_{k_j} a^j_{k_j} \eta^{k_j}) \wedge (\sum_{k_1} a^j_{k_1} \eta^{k_1}) \wedge \cdots \wedge (\widehat{\sum_{k_j} a^j_{k_j} \eta^{k_j}}) \wedge \cdots \wedge \sum_{k_q} a^{k_q}_q \eta^{k_q}$$

or as

$$\sum_{k_1,\ldots,k_q} \nabla_q(a^1_{k_1} \ldots a^q_{k_q} \eta^{k_1} \wedge \ldots \eta^{k_q}) =$$
$$\sum_{k_1,\ldots,k_q} \sum_j (-1)^{j+1} \nabla_1 (a^1_{k_1} \ldots a^q_{k_q} \eta^{k_j}) \wedge \eta^{k_1} \cdots \wedge \hat{\eta}^{k_j} \wedge \cdots \wedge \eta^{k_q}$$

yields the same result. So ∇_q is well defined.

If now $\theta^1, \ldots, \theta^m$ is a local frame of $\overline{\mathcal{V}}^*$ and $\nabla_1 \theta^j = \sum_k \omega^j_k \otimes \theta^k$, then the definition of ∇_q yields

$$\nabla_q(\theta^{j_1} \wedge \cdots \wedge \theta^{j_q}) = \sum_{k,\ell} \omega^{j_\ell}_k \otimes \theta^{j_1} \wedge \cdots \wedge \theta^{j_{\ell-1}} \wedge \theta^k \wedge \theta^{j_{\ell+1}} \wedge \cdots \wedge \theta^{j_q}.$$

Writing in general $\mathrm{Alt} : \bigotimes^q \overline{\mathcal{V}}^* \to \bigwedge^q \overline{\mathcal{V}}^*$ for the canonical projection, define

$$\begin{aligned}\mathbb{D}_0 &= \iota^* \circ d : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}; \overline{\mathcal{V}}^*), \\ \mathbb{D}_q &= \mathrm{Alt} \circ (\iota^* \otimes \mathrm{Id}) \circ \nabla_q : C^\infty(\mathcal{M}; \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; \textstyle\bigwedge^{q+1} \overline{\mathcal{V}}^*).\end{aligned} \tag{2.1}$$

It is easy to check that if $f \in C^\infty(\mathcal{M})$ and $\theta \in C^\infty(\mathcal{M}; \bigwedge^q \overline{\mathcal{V}}^*)$, then

$$\mathbb{D}_q(f\theta) = f\mathbb{D}_q\theta + \mathbb{D}_0 f \wedge \theta. \tag{2.2}$$

More generally, the operators $\mathbb{D}_q$ satisfy the graded Leibniz rule:

$$\mathbb{D}_{q+q'}(\psi \wedge \theta) = \mathbb{D}_q(\psi) \wedge \theta + (-1)^q \psi \wedge \mathbb{D}_{q'}\theta, \tag{2.3}$$

for sections ψ and θ of degrees q and q', resp.

Somewhat more generally, we get examples of operators

$$\mathbb{D}_q : C^\infty(\mathcal{M}; E \otimes \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; E \otimes \textstyle\bigwedge^{q+1} \overline{\mathcal{V}}^*)$$

where $E \to \mathcal{M}$ is an arbitrary complex vector bundle using the same scheme, as follows. Let again ∇ be a connection on $\overline{\mathcal{V}}^*$, denote also by ∇ a connection on E, thought of as

$$\nabla : C^\infty(\mathcal{M}; E) \to C^\infty(\mathcal{M}; E \otimes \mathbb{C}T^*\mathcal{M}).$$

If ψ is a section of E, define

$$\mathbb{D}_0\psi = (\mathrm{Id} \otimes \iota^*) \circ \nabla\psi \in C^\infty(\mathcal{M}; E \otimes \overline{\mathcal{V}}^*),$$

and if θ is a section of $\bigwedge^q \overline{\mathcal{V}}^*$ then

$$\mathbb{D}_q(\psi \otimes \theta) = \mathbb{D}_0\psi \wedge \theta + \psi \otimes \mathbb{D}_q\theta \tag{2.4}$$

with $\mathbb{D}_q\theta$ as defined earlier.

If $\phi \in C^\infty(\mathcal{M}; E \otimes \bigwedge^q \overline{\mathcal{V}}^*)$ and f is a function, we have $\mathbb{D}_q(f\phi) = f\mathbb{D}_q\phi + (-1)^q \phi \wedge \mathbb{D}_0 f$, which gives

$$\lim_{\tau\to\infty} \frac{1}{\tau} e^{-i\tau g} \mathbb{D}_q(e^{i\tau g}\phi) = (-1)^q i\phi \wedge \mathbb{D}_0 g.$$

Since $\mathbb{D}_0 g = \iota^* dg$,

$$\begin{aligned}\boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})(\alpha \otimes \eta) &= i\alpha \otimes (\iota^*\boldsymbol{\xi}) \wedge \eta,\\ &\qquad \boldsymbol{\xi} \in T^*\mathcal{M},\ \alpha \in E_{\pi(\boldsymbol{\xi})},\ \eta \in \textstyle\bigwedge^q \overline{\mathcal{V}}^*_{\pi(\boldsymbol{\xi})}.\end{aligned} \tag{2.5}$$

It follows that $\mathbb{D}_{q+1} \circ \mathbb{D}_q$ is of order 1 since the principal symbol of this composition as an operator of order 2 vanishes.

It is again the case that the graded Leibniz rule holds,

$$\mathbb{D}_{q+q'}(\phi \wedge \theta) = \mathbb{D}_q(\phi) \wedge \theta + (-1)^q \phi \wedge \mathbb{D}_{q'}\theta, \tag{2.6}$$

for sections ϕ of $C^\infty(\mathcal{M}; E \otimes \bigwedge^q \overline{\mathcal{V}}^*)$ and θ of $C^\infty(\mathcal{M}; \bigwedge^{q'} \overline{\mathcal{V}}^*)$.

Sequences of operators with principal symbol (2.5) are particularly well behaved. Because $\mathbb{D}_q$ is a first order differential operator, $\boldsymbol{\xi} \mapsto \boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})$ is linear in $\boldsymbol{\xi}$ (on fibers of $T^*\mathcal{M}$) so has a linear extension to $\mathbb{C}T^*\mathcal{M}$. The resulting homomorphism

$$d_q : \mathbb{C}T^*\mathcal{M} \to \operatorname{Hom}(E \otimes \textstyle\bigwedge^q \overline{\mathcal{V}}^*, E \otimes \bigwedge^{q+1} \overline{\mathcal{V}}^*)$$

(the extension of the homomorphism (2.5) to $\mathbb{C}T^*\mathcal{M}$) is generic, independent of the connections used to define the $\mathbb{D}_q$, because principal symbols of connections are themselves canonical. We explore such operators in the next section.

3 A sufficient condition for involutivity

The following proposition and its proof are motivated by the arguments of Folland and Kohn in [2, Ch.1, §6] concerning almost complex structures.

Proposition 3.1. *Suppose the operators*

$$\cdots \xrightarrow{\mathbb{D}_{q-1}} C^\infty(\mathcal{M}; E \otimes \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_q} C^\infty(\mathcal{M}; E \otimes \bigwedge^{q+1} \overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_{q+1}} \cdots$$

have principal symbol (2.5)*. If* $\mathbb{D}_{q_0+1} \circ \mathbb{D}_{q_0}$ *is of order* 0 *for some* $q_0 < m-1$*, then* $\mathcal{V}$ *is involutive. Furthermore, if the operators* $\mathbb{D}_q$ *are defined using connections as in the previous section but using* (1.2) *instead of* (2.1) *then all these compositions are of order* 0*.*

In the second part of the proposition we use that $\overline{\mathcal{V}}$ is involutive, so the definition using Cartan's formula is valid. The validity of the conclusion is essentially that the 'curvature' is of order 0.

Proof. Denote by $\mathbb{D}'_q$ the operators defined in (2.4) using connections on E and $\overline{\mathcal{V}}^*$. Since $\boldsymbol{\sigma}(\mathbb{D}_q) = \boldsymbol{\sigma}(\mathbb{D}'_q)$, there are homomorphisms $\alpha_q : E \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E \otimes \bigwedge^{q+1} \overline{\mathcal{V}}^*$ such that

$$\mathbb{D}_q \phi = \mathbb{D}'_q \phi + \alpha_q \phi.$$

We will pass to local computations using frames. Let the η_μ denote a frame of E, let $\overline{L}_1, \dots, \overline{L}_m$ denote one for $\overline{\mathcal{V}}$ and let $\theta^1, \dots, \theta^m$ be the frame dual to the latter, all over some open set $U \subset \mathcal{M}$. An arbitrary section ϕ of $E \otimes \bigwedge^q \overline{\mathcal{V}}^*$ has the form

$$\phi = \sideset{}{'}\sum_{|I|=q} f_I^\mu \eta_\mu \otimes \theta^I$$

where $I = (i_1, \dots, i_q)$, $i_1 < i_2 < \dots < i_q$ and $\theta^I = \theta^{i_1} \wedge \dots \wedge \theta^{i_q}$. Then (2.4) gives

$$\mathbb{D}'_q \eta_\mu \otimes \theta^I = \sideset{}{'}\sum_{|J|=q+1} \sum_\nu \beta_{\mu,J}^{\nu,I} \eta_\nu \otimes \theta^i \wedge \theta^J$$

with some $\beta_{\mu,J}^{\nu,I}$. Together with

$$\alpha_q(\eta_\mu \otimes \theta^I) = \sideset{}{'}\sum_{|J|=q+1} \sum_\nu \alpha_{\mu,J}^{\nu,I}\, \eta_\nu \otimes \wedge \theta^J$$

we thus get

$$\mathbb{D}_q \phi = \sideset{}{'}\sum_{|I|=q} \sum_{i \notin I} \overline{L}_i f_I^\mu\, \eta_\mu \otimes \theta^i \wedge \theta^I + \sideset{}{'}\sum_{\substack{|I|=q \\ |J|=q+1}} \sum_{\mu,\nu} \gamma_{\mu,J}^{\nu,I} f_I^\mu\, \eta_\nu \otimes \wedge \theta^J$$

where $\gamma_{\mu,J}^{\nu,I} = \alpha_{\mu,J}^{\nu,I} + \beta_{\mu,J}^{\nu,I}$. The details of these functions are not important. This gives the formula

$$\begin{aligned} \mathbb{D}_{q+1} \mathbb{D}_q \phi = \sideset{}{'}\sum_{|I|=q} \sum_{i \notin I} \sum_{j \notin I, j \neq i} \overline{L}_j \overline{L}_i f_I^\mu\, \eta_\mu \otimes \theta^j \wedge \theta^i \wedge \theta^I \\ + \sideset{}{'}\sum_{\substack{|I|=q \\ |K|=q+2}} \sum_k \delta_{K,\mu}^{\nu,k,I} \overline{L}_k f_I^\mu \eta_\nu \otimes \theta^K + \text{z.o.t.} \end{aligned}$$

with suitable coefficients $\delta^{\nu,k,I}_{K,\mu}$ whose particular form, as well as those of order zero, are unimportant. We will now arbitrarily fix some indices i, j, μ, then choose I of length $|I| = q$ with $i_i, j \notin I$ and consider $\phi = f\,\eta_\mu \otimes \theta^I$ with an arbitrary smooth function f. In the right hand side of the above formula applied to such ϕ, the coefficient of $\eta_\mu \otimes \theta^j \wedge \theta^i \wedge \theta^I$ is

$$\frac{1}{2}(\overline{L}_j\overline{L}_i - \overline{L}_i\overline{L}_j)f + \sum_k \delta^{\mu,k,I}_{\langle i,j,I\rangle,\mu}\overline{L}_k f + zf$$

with some function z; the index $K = \langle i, j, I\rangle$ consists of the indices i, j and those of I in increasing order. Thus, if for some q we have that $\mathbb{D}_{q+1}\mathbb{D}_q$ is of order zero, then we must have

$$[\overline{L}_j, \overline{L}_i] = -2\sum_k \delta^{\mu,k,I}_{\langle i,j,I\rangle,\mu}\overline{L}_k$$

for any i, j, which proves that $\overline{\mathcal{V}}$ is involutive.

To finish the proof of the proposition, suppose now that $\overline{\mathcal{V}}$ is involutive and that the operators $\mathbb{D}_q$ are defined using a connection on E and Cartan's formula for sections of $\bigwedge^q\overline{\mathcal{V}}^*$. That is, $\mathbb{D}_q$ on sections of $E \otimes \bigwedge^q\overline{\mathcal{V}}^*$ is given by (2.4) with $\mathbb{D}_q$ on the right hand side given by (1.2). We show that then $\mathbb{D}_{q+1} \circ \mathbb{D}_q$ is of order 0 in any degree.

With a function f and ϕ of arbitrary degree q we have

$$\begin{aligned}\mathbb{D}_{q+1}(\mathbb{D}_q(f\phi)) &= \mathbb{D}_{q+1}(f\mathbb{D}_q\phi + (-1)^q\phi \wedge \mathbb{D}_0 f)\\ &= f\mathbb{D}_{q+1}(\mathbb{D}_q\phi) + (-1)^{q+1}\mathbb{D}_q\phi \wedge \mathbb{D}_0 f\\ &\qquad + (-1)^q\mathbb{D}_q\phi \wedge \mathbb{D}_0 f + \phi \wedge \mathbb{D}_1(\mathbb{D}_0(f))\\ &= f\mathbb{D}_{q+1}(\mathbb{D}_q\phi) + \phi \wedge \mathbb{D}_1(\mathbb{D}_0(f)).\end{aligned}$$

Thus $\phi \mapsto \mathbb{D}_{q+1}(\mathbb{D}_q(f\phi)) - f\mathbb{D}_{q+1}(\mathbb{D}_q\phi) = \phi\wedge\mathbb{D}_1(\mathbb{D}_0 f) = 0$ for all f and ϕ. But $\mathbb{D}_1(\mathbb{D}_0 f) = 0$ since (1.1) is a complex. Thus $\mathbb{D}_{q+1}\mathbb{D}_q$ is of order 0. $\square$

Involutivity of $\mathcal{V}$ also follows trivially if one already has $\mathbb{D}_1\mathbb{D}_0 f = 0$.

The converse of the proposition does not hold: it may be the case that $\mathcal{V}$ is involutive but $\mathbb{D}_{q+1} \circ \mathbb{D}_q \neq 0$. Indeed, if $\mathcal{V}$ is involutive we may define

$\tilde{\mathbb{D}}_q$ using (1.2) which gives $\tilde{\mathbb{D}}_{q+1} \circ \tilde{\mathbb{D}}_q = 0$ for all q, then with arbitrary non-trivial sections α_q of $\overline{\mathcal{V}}^*$ define

$$\mathbb{D}_q \phi = \tilde{\mathbb{D}}_q \phi + \alpha_q \wedge \phi$$

for each q. Then

$$\mathbb{D}_{q+1}(\mathbb{D}_q \phi) = (\alpha_{q+1} - \alpha_q) \wedge \tilde{\mathbb{D}}_q \phi + (\tilde{\mathbb{D}}_1 \alpha_q + \alpha_{q+1} \wedge \alpha_q) \wedge \phi$$

so $\mathbb{D}_{q+1} \circ \mathbb{D}_q$ is a first order operator, nonzero as such if $\alpha_{q+1} \neq \alpha_q$.

We end the section by noting:

Lemma 3.2. *The subbundle $\overline{\mathcal{V}}$ is an almost elliptic structure if and only if the operators $\mathbb{D}_q$ define an elliptic sequence.*

4 Standard symbol

Let $\iota : \overline{\mathcal{V}} \hookrightarrow \mathbb{C}T\mathcal{M}$ be a subbundle, let $\overline{\mathcal{V}}^*$ be its dual, $\iota^* : \mathbb{C}T^*\mathcal{M} \to \overline{\mathcal{V}}^*$ the dual map, and let $E \to \mathcal{M}$ be a complex vector bundle. Let $\pi : \mathbb{C}T^*M \to \mathcal{M}$ be the canonical projection, define

$$d_q : \mathbb{C}T^*M \to \operatorname{Hom}(E \otimes {\textstyle\bigwedge}^q \overline{\mathcal{V}}^*, E \otimes {\textstyle\bigwedge}^{q+1} \overline{\mathcal{V}}^*)$$

by letting

$$d_q(\zeta)(\alpha \otimes \eta) = i\alpha \otimes \iota^*(\zeta) \wedge \eta.$$

Thus

$$d_{q+1}(\zeta) \circ d_q(\zeta) = 0 \tag{4.1}$$

for all ζ. Also

$$\ker d_q = \overline{\mathcal{V}}^\perp, \tag{4.2}$$

and, moreover,

$$\text{the induced map } \mathbb{C}T^*\mathcal{M} \otimes E \otimes {\textstyle\bigwedge}^q \overline{\mathcal{V}}^* \to E \otimes {\textstyle\bigwedge}^{q+1} \overline{\mathcal{V}}^* \text{ is surjective} \tag{4.3}$$

We refer to the maps d_q as standard symbols. Properties (4.1)–(4.3) plus a rank condition (which can be obtained from the analogue of (4.3) for the dual maps) characterize such symbols:

Proposition 4.1. *Let $\mathcal{V} \subset \mathbb{C}T\mathcal{M}$ be a subbundle of rank m, and let E^0, $E^1, \dots, E^m$ be complex vector bundles over $\mathcal{M}$ with $\operatorname{rk} E^0 = \operatorname{rk} E^m$. Suppose the homomorphisms*

$$p_q : \mathbb{C}T^*\mathcal{M} \to \operatorname{Hom}(E^q, E^{q+1}), \quad q = 0, 1, \dots, m-1$$

satisfy

1. *for each $\zeta \in \mathbb{C}T^*\mathcal{M}$, $p_{q+1}(\zeta) \circ p_q(\zeta) = 0$*
2. $\ker p_q = \overline{\mathcal{V}}^\perp$.
3. *For $q \geq 0$, the map $E^q \otimes \mathbb{C}T^*\mathcal{M} \to E^{q+1}$ induced by the bilinear map $(\alpha, \zeta) \mapsto p_q(\zeta)(\alpha)$ is surjective.*

Then there are unique isomorphisms $\Phi_q : E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^q$, $\Phi_0 = \mathrm{Id}$, such that

$$\Phi_{q+1}^{-1} \circ p_q(\zeta) \circ \Phi_q = d_q(\zeta) \quad \zeta \in \mathbb{C}T^*\mathcal{M} \tag{4.4}$$

with $d_q(\zeta)(\alpha \otimes \eta) = i\alpha \otimes \iota^*\zeta \wedge \eta$.

Note that if $\overline{\mathcal{V}}$ is elliptic, then $\overline{\mathcal{V}}^\perp \cap (T^*\mathcal{M}\backslash 0) = \emptyset$, so if (2) holds for each q, then the sequence

$$\cdots \xrightarrow{p_{q-1}(\boldsymbol{\xi})} E^q_{\pi(\boldsymbol{\xi})} \xrightarrow{p_q(\boldsymbol{\xi})} E^{q+1}_{\pi(\boldsymbol{\xi})} \xrightarrow{p_{q+1}(\boldsymbol{\xi})} \cdots$$

is exact for every $\boldsymbol{\xi} \in T^*\mathcal{M}\backslash 0$.

Proof. Taking advantage of Condition (2) and the fact that $\overline{\mathcal{V}}^*$ is canonically isomorphic to $\mathbb{C}T^*\mathcal{M}/\overline{\mathcal{V}}^\perp$ we view the p_q as homomorphisms $\overline{\mathcal{V}}^* \to \operatorname{Hom}(E^q, E^{q+1})$. Define

$$\hat{\Phi}_{q,\ell} : \prod_{i=1}^{\ell} \overline{\mathcal{V}}^* \to \operatorname{Hom}(E^q, E^{q+\ell}), \quad \ell \geq 1$$

inductively by

$$\hat{\Phi}_{q,1}(\zeta_1)(\alpha) = p_q(\zeta_1)(\alpha),$$
$$\hat{\Phi}_{q,\ell}(\zeta_\ell, \dots, \zeta_1)(\alpha) = p_{q+\ell-1}(\zeta_\ell)(\hat{\Phi}_{q,\ell-1}(\zeta_{\ell-1}, \dots, \zeta_1)(\alpha)) \qquad \text{if } \ell > 0,$$

for any $p \in \mathcal{M}$, $\alpha \in E^q_p$ and $\zeta_j \in \overline{\mathcal{V}}^*_p$. That is,

$$\hat{\Phi}_{q,\ell}(\zeta_\ell, \dots, \zeta_1) = p_{q+\ell-1}(\zeta_\ell) \circ p_{q+\ell-2}(\zeta_{\ell-1}) \circ \dots \circ p_q(\zeta_1). \tag{4.5}$$

Since $\hat{\Phi}_{q,\ell}$ is linear in each of of its arguments, it determines a bundle homomorphism

$$\Phi_{q,\ell} : \bigotimes_{i=1}^{\ell} \overline{\mathcal{V}}^* \to \operatorname{Hom}(E^q, E^{q+\ell}).$$

If $\ell \geq 2$ then Condition (1) of the proposition and (4.5) implies that

$$\hat{\Phi}_{q,\ell}(\zeta_\ell, \zeta_{\ell-1}, \dots, \zeta_1) = 0$$

whenever two successive elements ζ_j and ζ_{j+1} are equal, which in turn implies

$$\Phi_{q,\ell}(\operatorname{Alt}(\zeta)) = \Phi_{q,\ell}(\zeta), \quad \zeta = (\zeta_\ell, \zeta_{\ell-1}, \dots, \zeta_1),$$

where $\operatorname{Alt} : \bigotimes_{i=1}^q \overline{\mathcal{V}}^* \to \bigwedge^q \overline{\mathcal{V}}^*$ is the canonical projection. From the definition in (4.5) we get,

$$\hat{\Phi}_{q,\ell+m}(\zeta, \eta) = \hat{\Phi}_{q+m,\ell}(\zeta) \circ \hat{\Phi}_{q,m}(\eta), \quad \eta = (\eta_m, \dots, \eta_1),\ \zeta = (\zeta_\ell, \dots, \zeta_1)$$

which implies

$$\Phi_{q,\ell+m}(\zeta \wedge \eta) = \Phi_{q+m,\ell}(\zeta) \circ \Phi_{q,m}(\eta), \quad \eta = \eta_m \wedge \dots \wedge \eta_1,\ \zeta = \zeta_\ell \wedge \dots \wedge \zeta_1.$$

In particular

$$\Phi_{0,q+1}(\zeta \wedge \eta) = \Phi_{q,1}(\zeta) \circ \Phi_{0,q}(\eta), \quad \eta = \eta_q \wedge \dots \wedge \eta_1,\ \zeta = \zeta_1.$$

By way of the definition this says

$$\Phi_{0,q+1}(i\alpha \otimes \zeta \wedge \eta) = p_q(\zeta)(\Phi_{0,q}(\alpha \otimes \eta)),$$

that is,

$$\Phi_{0,q+1} \circ d_q(\zeta) = p_q(\zeta) \circ \Phi_{0,q}, \quad \zeta \in \overline{\mathcal{V}}^*. \tag{4.6}$$

The proof will be complete once we show that the maps $\Phi_{0,q}$ are isomorphisms.

An induction argument starting with the fact that, by Condition (3), $\Phi_{q,1}$ is surjective as a map $E^q \otimes \overline{\mathcal{V}}^* \to E^{q+1}$ gives that

$$\Phi_{q,\ell} : E^q \otimes \bigwedge^{\ell} \overline{\mathcal{V}}^* \to E^{q+\ell} \quad \text{is surjective.}$$

In particular, $\Phi_{0,m} : E^0 \otimes \bigwedge^m \overline{\mathcal{V}}^* \to E^m$ is surjective. Since $\operatorname{rk} E^0 = \operatorname{rk} E^m$ and $\bigwedge^m \overline{\mathcal{V}}^*$ is a line bundle, $\Phi_{0,m}$ must be an isomorphism.

To see that $\Phi_{0,q}$ is an isomorphism as a map $E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^q$, it only remains to verify that it is injective. To this end, suppose $w \in \ker \Phi_{0,q}$. This element lies over some point $p \in \mathcal{M}$. Using a basis $\zeta_1, \dots, \zeta_m$ of $\overline{\mathcal{V}}^*_p$ we write

$$w = \sum_{|I|=q}{}' w_I \otimes \zeta^I$$

with suitable $w_I \in E^0_p$. Fix some index I_0, let I_0^c be the complementary index to I_0. Then

$$0 = \Phi_{q,m-q}(\zeta^{I_0^c})(\Phi_{0,q}(w)) = \Phi_{0,m}(w_{I_0} \otimes \zeta^{I_0^c} \wedge \zeta^{I_0}).$$

Since $\Phi_{0,m}$ is an isomorphism, $w_{I_0} \otimes \zeta^{I_0^c} \wedge \zeta^{I_0} = 0$, so $w_{I_0} = 0$. Since I_0 is arbitrary, $w = 0$. Therefore $\Phi_{0,q}$ is injective.

In view of (4.6), the maps $\Phi_q = \Phi_{0,q}$ satisfy the assertion of the Proposition. □

We showed that $\Phi_{0,m} : E^0 \otimes \bigwedge^m \overline{\mathcal{V}}^* \to E^m$ is surjective without assuming that $\operatorname{rk} E^0 = \operatorname{rk} E^m$. Instead of this assumption one can assume that with some (hence any) Hermitian metrics on the E^q, the adjoints $p_q^\star(\zeta) \in \operatorname{Hom}(E^{q+1}, E^q)$ of the $p_q(\zeta)$ satisfy the corresponding version of (3). Since these adjoints satisfy (1) and (2), we conclude that there is a surjective map $\Psi_{0,m} : E^m \otimes \bigwedge^m \overline{\mathcal{V}}^* \to E^0$ which together with the surjectivity of $\Phi_{m,0}$ gives the equality of the ranks.

An example of a sequence not satisfying (3) is the following, again starting with a subbundle $\mathcal{V}$. Let $E^0 = \mathcal{M} \times \mathbb{C}$, let $E^q = \bigwedge^q \overline{\mathcal{V}}^* \oplus \bigwedge^{q-1} \overline{\mathcal{V}}^*$ for $q > 0$. Define $p_0 = d_0 \oplus 0$ and $p_q = d_q \oplus d_{q-1}$ for $q > 0$. Then conditions (1) and (2) are satisfied, but (3) fails for $q = 0$.

The hypotheses of Theorem 1.1 yield isomorphisms $\Phi_q : E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^q$ via Proposition 4.1 such that with $p_q = \boldsymbol{\sigma}(\mathbb{P}_q)$, $\Phi_{q+1}^{-1} p_q(\zeta)\Phi_q = d_q(\zeta)$. Thus with Φ_q also denoting the induced maps $C^\infty(\mathcal{M}; E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; E^q)$, the operators

$$\Phi_{q+1}^{-1} \circ \mathbb{P}_q \circ \Phi_q : C^\infty(\mathcal{M}; E^0 \otimes \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; E^0 \otimes \textstyle\bigwedge^{q+1} \overline{\mathcal{V}}^*) \tag{4.7}$$

have standard principal symbol. Thus if the operators

$$\mathbb{D}_q : C^\infty(\mathcal{M}; E^0 \otimes \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(\mathcal{M}; E^0 \otimes \textstyle\bigwedge^{q+1} \overline{\mathcal{V}}^*)$$

are defined using connections as in Section 2, then

$$\Phi_{q+1}^{-1} \circ \mathbb{P}_q \circ \Phi_q - \mathbb{D}_q$$

has vanishing principal symbol of order 1, which means that this difference is of order 0. So

$$\Phi_{q+1}^{-1} \circ \mathbb{P}_q \circ \Phi_q = \mathbb{D}_q + \gamma_q$$

where γ_q is defined by a homomorphism $E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^0 \otimes \bigwedge^{q+1} \overline{\mathcal{V}}^*$. The exactness of the symbol sequence in Theorem 1.1 implies that of the principal symbols of (4.7), which is the same as for the $\mathbb{D}$-sequence. Thus:

Corollary 4.2. *In Theorem 1.1, $\overline{\mathcal{V}}$ is an almost elliptic structure, the vector bundles can be assumed to be $E^q = E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^*$, the operators given by*

$$\mathbb{P}_q = \mathbb{D}_q + \gamma_q$$

where $\mathbb{D}_q$ is defined using some fixed connections on E and $\overline{\mathcal{V}}^*$ as described in Section 2 and suitable homomorphisms $\gamma_q : E \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E \otimes \bigwedge^{q+1} \overline{\mathcal{V}}^*$.

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