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A short overview of Besse's conjecture

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Abstract. We call CPE metrics the critical points of the total scalar curvature functional restricted to the space of metrics with constant scalar curvature of unitary volume. In the 1980's Arthur L. Besse conjectured[a] that every CPE metrics must be Einstein. This paper briefly describes some recent advances on the Besse's Conjecture.

Keywords: Einstein metric, total scalar curvature, critical point equation, σ_2-curvature, vacuum static space.

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[a]We thank the anonymous reviewer for reporting that this conjecture appeared naturally during some seminars 75-80's in France.

1 Introduction

This expository paper is based on a lecture given at *XIV - ENAMA - Encontro Nacional de Análise Matemática e Aplicações*, held in on-line format in Brazil in November 2021 and the main results was published in [2].

Scalar curvature functional appears in the study of Einstein metrics, since this is a type of curvature closely related with the Ricci tensor. Recall

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that a Riemannian manifold is said to be Einstein if the Ricci tensor is a multiple of the metric g, i.e., $Ric_g = \lambda g$, where $\lambda : M \to \mathbb{R}$, in particular if (M^n, g), $n \geq 3$, is connected, then λ is constant. In other words, (M^n, g) is Einstein if its traceless tensor

$$\mathring{Ric}_g = Ric_g - \frac{R_g}{n} g$$

is identically zero, where Ric_g and R_g are Ricci and scalar curvatures, respectively.

Let (M^n, g) be an n-dimensional, connected, closed (compact without boundary) manifold with $n \geq 3$, $\mathcal{M}$ be the Riemannian metric space and $S_2(M)$ be the space of symmetric 2-tensors on M. Fischer and Marsden [16] consider the scalar curvature map $\mathcal{R} : \mathcal{M} \to C^\infty$ which associates to each metric $g \in \mathcal{M}$ its scalar curvature. If γ_g is the linearization of the map $\mathcal{R}$ and γ_g^* is its L^2-formal adjoint, then they stated that

$$\gamma_g h = -\Delta_g tr_g h + \delta_g^2 h - \langle Ric_g, h \rangle$$

and

$$\gamma_g^* f = \nabla_g^2 f - (\Delta_g f) g - f Ric_g,$$

where $\delta_g = -div_g$, $h \in S_2(M)$, $f \in C^\infty(M)$ and ∇_g^2 is the Hessian form on M^n, respectively.

In 80's, in the classical book "Einstein Manifolds", A. Besse pointed out that a natural way to prove the existence of Einstein metrics is to search for critical points of the total scalar curvature $\mathcal{S} : \mathcal{M} \to \mathcal{R}$ defined by

$$\mathcal{S}(g) = \int_M R_g dv_g,$$

also known as the Einstein-Hilbert functional.

It is well known that the solution of the Yamabe problem shows that any compact manifold M^n admits a Riemannian metric with constant scalar curvature. In particular, the set $\mathcal{C} = \{g \in \mathcal{M}; R_g \text{ is constant}\} \neq \emptyset$. Thus, we may consider the set $\mathcal{M}_1 = \{g \in \mathcal{C}; Vol_g(M) = 1\} \neq \emptyset$ and then, investigate the critical points of the scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{M}_1$. Here, it is important to observe that N. Koiso [20] proved that, under a generic condition, $\mathcal{M}_1$ is an infinite dimensional manifold, such a smoothness is essential to seek critical points.

In Remark 4.48 of [9], p.128, it was conjectured that critical points of the total scalar curvature functional $\mathcal{S}$ restricted to $\mathcal{M}_1$ must be Einstein. More precisely, the Euler-Lagrange equation of Hilbert-Einstein action restricted to $\mathcal{M}_1$ may be written as the following critical point equation (CPE)

$$\gamma_g^* f = \nabla_g^2 f - (\Delta_g f)g - fRic_g = \mathring{Ric}_g. \tag{1.1}$$

Following the notation adopted in [8, 23], we consider the following definition.

Definition 1.1. A CPE metric is a triple (M^n, g, f), where (M^n, g) is a closed oriented Riemannian manifold of dimension $n \geq 3$ with constant scalar curvature, volume 1 and f is a smooth potential satisfying (1.1).

The Besse conjecture (or CPE conjeture) can be rewritten as:

Conjecture 1.2 ([9]). *A CPE metric is always Einstein.*

In their study of the surjectivity of the scalar curvature map $\mathcal{R}$, Fischer and Marsden considered the so-called vacuum static equation $\gamma_g^*(f) = 0$. Due to the similarity between this equation and (1.1), it is natural to consider the following definition.

Definition 1.3. Let (M^n, g) be a complete Riemannian manifold. We say that (M^n, g) is a vacuum static space if there is a (not identically zero) smooth function f solving the following vacuum static equation on M:

$$\gamma_g^* f = \nabla^2 f - \Delta_g f g - Ric_g f = 0. \tag{1.2}$$

We also refer (M^n, g, f) as a vacuum static space.

In the last few decades, numerous investigations have been made on these spaces. The classification problem is a fundamental question as well as rigidity results. For more details, see, e.g., [1, 28] and references therein.

Proceeding, observe that (1.1) is equivalent to

$$\mathring{Ric}_g = \nabla_g^2 f - \left(Ric_g - \frac{R_g}{n-1} g \right) f \tag{1.3}$$

which can be re-written in the form

$$(1+f)\mathring{Ric}_g = \nabla_g^2 f + \frac{R_g f}{n(n-1)} g. \tag{1.4}$$

Since our interest is in the context of the variational problem related to the Einstein-Hilbert functional, it is natural to assume that (M^n, g) is not scalar-flat ($R_g \not\equiv 0$). Hence, if (M^n, g, f) is a CPE metric and f is a constant function, then $f = 0$ and this implies that (M^n, g) is Einstein. Moreover, if (M^n, g, f) is a CPE metric with non-constant function f, then the set, sometimes called critical level set,

$$B = \{x \in M^n; f(x) = -1\}$$

has zero n-dimensional measure (see [23]). Thus, to prove that a CPE metric is Einstein, it is equivalent to show that (g, f) satisfies the equation

$$\nabla_g^2 f + \frac{R_g f}{n(n-1)} g = 0, \tag{1.5}$$

where f is not a constant function. Taking the trace of (1.5) one sees that

$$\Delta_g f = -\frac{R_g}{n-1} f. \tag{1.6}$$

Note that (1.6) follows directly from (1.2) and (1.1) as well. Now, multiplying the equation (1.6) by f and integrating over M, we obtain

$$\frac{R_g}{n-1} \int_M f^2 dv_g = -\int_M f \Delta_g f dv_g = \int_M |\nabla_g f|^2 dv_g > 0. \tag{1.7}$$

Since f is a non constant function, one concludes that $R_g > 0$.

If f satisfies the equation (1.5), then (M^n, g) is isometric to $\mathbb{S}^n(r)$ where $r = \left(\frac{R_g}{n(n-1)}\right)^{1/2}$ (see [24, 27]). In this way, considering f non constant, Conjecture 1.2 can be restated changing the Einstein property by the fact that the manifold is the round sphere.

In the next section we present a collection of partial answers for the Besse's conjecture.

2 Advances in Besse's conjecture

In this section we present some partial results for Conjecture 1.2. In fact, even before the publication of Besse's book, Lafontaine [21] proved that if the CPE metric is conformally flat, then the manifold is Einstein.

To more accurately present the latest advances in understanding the relationship between CPE and Einstein metrics, let us recall some important tensors. Let (M^n, g) be an n-dimensional Riemannian manifold with $n \geq 3$. It is well known that there exists an orthogonal decomposition of the curvature tensor Rm_g which is given by

$$Rm_g = W + A_g \odot g,$$

here $\odot$ is the Kulkarni-Nomizu product, W is the Weyl curvature tensor and A_g is the Schouten tensor defined as

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{1}{2(n-1)}R_g g\right).$$

The Weyl tensor has the same symmetries of the curvature tensor. It is well known that if $n = 3$, then $W = 0$, and for $n > 3$ M to be conformally flat is equivalent to $W = 0$.

Also, in coordinates, the Cotton tensor is defined as

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik})$$

and the Bach tensor, $n \geq 4$, is

$$B_{ij} = \frac{1}{n-3}\nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl}.$$

Moreover, the Weyl, the Cotton and the Bach tensors are totally trace-free. For more details about these tensors see [11].

The literature has presented some improvements of Lafontaine's result. In the work [3], Baltazar proved Besse's conjecture for compact manifolds with pinched Weyl curvature and satisfying the zero radial Weyl curvature condition. We remember that (M^n, g) has zero radial Weyl curvature when, for a suitable potential function f on M^n, the Weyl tensor satisfies $i_{\nabla f}W = W(\cdot, \cdot, \cdot, \nabla f) = 0$. It is clear that every locally conformally flat manifold is contained in this class.

Theorem 2.1 ([3], Theorem 1)**.** *The CPE conjecture is true for $n-$dimensional ($n \geq 3$) manifolds with nonnegative sectional curvature satisfying the zero radial Weyl curvature condition.*

In the same work was proved that the condition on sectional curvature can be replaced by a control on the norm of the traceless Ricci tensor.

Theorem 2.2 ([3], Theorem 2)**.** *Let (M^n, g, f), $n \geq 3$, be a CPE metric with zero radial Weyl curvature satisfying*

$$|\mathring{Ric}_g| \leq \frac{R_g^2}{n(n-1)}.$$

Then (M^n, g) is isometric to a round sphere.

Baltazar in 2020 [4] used a pointwise pinching condition on the Weyl tensor to provide one more positive answer to Conjecture 1.2.

Theorem 2.3 ([4], Theorem 1)**.** *The Besse conjecture is true for any CPE metrics satisfying*

$$|W| \leq \sqrt{\frac{n}{2(n-2)}}\left(\frac{R_g}{\sqrt{n(n-1)}} - 2|\mathring{Ric}_g|\right).$$

Since Einstein manifolds have parallel Ricci tensor, but the converse is not necessarily true, the following question arises: Is it possible to prove the Besse's conjecture under the Ricci parallel condition? In fact, this was answered by Chang, Hwang and Yun in [12] and posteriorly generalized by the same authors in [29] with the following result.

Theorem 2.4 ([29], Theorem 1.2)**.** *Let (g, f) be a non-trivial solution of the CPE on an n-dimensional compact Riemannian manifold M. If (M, g) has harmonic curvature, then M is isometric to a round sphere.*

There are particularities on different dimensions. Recently, Hwang and Yun has proved that the Conjecture 1.2 is true under the condition of positive isotropic curvature on (M^n, g) considering $n \geq 4$, see [19].

Theorem 2.5 ([19], Theorem 1.2)**.** *Let (M^n, g, f) CPE metric, with $n \geq 4$. If (M^n, g) has positive isotropic curvature, then (M^n, g) is isometric to a round sphere.*

For dimensions greater than 4, Baltazar et al. [5] proved that Besse's Conjecture is true if (M^n, g) has zero radial Weyl curvature.

Theorem 2.6 ([5], Theorem 4)**.** *Conjecture 1.2 is true for compact n-dimensional manifolds, $n \geq 5$ satisfying $i_{\nabla f} W = 0$.*

On the other hand, in dimension $n = 4$ there is a decomposition

$$W = W^+ \oplus W^-,$$

where $W^\pm : \Lambda^2_\pm \to \Lambda^2_\pm$ are called the self-dual and anti-self-dual parts of W. Here W is understood as an endomorphism of the bundle of 2-forms $\Lambda^2 = \Lambda^2_- \oplus \Lambda^2_+$. A metric is called half conformally flat if it is selfdual or antiselfdual, namely if $W^- = 0$ or $W^+ = 0$.

In 2014, Barros and Ribeiro Jr. [8] motivated by ideas on the study of Ricci solitons, proved that the conjecture is true for 4-dimensional half conformally flat manifolds.

Theorem 2.7 ([8], Theorem 1.3)**.** *Conjecture 1.2 is true for 4-dimensional half conformally flat manifolds.*

We note that $\mathbb{C}P^2$ endowed with the Fubini-Study metric shows that the half conformally flat condition is weaker than locally conformally flat condition in dimension 4. Which in turn can be further weakened by considering that M^4 has harmonic tensor W^+, i.e., $\delta W^+ = 0$ where δ is the formal divergence defined for any $(0,4)$-tensor T by

$$\delta T(X_1, X_2, X_3) = trace_g\{(Y, Z) \mapsto \nabla_Y T(Z, X_1, X_2, X_3)\},$$

as proved by Barros, Leandro and Ribeiro in [7].

Theorem 2.8 ([7], Theorem 1.3)**.** *Conjecture 1.2 is true for* 4*-dimensional with harmonic tensor* W^+*.*

In dimension $n = 3$, we will present four conditions that guarantee that a CPE metric is necessarily Einstein. First, since for $n = 3$ the condition $i_{\nabla f} W = 0$ is obviously satisfied, Theorem 2.1 implies the following.

Corollary 2.9 ([3], Corollary 1)**.** *The CPE conjecture is true for 3-dimensional manifolds with nonnegative sectional curvature.*

In 2013, Hwang [18] proved Besse's conjecture in dimension 3 under the condition $\ker({\gamma^*}_g) \neq 0$, as shown below.

Theorem 2.10 ([18], Theorem 1.1)**.** *Let* (M^3, g, f) *be an* 3*-dimensional CPE metric. If* $ker(\gamma_g^*) \neq 0$, *then* (M^3, g) *is isometric to a round sphere* $\mathbb{S}^3$.

More recently, Baltazar e Da Silva [6] obtained an affirmative answer to the CPE conjecture under the cyclic parallel Ricci tensor condition, i.e., $\nabla_i R_{jk} + \nabla_j R_{ki} + \nabla_k R_{ij} = 0$.

Theorem 2.11 ([6], Theorem 2)**.** *The Conjecture 1.2 is true for* 3*-dimensional manifolds with cyclic parallel Ricci tensor.*

Availing the similarity between CPE metrics and vacuum static spaces, Qing and Yuan [25] made progress about Besse's conjecture and the classification vacuum static spaces. Specifically, for $n = 3$ they showed:

Theorem 2.12 ([25], Theorem 1.3)**.** *Conjecture 1.2 holds for a compact Riemannian* 3*-manifold with no boundary with nonnegative complete divergence* $\mathcal{C} = C_{ijk},{}^{ijk}$ *of the Cotton tensor.*

In general dimensions they proved that the Conjecture 1.2 is true for Bach flat manifolds.

Theorem 2.13 ([25], Theorem 3.10). *Suppose that* $(M^n, g), n \geq 3$, *is a Bach flat CPE manifold. Then* (M^n, g) *is isometric to a round sphere.*

In Silva Filho [14], it was proved that a CPE metric admitting a non-trivial closed conformal vector field must be isometric to a round sphere metric, which provides another partial answer to the CPE conjecture. More precisely, he proved the following result.

Theorem 2.14 ([14], Theorem 1.5). *Let* (M^n, g, f) *be a CPE metric which admits a non-trivial closed conformal vector field. Then* (M^n, g) *is isometric to a round sphere* $\mathbb{S}^n$*.*

Regarding conditions on the potential function f of the CPE metric (M^n, g, f) we highlight three results. The first one by Hwang [17].

Theorem 2.15 ([17], Lemma 1). *A CPE metric* (M^n, g, f) *with* $f > -1$ *is Einstein.*

Then, in 2015, Neto [23] provided a necessary and sufficient condition on the norm of the gradient of the potential for a CPE metric to be Einstein as follows.

Theorem 2.16 ([23], Theorem 1). *Let* (M^n, g, f) *be an n-dimensional CPE metric. Then* (M^n, g) *is Einstein if and only if*

$$|\nabla f|^2 + \frac{Rf^2}{n(n-1)} = \alpha,$$

where α *is a constant.*

In the same year, Filho [15] gave sufficient integral conditions on f for a CPE metric to be isometric to the round sphere.

Theorem 2.17 ([15], Theorem 1). *Let* (M^n, g, f) *be CPE metric. Then* (M^n, g) *is isometric to round sphere and* f *is a first eigenfunction of the Laplacian, provided that*

1. $\int_M |\nabla f|^4 dM_g = \frac{(n+2)R_g^2}{3n(n-1)^2} \int_M f^4 dM_g$, *and*

2. $\int_M f^3 dM_g \geq 0$.

The list of results presented above is not intended to be an exhaustive one. Indeed, this research area is very fruitful. Finally, it is worth mentioning the recent work by Colombo, Mari and Rigoli [13] in which they studied an extension of the CPE conjecture to manifolds which support a structure relating curvature to the geometry of a smooth function φ into a target Riemannian space.

For proofs of mentioned results and more details we refer the reader to the original sources. Next section we mention a new approach on this matter made by the first author.

3 Besse's Conjecture with a new geometric point of view

Motivated by these works and others, in [2] the first author provided a necessary and sufficient condition for a CPE metric to be Einstein for $n \geq 3$, improving the understanding about CPE metrics and Besse's conjecture with a new geometric point of view that involves the potential function f inherited by the CPE condition. We recall some definitions and results.

Let (M^n, g) be an n-dimensional Riemanian manifold, $n \geq 3$. The σ_2-curvature, which will be denoted by $\sigma_2(g)$, is as a nonlinear map $\sigma_2 : \mathcal{M} \to C^\infty(M)$, defined as the second elementary symmetric function of the eigenvalue of the Schouten tensor. In this case, we obtain that

$$\sigma_2(g) = -\frac{1}{2}|Ric_g|^2 + \frac{n}{8(n-1)}R_g^2. \tag{3.1}$$

Motivated by works of Fischer and Marsden [16] and Lin and Yuan [22], in [26] was proved that the linearization of the σ_2-curvature at the metric g,

$$\Lambda_g : S_2(M) \to C^\infty(M),$$

is given by

$$\begin{aligned}\Lambda_g(h) &= \frac{1}{2}\left\langle Ric_g, \Delta_g h + \nabla^2 tr_g h + 2\delta^*\delta h + 2\mathring{R}(h)\right\rangle \\ &-\frac{n}{4(n-1)}R_g\left(\Delta_g tr_g h - \delta^2 h + \langle Ric, h\rangle\right),\end{aligned}$$

where δ^* is the L^2-formal adjoint of δ and $\mathring{R}(h)_{ij} = g^{kl}g^{st}R_{kijs}h_{lt}$.

Thus, its L^2-formal adjoint, $\Lambda_g^* : C^\infty(M) \to S_2(M)$, is

$$\begin{aligned}\Lambda_g^*(f) &= \frac{1}{2}\Delta_g(fRic_g) + \frac{1}{2}\delta^2(fRic_g)g + \delta^*\delta(fRic_g) + f\mathring{R}(Ric_g) \\ &-\frac{n}{4(n-1)}\left(\Delta_g(fR_g)g - \nabla^2(fR_g) + fR_gRic_g\right).\end{aligned}$$

This implies that

$$tr_g\Lambda_g^*(f) = \frac{2-n}{4}R_g\Delta_g f + \frac{n-2}{2}\langle\nabla^2 f, Ric_g\rangle - 2\sigma_2(g)f. \tag{3.2}$$

Note that,

$$\begin{aligned}\Lambda_g^*(1) &= \frac{1}{2}\Delta_g Ric_g - \frac{1}{4(n-1)}(\Delta_g R_g)g + \frac{2-n}{4(n-1)}\nabla^2 R_g \\ &+ \mathring{R}(Ric_g) - \frac{n}{4(n-1)}R_gRic_g.\end{aligned} \tag{3.3}$$

Then, by (3.2) and (3.3) we obtain

$$tr_g\Lambda_g^*(1) = -2\sigma_2(g) \tag{3.4}$$

and

$$div_g\Lambda_g^*(1) = -\frac{1}{2}d\sigma_2(g). \tag{3.5}$$

The relations (3.4) and (3.5) are similar to the relations between the Ricci tensor and the scalar curvature, namely $R_g = tr_gRic_g$ and $div_gRic_g = \frac{1}{2}dR_g$.

In [26] was introduced the notion of σ_2-singular space, which has the L^2-formal adjoint of the linearization of the σ_2-curvature map with non-trivial kernel, and under certain hypotheses it was proved rigidity and other results.

Definition 3.1. A complete Riemannian manifold (M, g) is σ_2-singular if

$$\ker \Lambda_g^* \neq \{0\},$$

where $\Lambda_g^* : C^\infty(M) \to S_2(M)$ is the L^2-formal adjoint of Λ_g. We will call the triple (M^n, g, f) as a σ_2-singular space if f is a nontrivial function in $\ker \Lambda_g^*$.

On the other hand, the first author together with Silva Santos obtained the following rigidity result for σ_2-singular Einstein manifolds with positive σ_2-curvature.

Theorem 3.2 ([26], Theorem 3). *Let (M^n, g, f) be a closed σ_2-singular Einstein manifold with positive σ_2-curvature. Then (M^n, g) is isometric to the round sphere with radius $r = \left(\frac{n(n-1)}{R_g}\right)^{\frac{1}{2}}$ and f is an eigenfunction of the Laplacian associated to the first eigenvalue $\frac{R_g}{n-1}$ on $\mathbb{S}^n(r)$. Hence dim $\ker \Lambda_g^* = n+1$ and $\int_M f dv_g = 0$.*

The next lemma is crucial for the result in [2].

Lemma 3.3. *Let (M^n, g, f) be an n-dimensional CPE metric, then*

$$tr_g \Lambda_g^*(f) = \left(\frac{n-2+nf}{2}\right) |\mathring{Ric}|^2.$$

Proof. Since (M^n, g, f) is an n-dimensional CPE metric, then f satisfies the equation (1.3)

$$\nabla_g^2 f = \mathring{Ric}_g + \left(Ric_g - \frac{R_g}{n-1} g\right) f. \tag{3.6}$$

Thus, by equations (1.6), (3.1) and (3.6), we get

$$\begin{aligned} tr_g\Lambda_g^*(f) &= \frac{2-n}{4}R_g\left(\frac{-R_g}{(n-1)}\right)f \\ &+ \frac{n-2}{2}\left\langle \mathring{Ric}_g + \left(Ric_g - \frac{R_g}{n-1}g\right)f, Ric_g\right\rangle \\ &+ \left(|Ric_g|^2 - \frac{n}{4(n-1)}R_g^2\right)f \\ &= \frac{n-2}{2}|\mathring{Ric}_g|^2 + \frac{n}{2}\left(|Ric_g|^2 - \frac{R_g^2}{n}\right)f \\ &= \left(\frac{n-2+nf}{2}\right)|\mathring{Ric}_g|^2. \end{aligned}$$

This proves the result. □

We recall Lemma 2 in [26]: if (M^n, g) is Einstein, then

$$\Lambda_g^*(f) = \frac{R_g(n-2)^2}{4n(n-1)}\left(\nabla^2 f - (\Delta_g f)g - \frac{R_g}{n}fg\right). \tag{3.7}$$

Now, we can announced the first result in this section.

Theorem 3.4. *Let (M^n, g, f) be an n-dimensional CPE metric with non-constant potential function f and $n \geq 3$. Then (M^n, g) is Einstein if and only if $f \in \ker \Lambda_g^*$, where $\Lambda_g : S_2(M) \to C^\infty(M)$ is the linearization of the σ_2-curvature and Λ_g^* is the L^2-formal adjoint of the operator Λ_g, i.e., (M^n, g, f) is a σ_2-singular space.*

As an immediate consequence, we obtain the following result.

Corollary 3.5. *Let (M^n, g, f) be an n-dimensional CPE metric with non-constant potential function f and $n \geq 3$. If $f \in \ker \Lambda_g^*$, then (M^n, g) is isometric to the round sphere with radius $r = \left(\frac{n(n-1)}{R_g}\right)^{\frac{1}{2}}$ and f is an eigenfunction of the Laplacian associated to the first eigenvalue $\frac{R_g}{n-1}$ on $\mathbb{S}^n(r)$. Hence, $\dim\ker \Lambda_g^* = n+1$ and $\int_M f dv_g = 0$.*

As already mentioned, the CPE equation is related to vacuum static space. In particular, the local scalar curvature rigidity is an interesting problem related to the vacuum static equation (1.2). This problem is closed related with the Conjecture of Min-Oo on $\mathbb{S}^m_+$, for more details in this sense see [10].

Moreover, we observe that if (M^n, g) is a closed Riemannian manifold and $\ker \Lambda_g^* \cap \ker \gamma_g^* \neq \{0\}$, then (M^n, g) is an Einstein manifold. Thus, it is isometric to the standard sphere $\mathbb{S}^n$.

Theorem 3.6. *Let (M^n, g, f) be an n-dimensional closed static vacuum space, $n \geq 3$. Then (M^n, g) is Einstein if and only if the space (M^n, g, f) is σ_2-singular. If f is a non-constant function (M^n, g) has to be isometric to the standard sphere $\mathbb{S}^n$, in the other case (M^n, g) has to be Ricci flat.*

Lin and Yuan have proved in [22] other results about deformation of Q-curvature involving the kernel of Γ_g^*, where $\Gamma_g : S_2(M) \to C^\infty(M)$ is the linearization of the Q-curvature and Γ_g^* is the L^2-formal adjoint of the operator Γ_g. In particular, they proved rigidity and other results, including an analogous result to our Theorem 3.6 in the context of Q-singular spaces.

We can summarize the above results in the following assertions.

Corollary 3.7. *Let (M^n, g) be an n-dimensional closed oriented manifold with $n \geq 3$ and f be a non-constant function defined in M. We consider the following statements:*

i) (M^n, g, f) *is a CPE metric;*

ii) (M^n, g, f) *is a vacuum static space;*

iii) (M^n, g, f) *is a σ_2-singular space.*

If any two of the statements hold, we have that (M^n, g) is isometric to the standard sphere $\mathbb{S}^n$. In particular, if any two statements are true, the other one is also true.

We can observe that if (M^n, g) is a closed Einstein manifold with negative scalar curvature, then (M^n, g) can not be a σ_2-singular space. In fact, if $f \in \ker(\Lambda_g^*)$ is a non-constant function and $R_g \neq 0$, we obtain from equations (1.5), (1.6) and (1.7), that $R_g > 0$. In this way the we can rewrite Corollary 3.7 as the following.

Corollary 3.8. *Let (M^n, g) be an n-dimensional closed oriented manifold with $n \geq 3$ and f be a non-constant function defined in M. We consider the following statements:*

i) (M^n, g, f) is a CPE metric.

ii) (M^n, g, f) is a vacuum static space.

iii) (M^n, g, f) is a σ_2-singular space.

iv) (M^n, g) is an Einstein manifold with $R_g > 0$.

If any two of the statements hold, we have that (M^n, g) is isometric to the standard sphere $\mathbb{S}^n$. In particular, if any two statements are true, the others are also true.

All proofs for these results in this section can be found in [2].

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