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Border control for the Bresse system by Carleman estimation

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. This work deals with border inspection for the Bresse system whose control is in $\{L\}$. The control is obtained through the inequality of Carleman and the HUM (Hilbert Uniqueness Method) due to Lions [18] e [17].
This paper is concerned with the exact controllability at the frontier for the Bresse system, where the control functions act on $\{L\}$. The inequality of observability is an important result to obtain the controls. In order to obtain such an inequality of observability we will make use of a Carleman inequality.
The main result is obtained by applying the Hilbert Uniqueness Method proposed by Lions, without using the Hölmgren's uniqueness theorem or the hypothesis of equal-speed waves of propagation.

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1 Introduction

Consider the Bresse system given by

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l[\omega_x - l\varphi] = f_1 \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = f_2 \\ \rho_1\omega_{tt} - k_0[\omega_x - l\varphi]_x + kl(\varphi_x + \psi + l\omega) = f_3 \end{cases} \tag{1.1}$$

in $Q = (0, L) \times (0, T)$. Assume Dirichlet boundary conditions that is,

$$\begin{aligned} &\varphi(0, t) = \psi(0, t) = \omega(0, t) = 0, \\ &\varphi(L, t) = g_1(t) \quad \psi(L, t) = g_2(t) \quad \omega(L, t) = g_3(t) \end{aligned} \tag{1.2}$$

for $t \in (0, T)$, and initial conditions given by

$$\begin{cases} \varphi(\cdot, 0) = \varphi_0, \quad \varphi_t(\cdot, 0) = \varphi_1, \\ \psi(\cdot, 0) = \psi_0, \quad \psi_t(\cdot, 0) = \psi_1, \\ \omega(\cdot, 0) = \omega_0, \quad \omega_t(\cdot, 0) = \omega_1. \end{cases} \tag{1.3}$$

The problem of exact controllability of (1.1)-(1.3) is to find controls g_1, g_2, g_3 such that at time $T > 0$ we have

$$\varphi(x, T) = \varphi_t(x, T) = \psi(x, T) = \psi_t(x, T) = \omega(x, T) = \omega_t(x, T) = 0$$

In the usual case, the positive constants $\rho_1, \rho_2, k, k_0, l$ and b are related to composition of the material, ω, φ and ψ are denoted the longitudinal, vertical, and shear angle displacements, respectively.

The equality of the propagation velocity of waves

$\dfrac{\rho_1}{\rho_2} = \dfrac{k}{b}$, and $k = k_0$.

is not considered here. This relation was used in several works, as in [2, 10, 11, 23] and recently this hypothesis was withdrawn in [8, 9].

The key point to obtain the control is the inequality of observability given by theorem4.1. In order to obtain the inequality of observability in the theorem4.1, one makes used of the inequality of Carleman given by the theorem3.1. Finally having the inequality of observability applies the Hilbert Uniqueness Method to obtain the desired.

This is paper is organized as follows in section 2 definitions and some results, see [5, 7, 12, 13, 20] in section 3 Carleman's inequality and inequality of observability [14, 16, 17, 19, 21] in section 4 border control for the Bresse system using the Hilbert Uniqueness Method [17, 18, 22].

2 Definitions:

Let

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l[\omega_x - l\varphi] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_1\omega_{tt} - k_0[\omega_x - l\varphi]_x + kl(\varphi_x + \psi + l\omega) = 0, \quad in \quad (0,L)\times(0,T) \\ \varphi(0,t) = \psi(0,t) = \omega(0,t) = 0, \quad t\in(0,T) \\ \varphi(L,t) = g_1(t), \quad \phi(L,t) = g_2(t), \quad \omega(L,t) = g_3(t), \quad t\in(0,T) \\ \varphi(.,0) = \varphi_0, \quad \varphi_t(.,0) = \varphi_1, \quad in \quad (0,L) \\ \psi(.,0) = \psi_0, \quad \psi_t(.,0) = \psi_1, \quad in \quad (0,L) \\ \omega(.,0) = w_0, \quad \omega_t(.,0) = w_1, \quad in \quad (0,L) \end{cases} \tag{2.1}$$

Definition 2.1. We say that $\{\varphi,\psi,\omega\}$ is an ultraweak solution of (2.1) if it satifies

$$\begin{aligned} &\int_Q \varphi F_1 + \psi F_2 + \omega F_3 \; dx \; dt + \rho_1(\varphi_0, u_t(0)) - \rho_1 \langle \varphi_1, u(0)\rangle_{H^{-1},H_0^1} \\ &+\rho_2(\psi_0, v_t(0)) - \rho_2 \langle \psi_1, v(0)\rangle_{H^{-1},H_0^1} + \rho_1(\omega_0, z_t(0)) \\ &-\rho_1 \langle \omega_1, z(0)\rangle_{H^{-1},H_0^1} + k\int_Q g_1(t)u_x(L)dt + b\int_Q g_2(t)v_x(L)dt \\ &+k_0\int_Q g_3(t)z_x(L)dt = 0 \end{aligned}$$

where $\{u,v,z\}$ is the solution of

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v + lz)_x - k_0 l[z_x - lu] = F_1, \\ \rho_2 v_{tt} - bv_{xx} + k(u_x + v + lz) = F_2, \\ \rho_1 z_{tt} - k_0[z_x - lu]_x + kl(u_x + v + lz) = F_3, \\ u(x,T) = u_t(x,T) = v(x,T) = v_t(x,T) = z(x,T) = z_t(x,T) = 0, \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = z(0,t) = z(L,t) = 0, \end{cases} \tag{2.2}$$

in the class $C([0,T];H^1(0,L))\cap C^1([0,T];L^2(0,L))$
with $F_1,F_2,F_3\in L^1(0,T;L^2(0,L))$.

From this definition comes the following:

Theorem 2.2. *Given* $T>0$, $\varphi_0,\psi_0,\omega_0\in L^2(0,L)$, $\varphi_1,\psi_1,\omega_1\in H^{-1}(0,L)$, $F_1,F_2,F_3\in L^1(0,T;H^{-1}(0,L))$ *and* $g_1,g_2,g_3\in L^2(0,T)$ *there is an unique ultraweak solution*

$$\{\varphi,\psi,\omega\}\in C([0,T];L^2(0,L))\cap C^1([0,T];H^{-1}(0,L)),$$

of 2.1. Moreover exists a constant, $C>0$, *such that*

$$\begin{aligned}\rho_1\|\varphi_t\|_{C(0,T;H^{-1}(0,L))}+\rho_2\|\psi_t\|_{C(0,T;H^{-1}(0,L))}+\rho_1\|\omega_t\|_{C(0,T;H^{-1}(0,L))}\\+\rho_1\|\varphi\|_{C(0,T;L^2(0,L))}+\rho_2\|\psi\|_{C(0,T;L^2(0,L))}+\rho_1\|\omega\|_{C(0,T;L^2(0,L))}\\\leq C[k|\varphi_0|+b|\psi_0|+k_0|\omega_0|++\rho_1\|\varphi_1\|+\rho_2\|\psi_1\|+\rho_1\|\omega_1\|\\+|g_1|+|g_2|+|g_3|]\end{aligned}\tag{2.3}$$

Demonstrated in [15]

For the ultraweak solution of (2.1) we have the inequalities

$$\begin{aligned}&\|\varphi\|_{L^\infty(0,T;L^2(0,L))}+\|\psi\|_{L^\infty(0,T;L^2(0,L))}+\|\omega\|_{L^\infty(0,T;L^2(0,L))}\\&\leq C\{k|\varphi_0|+b|\psi_0|+k_0|\omega_0|+\rho_1\|\varphi_1\|_{H^{-1}(0,L)}+\rho_2\|\psi_1\|_{H^{-1}(0,L)}\\&+\rho_1\|\omega_1\|_{H^{-1}(0,L)}+|g_1|+|g_2|+|g_3|\}\end{aligned}\tag{2.4}$$

$$\begin{aligned}&\|(\varphi_t,\psi_t,\omega_t)\|_{(L^\infty(0,T,H^{-1}(0,L)))^3}=\\&\|\varphi_t\|_{L^\infty(0,T,H^{-1}(0,L))}+\|\psi_t\|_{L^\infty(0,T,H^{-1}(0,L))}+\|\omega_t\|_{L^\infty(0,T,H^{-1}(0,L))}\\&\leq C\{k|\varphi_0|+b|\psi_0|+k_0|\omega_0|+\rho_1\|\varphi_1\|+\rho_2\|\psi_1\|+\rho_1\|\omega_1\|\\&+|g_1|+|g_2|+|g_3|\}.\end{aligned}\tag{2.5}$$

3 Carleman's inequality

We consider the functions $u, v, z \in L^2(-T, T; H^1(0, L))$, such that

$$\begin{array}{ll} L_1(u,v,z) = \rho_1 u_{tt} - k u_{xx} & \in L^2(-T,T;L^2(0,L)) \\ L_2(u,v,z) = \rho_2 v_{tt} - b v_{xx} & \in L^2(-T,T;L^2(0,L)) \\ L_3(u,v,z) = \rho_1 z_{tt} - k_0 z_{xx} & \in L^2(-T,T;L^2(0,L)). \end{array} \tag{3.1}$$

be

$$\begin{array}{l} L_{1,1}(u,v,z) = L_1(u,v,z) - k(v+lz)_x - k_0 l[z_x - lu] \\ L_{1,2}(u,v,z) = L_2(u,v,z) + k(u_x + v + lz) \\ L_{1,3}(u,v,z) = L_3(u,v,z) + k_0 l u_x + kl(u_x + v + lz); \end{array} \tag{3.2}$$

we have to

$$L_{1,1}(u,v,z), L_{1,2}(u,v,z), L_{1,3}(u,v,z) \in L^2(-T,T,L^2(0,L)).$$

We consider the functions $u, v, z \in L^2(-T, T; H^1(0, L))$, such that
$L_1(u,v,z), L_2(u,v,z), L_3(u,v,z) \in L^2(-T,T;L^2(0,L))$,
$u(0,t) = u(L,t) = v(0,t) = v(L,t) = z(0,t) = z(L,t) = 0 \ \ in \ \ (-T,T)$
$u(x,-T) = u(x,T) = v(x,-T) = v(x,T) = z(x,-T) = z(x,T) = 0 \ \ in \ \ (0,L)$
$u_t(x,-T) = u_t(x,T) = v_t(x,-T) = v_t(x,T) = z_t(x,-T) = z_t(x,T) = 0 \ \ in \ \ (0,L)$.
Let us now define, for $x_0 < 0$

$$\phi(x,t) = |x - x_0|^2 - \beta t^2 + M_0 \tag{3.3}$$

where $\beta > 0$ will be chosen later and M_0 is chosen in such that a way that

$$\forall (x,t) \in (0,L) \times (-T,T), \quad \phi(x,t) \geq 1, \tag{3.4}$$

to $\lambda > 0$ we now define

$$\varphi_\lambda(x,t) = e^{\lambda \phi(x,t)}. \tag{3.5}$$

Now how u, v, z are defined in $(0,L) \times (-T,T)$ be $s > 0$ define
$w_1 = e^{s\varphi_\lambda} u$, $w_2 = e^{s\varphi_\lambda} v$, $w_3 = e^{s\varphi_\lambda} z$.

Be (doing the calculations formally)

$$\frac{\partial}{\partial t}(e^{-s\varphi_\lambda}w_i) = e^{-s\varphi_\lambda}(\frac{\partial w_i}{\partial t} - s\lambda\frac{\partial \phi}{\partial t}\varphi_\lambda w_i)$$

$$\frac{\partial^2}{\partial t^2}(e^{-s\varphi_\lambda}w_i) = e^{-s\varphi_\lambda}(\frac{\partial^2 w_i}{\partial t^2} - s\lambda^2|\frac{\partial \phi}{\partial t}|^2\varphi_\lambda w_i - s\lambda\frac{\partial^2 \phi}{\partial t^2}\varphi_\lambda w_i$$
$$- 2s\lambda\varphi_\lambda\frac{\partial \phi}{\partial t}\frac{\partial w_i}{\partial t} + s^2\lambda^2\varphi_\lambda^2|\frac{\partial \phi}{\partial t}|^2 w_i)$$

$$\frac{\partial}{\partial x}(e^{-s\varphi_\lambda}w_i) = e^{-s\varphi_\lambda}(\frac{\partial w_i}{\partial x} - s\lambda\frac{\partial \phi}{\partial x}\varphi_\lambda w_i),$$

$$\frac{\partial^2}{\partial x^2}(e^{-s\varphi_\lambda}w_i) = e^{-s\varphi_\lambda}(\frac{\partial^2 w_i}{\partial x^2} - s\lambda^2|\frac{\partial \phi}{\partial x}|^2\varphi_\lambda w_i - s\lambda\frac{\partial^2 \phi}{\partial x^2}\varphi_\lambda w_i$$
$$- 2s\lambda\varphi_\lambda\frac{\partial \phi}{\partial x}\frac{\partial w_i}{\partial x} + s^2\lambda^2\varphi_\lambda^2|\frac{\partial \phi}{\partial x}|^2 w_i)$$

$i = 1, 2, 3.$

Therefore we can write

$$P_1(w_1, w_2, w_3) = P_1^1(w_1, w_2, w_3) + P_1^2(w_1, w_2, w_3) + R_1^0(w_1, w_2, w_3), \quad (3.6)$$

where (the operators ∇ and Δ will represent the first and second derivatives with respect to x.)

$$P_1^1(w_1, w_2, w_3) = \rho_1\frac{\partial^2 w_1}{\partial t^2} - k\Delta w_1 + s^2\lambda^2\varphi_\lambda^2(\rho_1|\frac{\partial \phi}{\partial t}|^2 - k|\nabla\phi|^2)w_1$$
$$P_1^2(w_1, w_2, w_3) = (M_1 - 1)s\lambda\varphi_\lambda(\rho_1\frac{\partial^2 \phi}{\partial t^2} - k\Delta\phi)w_1$$
$$-s\lambda^2\varphi_\lambda(\rho_1|\frac{\partial \phi}{\partial t}|^2 - k|\nabla\phi|^2)w_1 - 2s\lambda\varphi_\lambda(\rho_1\frac{\partial \phi}{\partial t}\frac{\partial w_1}{\partial t} - k\nabla\phi\nabla w_1),$$
$$R_1^0(w_1, w_2, w_3) = -M_1 s\lambda\varphi_\lambda(\rho_1\frac{\partial^2 \phi}{\partial t^2} - k\Delta\phi)w_1 \quad ,$$

$$P_2(w_1, w_2, w_3) = P_2^1(w_1, w_2, w_3) + P_2^2(w_1, w_2, w_3) + R_2^0(w_1, w_2, w_3), \quad (3.7)$$

where

$$P_2^1(w_1, w_2, w_3) = \rho_2\frac{\partial^2 w_2}{\partial t^2} - b\Delta w_2 + s^2\lambda^2\varphi_\lambda^2(\rho_2|\frac{\partial \phi}{\partial t}|^2 - b|\nabla\phi|^2)w_2,$$

$$P_2^2(w_1,w_2,w_3) = (M_2-1)s\lambda\varphi_\lambda(\rho_2\frac{\partial^2\phi}{\partial t^2} - b\Delta\phi)w_2$$
$$-s\lambda^2\varphi_\lambda(\rho_2|\frac{\partial\phi}{\partial t}|^2 - b|\nabla\phi|^2)w_2 - 2s\lambda\varphi_\lambda(\rho_2\frac{\partial\phi}{\partial t}\frac{\partial w_2}{\partial t} - b\nabla\phi\nabla w_2),$$

$$R_2^0(w_1,w_2,w_3) = -M_2 s\lambda\varphi_\lambda(\rho_2\frac{\partial^2\phi}{\partial t^2} - b\Delta\phi)w_2,$$

$$P_3(w_1,w_2,w_3) = P_3^1(w_1,w_2,w_3) + P_3^2(w_1,w_2,w_3) + R_3^0(w_1,w_2,w_3), \quad (3.8)$$

where

$$P_3^1(w_1,w_2,w_3) = \rho_1\frac{\partial^2 w_3}{\partial t^2} - k_0\Delta w_3 + s^2\lambda^2\varphi_\lambda^2(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k_0|\nabla\phi|^2)w_3$$
$$P_3^2(w_1,w_2,w_3) = (M_3-1)s\lambda\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k_0\Delta\phi)w_3$$
$$-s\lambda^2\varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k_0|\nabla\phi|^2)w_3 - 2s\lambda\varphi_\lambda(\rho_1\frac{\partial\phi}{\partial t}\frac{\partial w_2}{\partial t} - k_0\nabla\phi\nabla w_3),$$
$$R_3^0(w_1,w_2,w_3) = -M_3 s\lambda\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k_0\Delta\phi)w_3,$$

where M_1, M_2, M_3 are chosen in the estimation of Carleman.

Theorem 3.1. (**Carleman estimate**) *Be $x_0 < 0$ fixed and $0 < \beta < min\{\frac{15k}{16\rho_1}, \frac{15b}{16\rho_2}, \frac{15k_0}{16\rho_1}\}$ there exists $\lambda_0 > 0$ and $s_0 > 0$ and there exists a constant $C = C(s_0,\lambda_0,L,\beta,x_0)$ such that for each $u,v,z \in L^2(-T,T,L^2(0,L))$ with $u_x,v_x,z_x \in L^2(-T,T;L^2(0,L))$, $L_1(u,v,z), L_2(u,v,z), L_3(u,v,z) \in L^2(-T,T;L^2(0,L))$ and $u(0)=u(L)=v(0)=v(L)=z(0)=z(L)=0$, $u(T)=u(-T)=v(T)=v(-T)=z(T)=z(-T)=0$, $\frac{\partial u(T)}{\partial t} = \frac{\partial u(-T)}{\partial t} = \frac{\partial v(T)}{\partial t} = \frac{\partial v(-T)}{\partial t} = \frac{\partial z(T)}{\partial t} = \frac{\partial z(-T)}{\partial t} = 0$, implies that*

$$s\lambda\int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda(\rho_1|\frac{\partial u}{\partial t}|^2 + \rho_2|\frac{\partial v}{\partial t}|^2 + \rho_1|\frac{\partial z}{\partial t}|^2 + k|u_x|^2 + b|v_x|^2 +$$
$$k_0|z_x|^2 dxdt + s^3\lambda^3\int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda^3(|u|^2+|v|^2+|z|^2)dxdt$$
$$+\int_{-T}^{T}\int_0^L |P_1^1(w_1,w_2,w_3)|^2 dxdt + \int_{-T}^{T}\int_0^L |P_1^2(w_1,w_2,w_3)|^2 dxdt$$
$$+\int_{-T}^{T}\int_0^L |P_2^1(w_1,w_2,w_3)|^2 dxdt + \int_{-T}^{T}\int_0^L |P_2^2(w_1,w_2,w_3)|^2 dxdt$$

$$+\int_{-T}^{T}\int_{0}^{L}|P_3^1(w_1,w_2,w_3)|^2dxdt+\int_{-T}^{T}\int_{0}^{L}|P_3^2(w_1,w_2,w_3)|^2dxdt$$
$$\leq C\int_{-T}^{T}\int_{0}^{L}e^{2s\varphi_\lambda}|L_{1,1}(u,v,z)|^2dxdt+C\int_{-T}^{T}\int_{0}^{L}e^{2s\varphi_\lambda}|L_{1,2}(u,v,z)|^2dxdt$$
$$C\int_{-T}^{T}\int_{0}^{L}e^{2s\varphi_\lambda}|L_{1,3}(u,v,z)|^2dxdt+Cs\lambda\int_{-T}^{T}e^{2s\varphi_\lambda(L,t)}|u_x(L)|^2dt$$
$$+Cs\lambda\int_{-T}^{T}e^{2s\varphi_\lambda(L,t)}|v_x(L)|^2dt+Cs\lambda\int_{-T}^{T}e^{2s\varphi_\lambda(L,t)}|z_x(L)|^2dt.$$

for all $s>s_0$, *and* $\lambda>\lambda_0$.

Proof. It is sufficient to show the inequality of theorem with L_1, L_2, L_3, rather than $L_{1,1}, L_{1,2}, L_{1,3}$ since

$$|L_1(u,v,z)|^2\leq|L_{1,1}(u,v,z)|^2+C(|u|^2+|v|^2+|z|^2+|u_x|^2+|v_x|^2+|z_x|^2)$$

$$|L_2(u,v,z)|^2\leq|L_{1,2}(u,v,z)|^2+C(|u|^2+|v|^2+|z|^2+|u_x|^2+|v_x|^2+|z_x|^2)$$

$$|L_3(u,v,z)|^2\leq|L_{1,3}(u,v,z)|^2+C(|u|^2+|v|^2+|z|^2+|u_x|^2+|v_x|^2+|z_x|^2)$$

and the terms

$$\int_{0}^{T}\int_{0}^{L}e^{2s\varphi_\lambda}(|u|^2+|v|^2+|z|^2+|u_x|^2+|v_x|^2+|z_x|^2)\ dx\ dt,$$

can be absorbed by the terms

$$s\lambda\int_{0}^{T}\int_{0}^{L}e^{2s\varphi_\lambda}\varphi_\lambda(k|u_x|^2+b|v_x|^2+k_0|z_x|^2)\ dx\ dt$$
$$+s^3\lambda^3\int_{0}^{T}\int_{0}^{L}e^{2s\varphi_\lambda}\varphi_\lambda(|u|^2+|v|^2+|z|^2)\ dx\ dt$$

choosing s_0 large enough.

Now note that

$$P_1^1(w_1,w_2,w_3)+P_1^2(w_1,w_2,w_3)=[e^{s\varphi_\lambda}L_1(u,v,z)-R_1^0(w_1,w_2,w_3)]$$

$$P_2^1(w_1,w_2,w_3)+P_2^2(w_1,w_2,w_3)=[e^{s\varphi_\lambda}L_2(u,v,z)-R_2^0(w_1,w_2,w_3)]$$

$$P_3^1(w_1,w_2,w_3)+P_3^2(w_1,w_2,w_3)=[e^{s\varphi_\lambda}L_3(u,v,z)-R_3^0(w_1,w_2,w_3)].$$

Throughout or demonstration the operators ∇ and Δ will represent the first and second derivatives with respect to x respectively.

To

$$\begin{aligned}&\int_{-T}^{T}\int_0^L(|P_1^1(w_1,w_2,w_3)|^2+|P_1^2(w_1,w_2,w_3)|^2)dxdt\\&+2\int_{-T}^{T}\int_0^L P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt\\&=\int_{-T}^{T}\int_0^L|e^{s\varphi_\lambda}L_1(u,v,z)-R_1^0(w_1,w_2,w_3)|^2dxdt\\&\leq\int_{-T}^{T}\int_0^L 2e^{s\varphi_\lambda}|L_1(u,v,z)|^2\ dx\ dt\\&+\int_{-T}^{T}\int_0^L 2|R_0^1(w_1,w_2,w_3)|^2\ dx\ dt\end{aligned}\tag{3.9}$$

$$\begin{aligned}&\int_{-T}^{T}\int_0^L(|P_2^1(w_1,w_2,w_3)|^2+|P_2^2(w_1,w_2,w_3)|^2)dxdt\\&+2\int_{-T}^{T}\int_0^L P_2^1(w_1,w_2,w_3)P_2^2(w_1,w_2,w_3)dxdt\\&=\int_{-T}^{T}\int_0^L|e^{s\varphi_\lambda}L_2(u,v,z)-R_2^0(w_1,w_2,w_3)|^2dxdt\\&\leq\int_{-T}^{T}\int_0^L 2e^{s\varphi_\lambda}|L_2(u,v,z)|^2\ dx\ dt\\&+\int_{-T}^{T}\int_0^L 2|R_2^0(w_1,w_2,w_3)|^2\ dx\ dt\end{aligned}\tag{3.10}$$

$$\begin{aligned}&\int_{-T}^{T}\int_0^L(|P_3^1(w_1,w_2,w_3)|^2+|P_3^2(w_1,w_2,w_3)|^2)dxdt\\&+2\int_{-T}^{T}\int_0^L P_3^1(w_1,w_2,w_3)P_3^2(w_1,w_2,w_3)dxdt\\&=\int_{-T}^{T}\int_0^L|e^{s\varphi_\lambda}L_3(u,v,z)-R_3^0(w_1,w_2,w_3)|^2dxdt\\&\leq\int_{-T}^{T}\int_0^L 2e^{s\varphi_\lambda}|L_3(w_1,w_2,w_3)|^2\ dx\ dt\\&+\int_{-T}^{T}\int_0^L 2|R_3^0(w_1,w_2,w_3)|^2\ dx\ dt,\end{aligned}\tag{3.11}$$

to obtain Carleman's estimate we will use (3.9), (3.10) and (3.11), for this we must estimate

$$+\int_{-T}^{T}\int_{0}^{L} 2|R_1^0(w_1,w_2,w_3)|^2\ dx\ dt$$
$$+\int_{-T}^{T}\int_{0}^{L} 2|R_2^0(w_1,w_2,w_3)|^2\ dx\ dt$$
$$+\int_{-T}^{T}\int_{0}^{L} 2|R_3^0(w_1,w_2,w_3)|^2\ dx\ dt$$

and

$$\int_{-T}^{T}\int_{0}^{L} P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt$$
$$\int_{-T}^{T}\int_{0}^{L} P_2^1(w_1,w_2,w_3)P_2^2(w_1,w_2,w_3)dxdt$$
$$\int_{-T}^{T}\int_{0}^{L} P_3^1(w_1,w_2,w_3)P_3^2(w_1,w_2,w_3)dxdt.$$

We will first estimate the terms

$$\int_{-T}^{T}\int_{0}^{L} P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt$$
$$\int_{-T}^{T}\int_{0}^{L} P_2^1(w_1,w_2,w_3)P_2^2(w_1,w_2,w_3)dxdt$$
$$\int_{-T}^{T}\int_{0}^{L} P_3^1(w_1,w_2,w_3)P_3^2(w_1,w_2,w_3)dxdt,$$

We will call I_{ij}^1, $i,j=1,2,3$, the product of the terms of P_1^1 and P_1^2, i is the order of the terms in the sum in P_1^1 and j the order of the terms in the sum in P_1^2.

Doing the calculations,

$$I_{11}^1 = \rho_1(M_1-1)s\lambda\int_{-T}^{T}\int_{0}^{L} \frac{\partial^2 w_1}{\partial t^2}\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2}-k\Delta\phi)w_1dxdt$$
$$=\rho_1(1-M_1)s\lambda\int_{-T}^{T}\int_{0}^{L}|\frac{\partial w_1}{\partial t}|^2\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2}-k\Delta\phi)dxdt$$

$$
- \rho_1 \frac{(1-M_1)}{2} s\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda (\frac{\partial^2 \phi}{\partial t^2} + \lambda |\frac{\partial \phi}{\partial t}|^2)(\rho_1 \frac{\partial^2 \phi}{\partial t^2} - k\Delta\phi) dxdt,
$$

$$
I_{12}^1 = -\rho_1 s\lambda^2 \int_{-T}^{T}\int_0^L \frac{\partial^2 w_1}{\partial t^2}\varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) w_1 dxdt
$$

$$
= \rho_1 s\lambda^2 \int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) dxdt
$$

$$
- \rho_1 s\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda(\rho_1|\frac{\partial^2\phi}{\partial t^2}|^2) dxdt
$$

$$
- (2+\frac{1}{2})\rho_1 s\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda |\frac{\partial\phi}{\partial t}|^2 \rho_1 \frac{\partial^2\phi}{\partial t^2} dxdt
$$

$$
+ \frac{\rho_1 s\lambda^3}{2} \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda \frac{\partial^2\phi}{\partial t^2}(k|\nabla\phi|^2) dxdt
$$

$$
- \frac{\rho_1 s\lambda^4}{2} \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda |\frac{\partial\phi}{\partial t}|^2 (\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) dxdt,
$$

$$
I_{13}^1 = -2\rho_1 s\lambda \int_{-T}^{T}\int_0^L \frac{\partial^2 w_1}{\partial t^2}\varphi_\lambda(\rho_1 \frac{\partial\phi}{\partial t}\frac{\partial w_1}{\partial t} - k\nabla\phi\nabla w_1) dxdt
$$

$$
= \rho_1 s\lambda \int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda \rho_1 \frac{\partial^2\phi}{\partial t^2} dxdt
$$

$$
+ \rho_1 s\lambda^2 \int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda \rho_1 |\frac{\partial\phi}{\partial t}|^2 dxdt
$$

$$
+ \rho_1 s\lambda \int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda (k\Delta\phi + \lambda k|\nabla\phi|^2) dxdt
$$

$$
- 2\rho_1 s\lambda^2 \int_{-T}^{T}\int_0^L \frac{\partial w_1}{\partial t}\varphi_\lambda \frac{\partial\phi}{\partial t}(k\nabla\phi\nabla w_1) dxdt,
$$

$$
I_{21}^1 = -k(M_1-1)s\lambda \int_{-T}^{T}\int_0^L \Delta w_1 \varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi) w_1 dxdt
$$

$$
= k(M_1-1)s\lambda \int_{-T}^{T}\int_0^L |\nabla w_1|^2 \varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi) dxdt
$$

$$
- k\frac{(M_1-1)}{2}s\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda(\Delta\phi + \lambda|\nabla\phi|^2)(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi) dxdt,
$$

$$
I_{22}^1 = ks\lambda^2 \int_{T}^{-T}\int_0^L \Delta w_1 \varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) w_1 dxdt
$$

$$
= -ks\lambda^2 \int_{T}^{-T}\int_0^L |\nabla w_1|^2 \varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) dxdt
$$

$$+\frac{ks\lambda^3}{2}\int_T^{-T}\int_0^L |w_1|^2\varphi_\lambda(\lambda|\nabla\phi|^2+\Delta\phi)(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)dxdt$$
$$-2ks\lambda^3\int_T^{-T}\int_0^L |w_1|^2\varphi_\lambda|\nabla\phi|^2k\Delta\phi dxdt$$
$$-ks\lambda^2\int_T^{-T}\int_0^L |w_1|^2\varphi_\lambda k|\Delta\phi|^2dxdt,$$

$$I_{23}^1=2ks\lambda\int_{-T}^T\int_0^L \Delta w_1\varphi_\lambda(\rho_1\frac{\partial\phi}{\partial t}\frac{\partial w_1}{\partial t}-k\nabla\phi\nabla w_1)dxdt$$
$$=ks\lambda\int_{-T}^T\int_0^L |\nabla w_1|^2\varphi_\lambda(k\Delta\phi)dxdt+ks\lambda^2\int_{-T}^T\int_0^L |\nabla w_1|^2\varphi_\lambda(k|\nabla\phi|^2)dxdt$$
$$-2ks\lambda^2\int_{-T}^T\int_0^L \nabla w_1\nabla\phi\varphi_\lambda(\rho_1\frac{\partial\phi}{\partial t}\frac{\partial w_1}{\partial t})dxdt$$
$$+ks\lambda^2\int_{-T}^T\int_0^L |\nabla w_1|^2\varphi_\lambda\rho_1|\frac{\partial\phi}{\partial t}|^2dxdt$$
$$+ks\lambda\int_{-T}^T\int_0^L |\nabla w_1|^2\varphi_\lambda\rho_1\frac{\partial^2\phi}{\partial t^2}dxdt$$
$$-ks\lambda\int_{-T}^T |\nabla w_1|^2\varphi_\lambda(k\nabla\phi)\;|_0^L\;dt.$$

$$I_{31}^1=$$
$$s^3\lambda^3(M_1-1)\int_{-T}^T\int_0^L \varphi_\lambda^2(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)w_1(\rho_1\frac{\partial^2\phi}{\partial t^2}-k\Delta\phi)w_1\varphi_\lambda dxdt$$
$$=(M_1-1)s^3\lambda^3\int_{-T}^T\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)(\rho_1\frac{\partial^2\phi}{\partial t^2}-k\Delta\phi)dxdt,$$

$$I_{32}^1=-s^3\lambda^4\int_{-T}^T\int_0^L \varphi_\lambda^2(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)w_1(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)\varphi_\lambda w_1dxdt$$
$$=-s^3\lambda^4\int_{-T}^T\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)^2dxdt,$$

$$I_{33}^1=$$
$$-2s^3\lambda^3\int_{-T}^T\int_0^L \varphi_\lambda^2(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)w_1\varphi_\lambda(\rho_1\frac{\partial\phi}{\partial t}\frac{\partial w_1}{\partial t}-k\nabla\phi\nabla w_1)dxdt$$
$$=s^3\lambda^3\int_{-T}^T\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\frac{\partial^2\phi}{\partial t^2}-k\Delta\phi)(\rho_1|\frac{\partial\phi}{\partial t}|^2-k|\nabla\phi|^2)dxdt$$
$$+2s^3\lambda^3\int_{-T}^T\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\frac{\partial^2\phi}{\partial t^2}\rho_1|\frac{\partial\phi}{\partial t}|^2+k|\nabla\phi|^2k\Delta\phi)dxdt$$

$$+ 3s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)^2 dxdt,$$

Of the previous calculations we have

$$\int_{-T}^{T}\int_0^L P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt =$$
$$\rho_1(1-M_1)s\lambda\int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt$$
$$- \rho_1\frac{(1-M_1)}{2}s\lambda^2\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda(\frac{\partial^2\phi}{\partial t^2} + \lambda|\frac{\partial\phi}{\partial t}|^2)(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt$$
$$+ \rho_1 s\lambda^2\int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)dxdt$$
$$- \rho_1 s\lambda^2\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda(\rho_1|\frac{\partial^2\phi}{\partial t^2}|^2)dxdt$$
$$- (2+\frac{1}{2})\rho_1 s\lambda^3\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda|\frac{\partial\phi}{\partial t}|^2\rho_1\frac{\partial^2\phi}{\partial t^2}dxdt$$
$$+ \frac{\rho_1 s\lambda^3}{2}\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda\frac{\partial^2\phi}{\partial t^2}(k|\nabla\phi|^2)dxdt$$
$$- \frac{\rho_1 s\lambda^4}{2}\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda|\frac{\partial\phi}{\partial t}|^2(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)dxdt$$
$$+ \rho_1 s\lambda\int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda\rho_1\frac{\partial^2\phi}{\partial t^2}dxdt$$
$$+ \rho_1 s\lambda^2\int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda\rho_1|\frac{\partial\phi}{\partial t}|^2dxdt$$
$$+ \rho_1 s\lambda\int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda(k\Delta\phi + \lambda k|\nabla\phi|^2)dxdt$$
$$- 2\rho_1 s\lambda^2\int_{-T}^{T}\int_0^L \frac{\partial w_1}{\partial t}\varphi_\lambda\frac{\partial\phi}{\partial t}(k\nabla\phi\nabla w_1)dxdt$$
$$+ k(M_1-1)s\lambda\int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt$$
$$- k\frac{(M_1-1)}{2}s\lambda^2\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda(\Delta\phi + \lambda|\nabla\phi|^2)(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt$$
$$- ks\lambda^2\int_{T}^{-T}\int_0^L |\nabla w_1|^2\varphi_\lambda(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)dxdt$$
$$+ \frac{ks\lambda^3}{2}\int_{T}^{-T}\int_0^L |w_1|^2\varphi_\lambda(\lambda|\nabla\phi|^2 + \Delta\phi)(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)dxdt$$

$$
\begin{aligned}
&- 2ks\lambda^3 \int_{T}^{-T}\int_0^L |w_1|^2\varphi_\lambda|\nabla\phi|^2 k\Delta\phi dxdt \\
&- ks\lambda^2 \int_{T}^{-T}\int_0^L |w_1|^2\varphi_\lambda k|\Delta\phi|^2 dxdt \\
&+ ks\lambda \int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda(k\Delta\phi)dxdt + ks\lambda^2 \int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda(k|\nabla\phi|^2)dxdt \\
&- 2ks\lambda^2 \int_{-T}^{T}\int_0^L \nabla w_1\nabla\phi\varphi_\lambda(\rho_1\frac{\partial\phi}{\partial t}\frac{\partial w_1}{\partial t})dxdt \\
&+ ks\lambda^2 \int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda\rho_1|\frac{\partial\phi}{\partial t}|^2 dxdt \\
&+ ks\lambda \int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda\rho_1\frac{\partial^2\phi}{\partial t^2}dxdt \\
&- ks\lambda \int_{-T}^{T} |\nabla w_1|^2\varphi_\lambda(k\nabla\phi) \mid_0^L dt \\
&+ (M_1-1)s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt \\
&- s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)^2 dxdt \\
&+ s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)dxdt \\
&+ 2s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\frac{\partial^2\phi}{\partial t^2}\rho_1|\frac{\partial\phi}{\partial t}|^2 + k|\nabla\phi|^2 k\Delta\phi)dxdt \\
&+ 3s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1|\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)^2 dxdt,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-T}^{T}\int_0^L P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt = \\
&2\rho_1 s\lambda \int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda\rho_1\frac{\partial^2\phi}{\partial t^2}dxdt \\
&- \rho_1 M_1 s\lambda \int_{-T}^{T}\int_0^L |\frac{\partial w_1}{\partial t}|^2\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt \\
&+2s\lambda^2 \int_{-T}^{T}\int_0^L \varphi_\lambda[\rho_1^2|\frac{\partial w_1}{\partial t}|^2|\frac{\partial\phi}{\partial t}|^2 - 2\rho_1\frac{\partial w_1}{\partial t}\frac{\partial\phi}{\partial t}k\nabla w_1\nabla\phi + k^2|\nabla w_1|^2|\nabla\phi|^2 dxdt \\
&+ 2s\lambda k \int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda(k\Delta\phi)dxdt \\
&+ kM_1 s\lambda \int_{-T}^{T}\int_0^L |\nabla w_1|^2\varphi_\lambda(\rho_1\frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)dxdt
\end{aligned}
$$

$$
\begin{aligned}
&- ks\lambda \int_{-T}^{T} |\nabla w_1|^2 \varphi_\lambda (k\nabla\phi) \,|_0^L \, dt \\
&+ 2s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1 |\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2)^2 dxdt \\
&+ 2s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1 \frac{\partial^2\phi}{\partial t^2}\rho_1 |\frac{\partial\phi}{\partial t}|^2 + k|\nabla\phi|^2 k\Delta\phi) dxdt \\
&+ M_1 s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1 \frac{\partial^2\phi}{\partial t^2} - k\Delta\phi)(\rho_1 |\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) dxdt + X_1,
\end{aligned}
$$

where

$$
\begin{aligned}
&X_1 = \\
&- \rho_1 (\frac{1-M_1}{2}) s\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda (\frac{\partial^2\phi}{\partial t^2} + \lambda |\frac{\partial\phi}{\partial t}|^2)(\rho_1 \frac{\partial^2\phi}{\partial t^2} - k\Delta\phi) dxdt \\
&- \rho_1 s\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda (\rho_1 |\frac{\partial^2\phi}{\partial t^2}|^2) dxdt \\
&- \frac{5}{2} s\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda \rho_1 |\frac{\partial\phi}{\partial t}|^2 \frac{\partial^2\phi}{\partial t^2} dxdt \\
&+ \frac{\rho_1 s\lambda^3}{2} \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda \frac{\partial^2\phi}{\partial t^2} (k|\nabla\phi|^2) dxdt \\
&- \frac{\rho_1 s\lambda^4}{2} \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda |\frac{\partial\phi}{\partial t}| (\rho_1 |\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) dxdt \\
&- \frac{k(M_1-1)}{2} s\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda (\Delta\phi + \lambda|\nabla\phi|^2)(\rho_1 \frac{\partial^2\phi}{\partial t^2} - k\Delta\phi) dxdt \\
&+ \frac{ks\lambda^3}{2} \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda (\Delta\phi + \lambda|\nabla\phi|^2)(\rho_1 |\frac{\partial\phi}{\partial t}|^2 - k|\nabla\phi|^2) dxdt \\
&- ks\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda |\nabla\phi|^2 k\Delta\phi dxdt \\
&- ks\lambda^2 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda k|\nabla\phi|^2 dxdt.
\end{aligned}
$$

The terms grouped in X_1 are the terms that can be estimated by $s\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 \, dx \, dt$, and we will do so later.

Note that

$$2s\lambda^2\int_{-T}^{T}\int_0^L \varphi_\lambda[\rho_1^2|\frac{\partial w_1}{\partial t}|^2|\frac{\partial \phi}{\partial t}|^2 - 2\rho_1\frac{\partial w_1}{\partial t}\frac{\partial \phi}{\partial t}k\nabla w_1\nabla\phi + +k^2|\nabla w_1|^2|\nabla\phi|^2]dxdt$$
$$\geq 0,$$

and, from (3.3)

$$\phi(x,t) = |x-x_0|^2 - \beta t^2 + M_0$$
$$\frac{\partial \phi}{\partial t}(x,t) = -2\beta t$$
$$\frac{\partial^2 \phi}{\partial t^2}(x,t) = -2\beta$$
$$\nabla\phi(x,t) = 2(x-x_0)$$
$$\Delta\phi(x,t) = 2$$

to

$$\int_{-T}^{T}\int_0^L P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt$$
$$\geq -4\rho_1\beta s\lambda\int_{-T}^{T}\int_0^L \rho_1|\frac{\partial w_1}{\partial t}|^2\varphi_\lambda dxdt$$
$$+ 2M_1(\rho_1\beta + k)s\lambda\int_{-T}^{T}\int_0^L \rho_1|\frac{\partial w_1}{\partial t}|^2\varphi_\lambda dxdt$$
$$+ 4ks\lambda\int_{-T}^{T}\int_0^L k|\nabla w_1|^2\varphi_\lambda dxdt$$
$$- 2M_1 s\lambda(\rho_1\beta + k)\int_{-T}^{T}\int_0^L k|\nabla w_1|^2\varphi_\lambda dxdt$$
$$- k^2(L-x_0)s\lambda\int_{-T}^{T}|\nabla w_1|^2\varphi_\lambda(L)dt$$
$$+ 32s^3\lambda^4\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)^2dxdt$$
$$+ 16s^3\lambda^3\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(-\rho_1^2\beta^3t^2 + k^2(x-x_0)^2)dxdt$$
$$- 8M_1s^3\lambda^3\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta + k)(\rho_1\beta^2t^2 - k(x-x_0)^2)dxdt + X_1.$$

Using the definition of λ and φ_λ and $|a+b| \leq |a| + |b|$ we have to: $|X_1| \leq Cs\lambda^3\int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3 dxdt$ for some $C > 0$.

Doing $-4\rho_1\beta + 2M_1(\rho_1\beta + k) > 0$, then $M_1 > \dfrac{2\beta\rho_1}{(\rho_1\beta + k)}$ and from $4k - 2M_1(\rho_1\beta + k) > 0$, then $M_1 < \dfrac{2k}{(\rho_1\beta + k)}$.

Thus

$$\frac{2\beta\rho_1}{(\rho_1\beta + k)} < M_1 < \frac{2k}{(\rho_1\beta + k)}$$

if and only if, $\beta < \dfrac{k}{\rho_1}$.

Thus

$$\begin{aligned}
&-4\rho_1\beta s\lambda \int_{-T}^{T}\int_0^L \rho_1 |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda dxdt \\
&+ 2M_1(\rho_1\beta + k)s\lambda \int_{-T}^{T}\int_0^L \rho_1 |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda dxdt \\
&+ 4ks\lambda \int_{-T}^{T}\int_0^L k|\nabla w_1|^2 \varphi_\lambda dxdt \\
&- 2M_1 s\lambda(\rho_1\beta + k) \int_{-T}^{T}\int_0^L k|\nabla w_1|^2 \varphi_\lambda dxdt \\
&\geq cs\lambda \int_{-T}^{T}\int_0^L \rho_1 |\frac{\partial w_1}{\partial t}|^2 \varphi_\lambda dxdt + cs\lambda \int_{-T}^{T}\int_0^L k|\nabla w_1|^2 \varphi_\lambda dxdt,
\end{aligned}$$

for some $c > 0$.

Taking $\beta < \frac{15k}{16\rho_1}$ we have that $(16k) = (k + 15k) > (k + 16\rho_1\beta)$, we can estimate

$$\begin{aligned}
&32s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1\beta^2 t^2 - k(x - x_0)^2)^2 dxdt \\
&+ 16s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (-\rho_1^2\beta^3 t^2 - k^2(x - x_0)^2) dxdt \\
&- 8M_1 s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1\beta + k)(\rho_1\beta^2 t^2 - k(x - x_0)^2) dxdt \\
&= 32s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1\beta^2 t^2 - k(x - x_0)^2)^2 dxdt \\
&- 16\rho_1\beta s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2 \varphi_\lambda^3 (\rho_1\beta^2 t^2) dxdt
\end{aligned}$$

$$
\begin{aligned}
&+ 16ks^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(k(x-x_0)^2)dxdt \\
&- 8M_1\rho_1\beta s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)dxdt \\
&- 8M_1ks^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)dxdt \\
&= 32s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)^2dxdt \\
&- (16+8M_1)\rho_1\beta s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2)dxdt \\
&+ (16k+8M_1\rho_1\beta)s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(k(x-x_0)^2)dxdt \\
&- 8M_1ks^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)dxdt \\
&\geq 32s^3\lambda^4 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)^2dxdt \\
&- (16+8M_1)\rho_1\beta s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)dxdt \\
&- 8M_1ks^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3(\rho_1\beta^2t^2 - k(x-x_0)^2)dxdt \\
&+ ks^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3 k(x-x_0)^2dxdt \\
&= s^3\lambda^3 \int_{-T}^{T}\int_0^L |w_1|^2\varphi_\lambda^3 F(X)dxdt,
\end{aligned}
$$

with

$F(X) = 32\lambda X^2 - [(16+8M_1)\rho_1\beta + 8kM_1]X + k^2(x-x_0)^2$

and

$X = (\rho_1\beta^2t^2 - k(x-x_0)^2)$.

There is a sufficiently large $\lambda_1 > 0$ such that

$4\lambda X^2 - [(2+M_1)\rho_1\beta + kM_1]X \geq 0 \; \forall\lambda > \lambda_1$,

and as $k(x-x_0)^2 > C > 0$ for fixed $x_0 < 0$ and some $C > 0$,

soon

$F(X) > C$ for $\lambda > \lambda_1$.

Thus

$$s^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_1|^2\varphi_\lambda^3F(X)dxdt$$
$$\geq Cs^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_1|^2\varphi_\lambda^3dxdt.$$

Therefore

$$\int_{-T}^{T}\int_{0}^{L}P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt$$
$$\geq Cs\lambda\int_{-T}^{T}\int_{0}^{L}\rho_1|\frac{\partial w_1}{\partial t}|^2\varphi_\lambda dxdt+Cs\lambda\int_{-T}^{T}\int_{0}^{L}k|\nabla w_1|^2\varphi_\lambda dxdt$$
$$+Cs^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_1|^2\varphi_\lambda^3dxdt-k^2(L-x_0)s\lambda\int_{-T}^{T}|\nabla w_1|^2\varphi_\lambda(L)dt+X_1$$

Like $|X_1|\leq Cs\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_1|^2\varphi_\lambda^3dxdt$ for some $C>0$, your terms can be absorved for s large enough, and the term $\int_{-T}^{T}\int_{0}^{L}2|R_0^1(w_1,w_2,w_3)|^2\ dx\ dt$ can be absorved by $s^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_1|^2\varphi_\lambda^3dxdt$ for s *and* λ large enough, so there is a constant $C>0$ such that

$$\int_{-T}^{T}\int_{0}^{L}P_1^1(w_1,w_2,w_3)P_1^2(w_1,w_2,w_3)dxdt$$
$$-\int_{-T}^{T}\int_{0}^{L}2|R_1^0(w_1,w_2,w_3)|^2\ dx\ dt$$
$$\geq Cs\lambda\int_{-T}^{T}\int_{0}^{L}\rho_1|\frac{\partial w_1}{\partial t}|^2\varphi_\lambda dxdt+Cs\lambda\int_{-T}^{T}\int_{0}^{L}k|\nabla w_1|^2\varphi_\lambda dxdt$$
$$+Cs^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_1|^2\varphi_\lambda^3dxdt-(L-x_0)s\lambda\int_{-T}^{T}|\nabla w_1|^2\varphi_\lambda(L)dxdt.$$

For w_2 e w_3 is done in the same way as for w_1, since the equations similar and so we get

$$\int_{-T}^{T}\int_{0}^{L}P_2^1(w_1,w_2,w_3)P_2^2(w_1,w_2,w_3)dxdt$$
$$-\int_{-T}^{T}\int_{0}^{L}2|R_2^0(w_1,w_2,w_3)|^2\ dx\ dt$$

$$\geq Cs\lambda\int_{-T}^{T}\int_{0}^{L}\rho_2|\frac{\partial w_2}{\partial t}|^2\varphi_\lambda dxdt + Cs\lambda\int_{-T}^{T}\int_{0}^{L} b|\nabla w_2|^2\varphi_\lambda dxdt$$
$$+ Cs^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_2|^2\varphi_\lambda^3 dxdt - (L-x_0)s\lambda\int_{-T}^{T}|\nabla w_2|^2\varphi_\lambda(L)dxdt$$
$$\int_{-T}^{T}\int_{0}^{L} P_3^1(w_1,w_2,w_3)P_3^2(w_1,w_2,w_3)dxdt$$
$$-\int_{-T}^{T}\int_{0}^{L} 2|R_0^3(w_1,w_2,w_3)|^2\ dx\ dt$$
$$\geq Cs\lambda\int_{-T}^{T}\int_{0}^{L}\rho_1|\frac{\partial w_3}{\partial t}|^2\varphi_\lambda dxdt + Cs\lambda\int_{-T}^{T}\int_{0}^{L} k_0|\nabla w_3|^2\varphi_\lambda dxdt$$
$$+ Cs^3\lambda^3\int_{-T}^{T}\int_{0}^{L}|w_3|^2\varphi_\lambda^3 dxdt - (L-x_0)s\lambda\int_{-T}^{T}|\nabla w_3|^2\varphi_\lambda(L)dxdt.$$

Making the change $u = e^{-s\varphi}w_1$, $v = e^{-s\varphi}w_2$, $z = e^{-s\varphi}w_3$ we get

$$s\lambda\int_{-T}^{T}\int_{0}^{L} e^{2s\varphi_\lambda}\varphi_\lambda(\rho_1|\frac{\partial u}{\partial t}|^2 + \rho_2|\frac{\partial v}{\partial t}|^2 + \rho_1|\frac{\partial z}{\partial t}|^2$$
$$+ k|u_x|^2 + b|v_x|^2 + k_0|z_x|^2)dxdt$$
$$+s^3\lambda^3\int_{-T}^{T}\int_{0}^{L} e^{2s\varphi_\lambda}\varphi_\lambda^3(|u|^2+|v|^2+|z|^2)dxdt+\int_{-T}^{T}\int_{0}^{L}|P_1^1(w_1,w_2,w_3)|^2dxdt$$
$$+\int_{-T}^{T}\int_{0}^{L}|P_1^2(w_1,w_2,w_3)|^2dxdt + \int_{-T}^{T}\int_{0}^{L}|P_2^1(w_1,w_2,w_3)|^2dxdt$$
$$+\int_{-T}^{T}\int_{0}^{L}|P_2^2(w_1,w_2,w_3)|^2dxdt + \int_{-T}^{T}\int_{0}^{L}|P_3^1(w_1,w_2,w_3)|^2dxdt$$
$$+\int_{-T}^{T}\int_{0}^{L}|P_3^2(w_1,w_2,w_3)|^2dxdt$$
$$\leq C\int_{-T}^{T}\int_{0}^{L} e^{2s\varphi_\lambda}|L_{1,1}(u,v,z)|^2dxdt + C\int_{-T}^{T}\int_{0}^{L} e^{2s\varphi_\lambda}|L_{1,2}(u,v,z)|^2dxdt$$
$$+ C\int_{-T}^{T}\int_{0}^{L} e^{2s\varphi_\lambda}|L_{1,3}(u,v,z)|^2dxdt + Cs\lambda\int_{-T}^{T} e^{2s\varphi_\lambda(L,t)}|u_x(L)|^2dt$$
$$+ Cs\lambda\int_{-T}^{T} e^{2s\varphi_\lambda(L,t)}|v_x(L)|^2dt + Cs\lambda\int_{-T}^{T} e^{2s\varphi_\lambda(L,t)}|z_x(L)|^2dt.$$

□

Note: Note that by the Schwarz inequality

$$s\lambda\int_{-T}^{T}\int_{0}^{L} e^{2s\varphi_\lambda}\varphi_\lambda(\rho_1|\frac{\partial u}{\partial t}|^2 + \rho_2|\frac{\partial v}{\partial t}|^2 + \rho_1|\frac{\partial z}{\partial t}|^2$$
$$+ k|u_x|^2 + b|v_x|^2 + k_0|z_x|^2 + k|u_x + v + lz|^2 + k_0|z_x - lu|^2)dxdt$$

$$+ s^3\lambda^3 \int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda^3(|u|^2+|v|^2+|z|^2)dxdt$$
$$\leq Cs\lambda \int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda(\rho_1|\frac{\partial u}{\partial t}|^2+\rho_2|\frac{\partial v}{\partial t}|^2+\rho_1|\frac{\partial z}{\partial t}|^2$$
$$+ k|u_x|^2+b|v_x|^2+k_0|z_x|^2)dxdt$$
$$+ Cs^3\lambda^3 \int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda^3(|u|^2+|v|^2+|z|^2)dxdt,$$

so the previous theorem can be stated as follows.

In the following theorem, consider same observations those given for theorem 3.1.

Theorem 3.2. *(***Carleman estimate***) Be $x_0 < 0$ fixed and $0 < \beta < min\{\frac{15k}{16\rho_1}, \frac{15b}{16\rho_2}, \frac{15k_0}{16\rho_1}\}$ there exists $\lambda_0 > 0 and s_0 > 0$ and there exists a constant $C = C(s_0, \lambda_0, (0, L), \beta, x_0)$ such that for each $u, v, z \in L^2(-T, T, L^2(0, L))$ with $u_x, v_x, z_x \in L^2(-T, T; L^2(0, L))$, $L_1(u, v, z), L_2(u, v, z), L_3(u, v, z) \in L^2(-T, T; L^2(0, L))$ and $u(0) = u(L) = v(0) = v(L) = z(0) = z(L) = 0$, $u(T) = u(-T) = v(T) = v(-T) = z(T) = z(-T) = 0$, $\frac{\partial u(T)}{\partial t} = \frac{\partial u(-T)}{\partial t} = \frac{\partial v(T)}{\partial t} = \frac{\partial v(-T)}{\partial t} = \frac{\partial z(T)}{\partial t} = \frac{\partial z(-T)}{\partial t} = 0$, implies that*

$$s\lambda \int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda(\rho_1|\frac{\partial u}{\partial t}|^2+\rho_2|\frac{\partial v}{\partial t}|^2+\rho_1|\frac{\partial z}{\partial t}|^2$$
$$+ k|u_x|^2+b|v_x|^2+k_0|z_x|^2+k|u_x+v+lz|^2+k_0|z_x-lu|^2)dxdt$$
$$+s^3\lambda^3 \int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}\varphi_\lambda^3(|u|^2+|v|^2+|z|^2)dxdt + \int_{-T}^{T}\int_0^L |P_1^1(w_1,w_2,w_3)|^2dxdt$$
$$+ \int_{-T}^{T}\int_0^L |P_1^2(w_1,w_2,w_3)|^2dxdt + \int_{-T}^{T}\int_0^L |P_2^1(w_1,w_2,w_3)|^2dxdt$$
$$+ \int_{-T}^{T}\int_0^L |P_2^2(w_1,w_2,w_3)|^2dxdt + \int_{-T}^{T}\int_0^L |P_3^1(w_1,w_2,w_3)|^2dxdt$$
$$+ \int_{-T}^{T}\int_0^L |P_3^2(w_1,w_2,w_3)|^2dxdt$$
$$\leq C\int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}|L_{1,1}(u,v,z)|^2dxdt + C\int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}|L_{1,2}(u,v,z)|^2dxdt$$
$$+ C\int_{-T}^{T}\int_0^L e^{2s\varphi_\lambda}|L_{1,3}(u,v,z)|^2dxdt + Cs\lambda\int_{-T}^{T} e^{2s\varphi_\lambda(L,t)}|u_x(L)|^2dt$$

$$+ Cs\lambda \int_{-T}^{T} e^{2s\varphi_\lambda(L,t)} |v_x(L)|^2 dt + Cs\lambda \int_{-T}^{T} e^{2s\varphi_\lambda(L,t)} |z_x(L)|^2 dt.$$

for all $s > s_0$, *and* $\lambda > \lambda_0$.

4 Inequality of observability

We must remember the problem of exact controllability at the frontier for the Bresse system.
For any couple of initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1) \in L^2(0, L) \times H^{-1}(0, L)$ we consider the Bresse system

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) = 0, \quad in \quad (0, L) \times (0, T) \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = 0, \quad in \quad (0, L) \times (0, T) \\ \rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0, \quad in \quad (0, L) \times (0, T) \\ \varphi(0, t) = \psi(0, t) = \omega(0, t) = 0, \quad t \in (0, T) \\ \varphi(L, t) = g_1(t), \quad \phi(L, t) = g_2(t), \quad \omega(L, t) = g_3(t), \quad t \in (0, T) \\ \varphi(., 0) = \varphi_0, \quad \varphi_t(., 0) = \varphi_1, \quad in \quad (0, L) \\ \psi(., 0) = \psi_0, \quad \psi_t(., 0) = \psi_1, \quad in \quad (0, L) \\ \omega(., 0) = w_0, \quad \omega_t(., 0) = \omega_1, \quad in \quad (0, L) \end{cases} \tag{4.1}$$

where g_1, g_2, g_3 are controls taken in $L^2(0, T)$.

It has already been shown that problem (4.1) has an unique solution $u \in C([0, T]; L^2(0, L))$ with $u_t \in C([0, T]; H^{-1}(0, L))$.

The problem of exact controllability is then to find controls g_1, g_2, g_3 such that at time T we have

$$\varphi(x, T) = \varphi_t(x, T) = \psi(x, T) = \psi_t(x, T) = \omega(x, T) = \omega_t(x, T).$$

By Lions [17] is equivalent to proving an observability inequality for the adjoint problem

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v + lz)_x - k_0 l[z_x - lu] = 0, \\ \rho_2 v_{tt} - b v_{xx} + k(u_x + v + lz) = 0, \\ \rho_1 z_{tt} - k_0[z_x - lu]_x + kl(u_x + v + lz) = 0, \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = z(0,t) = z(L,t) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x) \\ z(x,0) = z_0(x), z_t(x,0) = z_1(x), \end{cases} \tag{4.2}$$

where $u_0, v_0, z_0 \in H_0^1(0,L)$ e $u_1, v_1, z_1 \in L^2(0,L)$.

We want to show that we have an inequality of the following type

$$\exists C_0 > 0, \quad such \quad that \quad E_0 \leq C_0 \int_0^T (|u_x(L)|^2 + |v_x(L)|^2 + |z_x(L)|^2)dt \tag{4.3}$$

where

$E(t) = \frac{1}{2}\int_0^L \rho_1|u_t(t)|^2 + \rho_2|v_t(t)|^2 + \rho_1|z_t(t)|^2 + k_0|z_x(t) - lu(t)|^2 + b|v_x(t)|^2 + k|u_x(t) + v(t) + lz(t)|^2 dx = E(0).$

and

$E_0 = E(0)$ is the initial energy namely

$\frac{1}{2}\int_0^L \rho_1|u_1|^2 + \rho_2|v_1|^2 + \rho_1|z_1|^2 + k_0|z_{0x} - lu_0|^2 + b|v_{0x}|^2 + k|u_{0x} + v_0 + lz_0|^2 dx.$

In the following theorem we will also use the functional

$\widetilde{E}(t) = \frac{1}{2}\int_0^L \rho_1|u_t(t)|^2 + \rho_2|v_t(t)|^2 + \rho_1|z_t(t)|^2 + k_0|z_x(t)|^2 + b|v_x(t)|^2 + k|u_x(t)|^2 dx,$

which is equivalent to the functional

$E(t) = \frac{1}{2}\int_0^L \rho_1|u_t(t)|^2 + \rho_2|v_t(t)|^2 + \rho_1|z_t(t)|^2 + k_0|z_x(t) - lu(t)|^2 + b|v_x(t)|^2 + k|u_x(t) + v(t) + lz(t)|^2 dx,$

and

$\widetilde{E}(0) = \frac{1}{2}\int_0^L \rho_1|u_1|^2 + \rho_2|v_1|^2 + \rho_1|z_1|^2 + k_0|z_{0x}|^2 + b|v_{0x}|^2 + k|u_{0x}|^2 dx.$

Theorem 4.1. *We assume that*

$x_0 < 0,\ T > 2\sup_{x\in[0,L]} |x - x_0| = 2(L - x_0)$.

Then there exists a constant C_0 *such that for every* $u_0, v_0, z_0 \in H_0^1(0, L)$ *and* $u_1, v_1, z_1 \in L^2(0, L)$ *we have*

$$E_0 \leq C_0 \int_0^T (|u_x(L)|^2 + |v_x(L)|^2 + |z_x(L)|^2)dt \tag{4.4}$$

Proof. For $u_0, v_0, z_0 \in H_0^1(0, L)$ and $u_1, v_1, z_1 \in L^2(0, L)$ the system (4.2) as an unique solution

$u, v, z \in C([0, T]; H_0^1(0, L)) \cap C^1([0, T]; L^2(0, L))$.

and the energy functional is

$E(t) = \dfrac{1}{2}\displaystyle\int_0^L \rho_1|u_t(t)|^2 + \rho_2|v_t(t)|^2 + \rho_1|z_t(t)|^2 + k_0|z_x(t) - lu(t)|^2 + b|v_x(t)|^2 + k|u_x(t) + v(t) + lz(t)|^2 dx = E(0)$.

Be

$$E_M(t) = E(\frac{T}{2}) = E(t) = E(0). \tag{4.5}$$

Chosen

$$\phi_M(x, t) = |x - x_0|^2 - \beta(t - \frac{T}{2})^2 + M_0 \tag{4.6}$$

where β is chosen such that

$$\frac{4}{T^2} \sup_{x\in[0,L]} |x - x_0|^2 < \beta < min\{\frac{15k}{16\rho_1}, \frac{15b}{16\rho_2}, \frac{15k_0}{16\rho_1}\} \tag{4.7}$$

and M_0 is chosen such that

$$\forall (x, t) \in (0, L) \times (-T, T), \phi_M(x, t) \geq 1 \tag{4.8}$$

we have to

$$\phi_M(x, \frac{T}{2}) = |x - x_0|^2 + M_0 \geq \phi_M(x, t), \quad \forall t \in [0, T] \tag{4.9}$$

and

$\phi_M(x, \frac{T}{2}) > M_0$,

so by continuity of ϕ_M there exists $\eta > 0$ such that,

$$\forall [\frac{T}{2} - \eta, \frac{T}{2} + \eta], \quad \forall x \in (0, L) \quad \phi_M(x, t) \geq M_0. \tag{4.10}$$

Now how $\phi_M(x,0)=\phi_M(x,T)=|x-x_0|^2-\beta\frac{T^2}{4}+M_0$.

Therefore, with the choice of β and the hypothesis on T

$\dfrac{4}{T^2}\sup_{x\in[0,L]}|x-x_0|^2<\beta$ implies $\sup_{x\in[0,L]}|x-x_0|^2<\beta\dfrac{T^2}{4}$ then

$\sup_{x\in[0,L]}|x-x_0|^2-\beta\dfrac{T^2}{4}<0$

soon

$\phi_M(x,0)=\phi_M(x,T)<M_0$

and by the continuity of ϕ_M

$$\exists\delta>0,\forall t\in[0,\delta]\cup[T-\delta,T]\quad \phi_M(x,t)\leq M_0 \tag{4.11}$$

chosen η and δ such that $\eta+\delta<\frac{T}{2}$.

Let us now define θ_δ as a function in $C_0^\infty([0,T])$ such that

$\forall t\in[0,T]\quad 0\leq\theta_\delta\leq 1,\quad and\quad \forall t\in[\delta,T-\delta]\quad \theta_\delta(t)=1.$

We set

$$\begin{aligned} U(x,t)&=\theta_\delta u(x,t)\\ V(x,t)&=\theta_\delta v(x,t)\\ Z(x,t)&=\theta_\delta z(x,t)\end{aligned} \tag{4.12}$$

doing the calculations

$$\begin{aligned} &U_x(x,t)=\theta_\delta u_x(x,t),\quad U_{xx}(x,t)=\theta_\delta u_{xx}(x,t)\\ &U_t(x,t)=\theta_{\delta t}u(x,t)+\theta_\delta u_t(x,t)\\ &U_{tt}(x,t)=\theta_{\delta tt}u(x,t)+2\theta_{\delta t}u_t(x,t)+\theta_\delta u_{tt}(x,t)\end{aligned} \tag{4.13}$$

analogous to V and Z.

$$\begin{aligned} &L_{1,1}(U,V,Z)=\rho_1\theta_{\delta tt}u+2\rho_1\theta_{\delta t}u_t\\ &L_{1,2}(U,V,Z)=\rho_2\theta_{\delta tt}v+2\rho_1\theta_{\delta t}v_t\\ &L_{1,3}(U,V,Z)=\rho_1\theta_{\delta tt}z+2\rho_1\theta_{\delta t}z_t\\ &U(0)=U(L)=V(0)=V(L)=Z(0)=Z(L)=0\\ &U(.,0)=U(.,T)=V(.,0)=V(.,T)=Z(.,0)=Z(.,T)=0\\ &U_t(.,0)=U_t(.,T)=V_t(.,0)=V_t(.,T)=Z_t(.,0)=Z_t(.,T)=0\end{aligned} \tag{4.14}$$

it we denote for $\lambda>0$

$\varphi(x,t) = e^{\lambda\phi_M(x,t)}$

we can apply Carleman's inequality to U, V, Z onthe interval $(0,T)$ (for that, just make a change of variable $\xi = t - \frac{T}{2}$ and $t \in (0,T)$ implies $\xi \in (-\frac{T}{2}, \frac{T}{2})$ and then return to the interval $(0,T)$ by changing the variable).

There exists $s_0, \lambda_0 > 0$ and $C > 0$ such that for $s \geq s_0$ and $\lambda \geq \lambda_0 > 0$ we have

$$\begin{aligned}
&s\lambda \int_0^T \int_0^L e^{2s\varphi_M} \varphi_M (\rho_1 |\frac{\partial U}{\partial t}|^2 + \rho_2 |\frac{\partial V}{\partial t}|^2 + \rho_1 |\frac{\partial Z}{\partial t}|^2 + k|U_x|^2 + b|V_x|^2 \\
&+k_0|Z_x|^2 + k|U_x + V + lZ|^2 + k_0|Z_x - lU|^2) dxdt \\
&+s^3\lambda^3 \int_0^T \int_0^L e^{2s\varphi_M} \varphi_M^3 (|U|^2 + |V|^2 + |Z|^2) dxdt \\
&\leq C \int_0^T \int_0^L e^{2s\varphi_M} |L_{1,1}(U,V,Z)|^2 dxdt \\
&+C \int_0^T \int_0^L e^{2s\varphi_M} |L_{1,2}(U,V,Z)|^2 dxdt \\
&+C \int_0^T \int_0^L e^{2s\varphi_M} |L_{1,3}(U,V,Z)|^2 dxdt + Cs\lambda \int_0^T e^{2s\varphi_M(L,t)} |U_x(L)|^2 dt \\
&+Cs\lambda \int_0^T e^{2s\varphi_M(L,t)} |V_x(L)|^2 dt + Cs\lambda \int_0^T e^{2s\varphi_M(L,t)} |Z_x(L)|^2 dt.
\end{aligned} \tag{4.15}$$

Using Schwarz and Poincaré and the fact that $\theta_\delta \in C_0^\infty([0,T])$ and its derivatives cancel out in $[\delta, T-\delta]$ we have

$$\begin{aligned}
&\int_0^T \int_0^L e^{2s\varphi_M} |L_{1,1}(U,V,Z)|^2 dxdt + \int_0^T \int_0^L e^{2s\varphi_M} |L_{1,2}(U,V,Z)|^2 dxdt \\
&+ \int_0^T \int_0^L e^{2s\varphi_M} |L_{1,3}(U,V,Z)|^2 dxdt \\
&\leq C_1 \int_0^\delta \int_0^L e^{2se^{\lambda M_0}} (\rho_1 |\frac{\partial u}{\partial t}|^2 + \rho_2 |\frac{\partial}{\partial t}|^2 + \rho_1 |\frac{\partial z}{\partial t}|^2 + k|u_x|^2 + b|v_x|^2 \\
&+k_0|z_x|^2 + k|u_x + v + lz|^2 + k_0|z_x - lu|^2) dxdt + \\
&C_1 \int_{T-\delta}^T \int_0^L e^{2se^{\lambda M_0}} (\rho_1 |\frac{\partial u}{\partial t}|^2 + \rho_2 |\frac{\partial}{\partial t}|^2 + \rho_1 |\frac{\partial z}{\partial t}|^2 \\
&+k|u_x|^2 + b|v_x|^2 + k_0|z_x|^2 + k|u_x + v + lz|^2 + k_0|z_x - lu|^2) dxdt
\end{aligned} \tag{4.16}$$

On the other hand $U=u, V=v, Z=z$ in $[\frac{T}{2}-\eta, \frac{T}{2}+\eta]$ then

$$\begin{aligned}
&s\lambda\int_0^T\int_0^L e^{2s\varphi_M}\varphi_M(\rho_1\|\frac{\partial U}{\partial t}\|^2+\rho_2\|\frac{\partial V}{\partial t}\|^2+\rho_1\|\frac{\partial Z}{\partial t}\|^2+k\|U_x\|^2\\
&+b\|V_x\|^2+k_0\|Z_x\|^2+k\|U_x+V+lZ\|^2+k_0\|Z_x-lU\|^2)dxdt\\
&\geq s\lambda\int_{\frac{T}{2}-\eta}^{\frac{T}{2}+\eta}\int_0^L e^{2se^{\lambda M_0}}(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial v}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2\\
&+k\|u_x\|^2+b\|v_x\|^2+k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt
\end{aligned} \tag{4.17}$$

from (4.16), (4.17) and the Carleman's inequality (4.15) dividing by $e^{2se^{\lambda M_0}}$ we have

$$\begin{aligned}
&s\lambda\int_{\frac{T}{2}-\eta}^{\frac{T}{2}+\eta}\int_0^L(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial v}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2\\
&+k\|u_x\|^2+b\|v_x\|^2+k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt\\
&\leq C_1\int_0^\delta\int_0^L(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2+k\|u_x\|^2+b\|v_x\|^2\\
&+k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt+\\
&C_1\int_{T-\delta}^T\int_0^L(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2\\
&+k\|u_x\|^2+b\|v_x\|^2+k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|U_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|V_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|Z_x(L)\|^2dt.
\end{aligned} \tag{4.18}$$

but

$$\begin{aligned}
&\int_0^\delta\int_0^L(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial v}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2+k\|u_x\|^2+b\|v_x\|^2+\\
&k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt\\
&+\int_{T-\delta}^T\int_0^L(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial v}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2\\
&+k\|u_x\|^2+b\|v_x\|^2+k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt\\
&\leq C(E(t)+\widetilde{E}(t))\leq C(E(0)+\widetilde{E}(0))\leq CE(0),
\end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
&s\lambda\int_{\frac{T}{2}-\eta}^{\frac{T}{2}+\eta}\int_0^L(\rho_1\|\frac{\partial u}{\partial t}\|^2+\rho_2\|\frac{\partial v}{\partial t}\|^2+\rho_1\|\frac{\partial z}{\partial t}\|^2\\
&+k\|u_x\|^2+b\|v_x\|^2+k_0\|z_x\|^2+k\|u_x+v+lz\|^2+k_0\|z_x-lu\|^2)dxdt\\
&\geq Cs\lambda\int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta}E(t)dt=Cs\lambda\int_{\frac{T}{2}-\delta}^{\frac{T}{2}+\delta}E(0)dt=C_2s\lambda E(0),
\end{aligned}
\tag{4.20}
$$

from (4.18), (4.19) and (4.20) for $s\lambda$ sufficiently large if one has to

$$
\begin{aligned}
&s\lambda E(0)\leq Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|U_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|V_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|Z_x(L)\|^2dt\\
&\leq Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|\theta_\delta u_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|\theta_\delta v_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|\theta_\delta z_x(L)\|^2dt\\
&\leq Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|u_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|v_x(L)\|^2dt\\
&+Cs\lambda\int_0^T e^{2s(\varphi_M(L,t)-e^{\lambda M_0})}\|z_x(L)\|^2dt\\
&\leq Cs\lambda\int_0^T \|u_x(L)\|^2dt\\
&+Cs\lambda\int_0^T \|v_x(L)\|^2dt+Cs\lambda\int_0^T \|z_x(L)\|^2dt
\end{aligned}
\tag{4.21}
$$

of previous inequality we have the desired. □

5 Border control for the Bresse system using the HUM method

Be arbitrary $x_0 < 0$, however fixed.

We will define:

$$\begin{aligned} &m : \mathbb{R} \to \mathbb{R} \\ &x \mapsto x - x_0. \end{aligned} \tag{5.1}$$

Consider the following problem

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l[\omega_x - l\varphi] = 0, \quad in \quad (0,L) \times (0,T) \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = 0, \quad in \quad (0,L) \times (0,T) \\ \rho_1 \omega_{tt} - k_0 [\omega_x - l\varphi]_x + kl(\varphi_x + \psi + l\omega) = 0, \quad in \quad (0,L) \times (0,T) \\ \varphi(0,t) = \psi(0,t) = \omega(0,t) = 0, \quad t \in (0,T) \\ \varphi(L,t) = g_1(t), \quad \phi(L,t) = g_2(t), \quad \omega(L,t) = g_3(t), \quad t \in (0,T) \\ \varphi(.,0) = \varphi_0, \quad \varphi_t(.,0) = \varphi_1, \quad in \quad (0,L) \\ \psi(.,0) = \psi_0, \quad \psi_t(.,0) = \psi_1, \quad in \quad (0,L) \\ \omega(.,0) = \omega_0, \quad \omega_t(.,0) = \omega_1, \quad in \quad (0,L). \end{array} \right. \tag{5.2}$$

Our goal is to find $T_0 > 0$ and a Hilbert space H, so that if $\{\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1\} \in H$ then there are controls g_1, g_2, g_3 in $]0,T[$ such that the solution of (5.2) verify:

$\varphi(x,T) = \varphi_t(x,T) = \psi(x,T) = \psi_t(x,T) = \omega(x,T) = \omega_t(x,T) = 0, \quad \forall T > T_0.$

Be $\{u_0, u_1, v_0, v_1, z_0, z_1\} \in (D(0,L) \times D(0,L))^3$ and consider the homogeneous system of Bresse:

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v + lz)_x - k_0 l[z_x - lu] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2 v_{tt} - b v_{xx} + k(u_x + v + lz) = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_1 z_{tt} - k_0[z_x - lu]_x + kl(u_x + v + lz) = 0, \quad in \quad (0,L)\times(0,T) \\ u(0,t) = v(0,t) = z(0,t) = 0, \quad t\in(0,T) \\ u(L,t) = 0 = v(L,t) = z(L,t) = 0, \quad t\in(0,T) \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad in \quad (0,L) \\ v(.,0) = v_0, \quad v_t(.,0) = v_1, \quad in \quad (0,L) \\ z(.,0) = z_0, \quad z_t(.,0) = z_1, \quad in \quad (0,L). \end{cases} \tag{5.3}$$

The system (5.3) as a single solution in the class:

$$\{u,v,z\} \in C([0,T]; H_0^1(0,L)\cap H^2(0,L)) \cap C^1([0,T]; H_0^1(0,L)). \tag{5.4}$$

In fact,we have the following result of elliptical regularity:

$$\{u,v,z\} \in C^\infty([0,L]\times(0,T)), \tag{5.5}$$

See [4], and still $u_x(L), v_x(L), z_x(L) \in L^2(0,T)$.

Consider the retrograde problem:

$$\begin{cases} \rho_1 \Phi_{tt} - k(\Phi_x + \Psi + lW)_x - k_0 l[W_x - l\phi] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2 \Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi + lW) = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_1 W_{tt} - k_0[W_x - l\Phi]_x + kl(\Phi_x + \Psi + lW) = 0, \quad in \quad (0,L)\times(0,T) \\ \Phi(0,t) = \Psi(0,t) = W(0,t) = 0, \quad t\in(0,T) \\ \Phi(L,t) = u_x(L), \quad \Psi(L,t) = v_x(L), \quad W(L,t) = z_x(L), \quad t\in(0,T) \\ \Phi(.,T) = \Phi_t(.,T) = \Psi(.,T) = \Psi_t(.,T) = W(.,T) = W_t(.,T) = 0, \\ in \quad (0,L). \end{cases} \tag{5.6}$$

where $\{u,v,z\}$ is solution of (5.3). Reversing in time the problem (5.6) admits an unique solution $\{\Phi,\Psi,W\}$ by transposition, in the class

$$\{\Phi,\Psi,W\} \in C([0,T]; L^2(0,L)) \cap C^1([0,T]; H^{-1}(0,L)), \tag{5.7}$$

see [15]. By virtue of the regularity of the functions $\{\Phi, \Psi, W\}$ and of the uniqueness of the problems (5.3), (5.6) we define:

$$\begin{aligned}
&\wedge : (D(0,L) \times D(0,L))^3 \to (H^{-1}(0,L) \times L^2(0,L))^3 \\
&\{u_0, u_1, v_0, v_1, z_0, z_1\} \mapsto \wedge\{(u_0,u_1),(v_0,v_1),(z_0,z_1)\} \\
&= \{(\rho_1\Phi_t(x,0), -\rho_1\Phi(x,0)), (\rho_2\Psi_t(x,0), -\rho_2\Psi(x,0)), \\
&(\rho_1 W_t(x,0), -\rho_1 W(x,0))\}.
\end{aligned} \tag{5.8}$$

We will now develop a rationale that will allow us to obtain a relation between the $\wedge$ application defined above and the derivatives $u_x(L)$, $v_x(L)$, $z_x(L)$ of problem (5.3).

As $H_0^2(0,T)$ is dense in $L^2(0,T)$ and $u_x(L), v_x(L), z_x(L) \in L^2(0,T)$, there are $(g_{1\mu})_{\mu\in\mathbb{N}}, (g_{2\mu})_{\mu\in\mathbb{N}}, (g_{3\mu})_{\mu\in\mathbb{N}} \subset H_0^2(0,T)$ such that

$$\begin{aligned}
g_{1\mu} &\to u_x(L) \\
g_{2\mu} &\to v_x(L) \\
g_{3\mu} &\to z_x(L)
\end{aligned} \tag{5.9}$$

consider the sequence of problems

$$\begin{cases}
\rho_1\Phi_{\mu tt} - k(\Phi_{\mu x} + \Psi_\mu + lW_\mu)_x - k_0 l[W_x - l\Phi] = 0, \\
in \quad (0,L)\times(0,T) \\
\rho_2\Psi_{\mu tt} - b\Psi_{\mu xx} + k(\Phi_{\mu x} + \Psi_\mu + lW_\mu) = 0, \quad in \quad (0,L)\times(0,T) \\
\rho_1 W_{\mu tt} - k_0[W_{x\mu} - l\Phi_\mu]_x + kl(\Phi_{\mu x} + \Psi_\mu + lW_\mu) = 0, \\
in \quad (0,L)\times(0,T) \\
\Phi_\mu(0,t) = \Psi_\mu(0,t) = W_\mu(0,t) = 0, \quad t\in(0,T) \\
\Phi_\mu(L,t) = g_{1\mu}(t), \quad \Psi_\mu(L,t) = g_{2\mu}(t), \quad W_\mu(L,t) = g_{3\mu}(t), \\
t \in (0,T) \\
\Phi_\mu(.,T) = \Phi_{\mu t}(.,T) = \Psi_\mu(.,T) = \Psi_{\mu t}(.,T) = W_\mu(.,T) \\
= W_{\mu t}(.,T) = 0 \;\; in \quad (0,L).
\end{cases} \tag{5.10}$$

Reversing in time, then for each $\mu \in \mathbb{N}$, its follows that problem (5.10) admits a single solution in class

$\{\Phi_\mu, \Psi_\mu, W_\mu\} \in C([0,T]; H^1(0,L)) \cap C^1([0,T]; L^2(0,L))$
check

$$\left\{\begin{array}{l} \rho_1 \Phi_{\mu tt} - k(\Phi_{\mu x} + \Psi_\mu + lW_\mu)_x - k_0 l[W_x - l\phi] = 0, \\ \rho_2 \Psi_{\mu tt} - b\Psi_{\mu xx} + k(\Phi_{\mu x} + \Psi_\mu + lW_\mu) = 0, \\ \rho_1 W_{\mu tt} - k_0[W_{x\mu} - l\Phi_\mu]_x + kl(\Phi_{\mu x} + \Psi_\mu + lW_\mu) = 0. \end{array}\right. \tag{5.11}$$

Futher, for each μ, the solution $\{\Phi_\mu, \Psi_\mu, W_\mu\}$ from (5.10) is also a solution by transposition. Its follows that $\{(\Phi_\mu - \Phi), (\Psi_\mu - \Psi), (W_\mu - W)\}$ (where $\{\Phi, \Psi, W\}$ is solution by transposition of (5.6)) is the only solution by transposition of:

$$\left\{\begin{array}{l} \rho_1(\Phi_\mu - \Phi)_{tt} - k((\Phi_\mu - \Phi)_x + (\Psi_\mu - \Psi) + l(W_\mu) - W)_x \\ -k_0 l[(W_\mu - W)_x - l(\Phi_\mu - \Phi)] = 0, \\ \rho_2(\Psi_\mu - \Psi)_{tt} - b(\Psi_\mu - \Psi)_{xx} \\ +kl((\Phi_\mu - \Phi)_x + (\Psi_\mu - \Psi) + l(W_\mu - W)) = 0, \\ \rho_1(W_\mu - W)_{tt} - k_0[(W_\mu - W)_x - l(\Phi_\mu - \Phi)]_x \\ +kl((\Phi_\mu - \Phi)_x + (\Psi_\mu - \Psi) + l(W_\mu - W)) = 0, \\ \Phi_\mu(0,t) - \Phi(0,t) = \Psi_\mu(0,t) - \Psi(0,t) = W_\mu(0,t) - W(0,t) = 0, \\ \Phi_\mu(L,t) - \Phi_\mu(L,t) = g_{1\mu}(t) - u_x(L), \\ \Psi_\mu(L,t) - \Psi(L,t) = g_{2\mu}(t) - v_x(L), \\ W_\mu(L,t) - W_\mu(L,t) = g_{3\mu}(t) - z_x(L), \\ \Phi_\mu(.,T) - \Phi(.,T) = \Phi_{\mu t}(.,T) - \Phi_t(.,T) \\ = \Psi_\mu(.,T) - \Psi(.,T) = \Psi_{\mu t}(.,T) - \Psi_t(.,T) \\ = W_\mu(.,T) - W(.,T) = W_{\mu t}(.,T) - W_t(.,T) = 0 \end{array}\right. \tag{5.12}$$

from (2.4)

$$\begin{array}{l} \|\Phi_\mu - \Phi\|_{C([0,T];L^2(0,L))} + \|\Psi_\mu - \Psi\|_{C([0,T];L^2(0,L))} \\ +\|W_\mu - W\|_{C([0,T];L^2(0,L))} \\ \leq C\{\|g_{1\mu} - u_x(L)\|_{L^2(0,L)} + \|g_{2\mu} - v_x(L)\|_{L^2(0,L)} \\ +\|g_{3\mu} - z_x(L)\|_{L^2(0,L)}\} \end{array} \tag{5.13}$$

and from (2.5), we get:

$$\begin{aligned}&\|\Phi_{\mu t}-\Phi_t\|_{C([0,T];H^{-1}(0,L))}+\|\Psi_{\mu t}-\Psi_t\|_{C([0,T];H^{-1}(0,L))}\\&+\|W_{\mu t}-W_t\|_{C([0,T];H^{-1}(0,L))}\leq C\{\|g_{1\mu}-u_x(L)\|_{L^2(0,L)}\\&+\|g_{2\mu}-v_x(L)\|_{L^2(0,L)}+\|g_{3\mu}-z_x(L)\|_{L^2(0,L)}\}.\end{aligned}\tag{5.14}$$

From (5.9), (5.13) and (5.14) it turns out that:

$$\begin{aligned}&\Phi_\mu\to\Phi\quad in\quad C([0,T];L^2(0,L)),\\&\Psi_\mu\to\Psi\quad in\quad C([0,T];L^2(0,L)),\\&W_\mu\to W\quad in\quad C([0,T];L^2(0,L)),\end{aligned}\tag{5.15}$$

and

$$\begin{aligned}&\Phi_{\mu t}\to\Phi_t\quad in\quad C([0,T];H^{-1}(0,L)),\\&\Psi_{\mu t}\to\Psi_t\quad in\quad C([0,T];H^{-1}(0,L)),\\&W_{\mu t}\to W_t\quad in\quad C([0,T];H^{-1}(0,L)).\end{aligned}\tag{5.16}$$

On the other hand, data $X_0, X_1, Y_0, Y_1, Z_0, Z_1 \in D(0,L)$ there is a solution $\{X, Y, Z\}$ of (5.3) in class (5.4) with initial data $\{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}$. Composing (5.11) with X, Y, Z it turns out that:

$$\left\{\begin{aligned}&\int_0^T\langle\rho_1\Phi_{\mu tt}-k(\Phi_{\mu x}+\Psi_\mu+lW_\mu)_x-k_0l(W_x-l\phi),X\rangle dt=0\\&\int_0^T\langle\rho_2\Psi_{\mu tt}-b\Psi_{\mu xx}+k(\Phi_{\mu x}+\Psi_\mu+lW_\mu),Y\rangle dt=0\\&\int_0^T\langle\rho_1W_{\mu tt}-k_0(W_{x\mu}-l\Phi_\mu)_x+kl(\Phi_{\mu x}+\Psi_\mu+lW_\mu),Z\rangle dt=0.\end{aligned}\right.\tag{5.17}$$

It follows that integranting by parts gives

$$\begin{aligned}&\rho_1(\Phi_\mu(.,0),X_1)-\rho_1\langle\Phi_{\mu t}(.,0),X_0\rangle+k\int_0^T\langle g_{1\mu}(t),X_x(L)\rangle dt\\&\rho_2(\Psi_\mu(.,0),Y_1)-\rho_2\langle\Psi_{\mu t}(.,0),Y_0\rangle+b\int_0^T\langle g_{2\mu}(t),Y_x(L)\rangle dt\\&\rho_1(W_\mu(.,0),Z_1)-\rho_1\langle W_{\mu t}(.,0),Z_0\rangle+k_0\int_0^T\langle g_{3\mu}(t),Z_x(L)\rangle dt=0,\end{aligned}\tag{5.18}$$

from (5.9), (5.15), (5.16) and (5.18) it turns out that

$$\begin{aligned}
&k\int_0^T \langle u_x(L), X_x(L)\rangle dt + b\int_0^T \langle v_x(L), Y_x(L)\rangle dt + k_0\int_0^T \langle z_x(L), Z_x(L)\rangle dt \\
&= -\rho_1(\Phi(.,0), X_1) + \rho_1\langle \Phi_t(.,0), X_0\rangle - \rho_2(\Psi(.,0), Y_1) \\
&+\rho_2\langle \Psi_t(.,0), Y_0\rangle - \rho_1(W(.,0), Z_1) + \rho_1\langle W_t(.,0), Z_0\rangle,
\end{aligned} \tag{5.19}$$

where $\{X, Y, Z\}$ is solution of (5.3) with data $\{X_1, X_1, Y_0, Y_1, Z_0, Z_1\} \in [D(0,L)\times D(0,L)]^3$ and $\{u, v, z\}$ is the unique solution of (5.3) with data $\{u_0, u_1, v_0, v_1, z_0, z_1\}$.

Note that (5.19) can be written as

$$\begin{aligned}
&k\int_0^T \langle u_x(L), X_x(L)\rangle dt + b\int_0^T \langle v_x(L), Y_x(L)\rangle dt + k_0\int_0^T \langle z_x(L), Z_x(L)\rangle dt \\
&= \langle\{\rho_1(\Phi_t(.,0), -\Phi(.,0)), \rho_2(\Psi_t(.,0), -\Psi(.,0)), \rho_1(W_t(.,0), -W(.,0))\}, \\
&\{(X_0, X_1), (Y_0, Y_1), (Z_0, Z_1)\}\rangle_{[H^{-1}(0,L)\times L^2(0,L)]^3, [H^{-1}(0,L)\times L^2(0,L)]^3} \\
&= \langle \wedge\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}\rangle_{(D'(0,L))^6, (D(0,L))^6}.
\end{aligned} \tag{5.20}$$

We defined

$$\begin{aligned}
&(.,.)_* : (D(0,L))^6 \times (D(0,L))^6 \to \mathbb{R} \\
&\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{X_0, X_1, Y_0, Y_1, Z_0, Z_1\} \mapsto \\
&k\int_0^T \langle u_x(L), X_x(L)\rangle dt + b\int_0^T \langle v_x(L), Y_x(L)\rangle dt + k_0\int_0^T \langle z_x(L), Z_x(L)\rangle dt.
\end{aligned} \tag{5.21}$$

Clearly the application in (5.21) is linear and positive.

To prove that (5.21) is an internal product in $(D(0,L))^6 \times (D(0,L))^6$, we must show that application is strictly positive. More precisely we will prove that:

$$\begin{aligned}
&(\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{u_0, u_1, v_0, v_1, z_0, z_1\})_* = 0 \\
&\Leftrightarrow u_0 = u_1 = v_0 = v_1 = z_0 = z_1 = 0
\end{aligned} \tag{5.22}$$

If

$$\begin{aligned}
&u_0 = u_1 = v_0 = v_1 = z_0 = z_1 = 0 \\
&\Rightarrow (\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{u_0, u_1, v_0, v_1, z_0, z_1\})_* = 0
\end{aligned} \tag{5.23}$$

it's immediate.

Suppose that

$$\begin{aligned}&(\{u_0,u_1,v_0,v_1,z_0,z_1\},\{u_0,u_1,v_0,v_1,z_0,z_1\})_*\\&=k\int_0^T|u_x(L)|^2dt+b\int_0^T|v_x(L)|^2dt+k_0\int_0^T|z_x(L)|^2dt=0\end{aligned}\tag{5.24}$$

and by inverse inequality $\forall\, T>2\sup_{x\in[0,L]}|x-x_0|=2(L-x_0),\quad x_0<0.$

$$0\le E(0)\le C_0\{k\int_0^T|u_x(L)|^2dt+b\int_0^T|v_x(L)|^2dt+k_0\int_0^T|z_x(L)|^2dt\}\tag{5.25}$$

which implies $u_0=u_1=v_0=v_1=z_0=z_1=0$.

It follows from the foregoing that the application

$$\begin{aligned}&\|.\|_*:(D(0,L))^6\to\mathbb{R}^+\\&\{u_0,u_1,v_0,v_1,z_0,z_1\}\mapsto\\&(k\int_0^T|u_x(L)|^2dt+b\int_0^T|v_x(L)|^2dt+k_0\int_0^T|z_x(L)|^2dt)^{\frac12}.\end{aligned}\tag{5.26}$$

defines a norm in $(D(0,L))^6$. Consider F the Hilbert space obtained by completing $(D(0,L))^6$ with the norm $\|.\|_*$, this is

$$F=\overline{(D(0,L)\times D(0,L))^3}^{\|.\|_*}.\tag{5.27}$$

By direct and inverse inequality there are $C_1,C_2>0$ such that

$$\begin{aligned}&C_1E(0)\le k\int_0^T|u_x(L)|^2dt+b\int_0^T|v_x(L)|^2dt+k_0\int_0^T|z_x(L)|^2dt\\&\le C_2E(0),\end{aligned}\tag{5.28}$$

next

$$\begin{aligned}&C_1'\|\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_{(H_0^1(0,L)\times L^2(0,L))^3}\\\le\;&\|\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_*\le C_2'\|\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_{(H_0^1(0,L)\times L^2(0,L))^3}\\&\{u_0,u_1,v_0,v_1,z_0,z_1\}\in(D(0,L)\times D(0,L))^3.\end{aligned}\tag{5.29}$$

Resuts of (5.26) and (5.29) that the norm $\|\{.\}\|_*$ is equivalent the norm $\|\{.\}\|_{(H_0^1(0,L)\times L^2(0,L))^3}$ in $(D(0,L)\times D(0,L))^3$.

Consequenty of (5.27) we get:

$$F = \overline{(D(0,L)\times D(0,L))^3}^{\|.\|_*} = \overline{(D(0,L)\times D(0,L))^3}^{\|.\|_{(H_0^1(0,L)\times L^2(0,L))^3}} \\ = (H_0^1(0,L)\times L^2(0,L))^3. \tag{5.30}$$

Providing $(D(0,L))^6$ of the topology given by the norm $\|\|_*$ we prove that the operator $\wedge$ given in (5.8), wich is obviously linear is continuous. In fact (5.20), (5.21) results

$$\begin{aligned} &|\langle \wedge\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}\rangle|_{F',F} \\ &= |(\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\})_*| \\ &\leq \|\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_*\|\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}\|_* \\ &\forall\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}\in D^6(0,L). \end{aligned} \tag{5.31}$$

By the density of $(D(0,L))^6$ in F follows that:

$$\begin{aligned} &|\langle \wedge\{u_0,u_1,v_0,v_1,z_0,z_1\},\{A_0,A_1,B_0,B_1,C_0,C_1\}\rangle|_{F',F} \\ &\leq \|\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_*\|\{A_0,A_1,B_0,B_1,C_0,C_1\}\|_* \\ &\forall\{u_0,u_1,v_0,v_1,z_0,z_1\}\in D^6(0,L), \quad e \quad \{A_0,A_1,B_0,B_1,C_0,C_1\}\in F, \end{aligned} \tag{5.32}$$

with implies that:

$$\begin{aligned} &\|\langle \wedge\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_{F'} \leq \|\{u_0,u_1,v_0,v_1,z_0,z_1\}\|_* \\ &\forall\{u_0,u_1,v_0,v_1,z_0,z_1\}\in D^6(0,L), \end{aligned} \tag{5.33}$$

With proves the continuity of the operator $\wedge$. Now by the density of $(D(0,L))^6$ in F we can extend $\wedge$, in a unique way, to a linear and continuous operator:

$$\begin{aligned} &\widetilde{\wedge} : F \to F' \\ &\{A_0,A_1,B_0,B_1,C_0,C_1\} \mapsto \widetilde{\wedge}\{A_0,A_1,B_0,B_1,C_0,C_1\} \\ &= \lim_{\mu\to+\infty} \wedge\{A_{0\mu},A_{1\mu},B_{0\mu},B_{1\mu},C_{0\mu},C_{1\mu}\} \end{aligned} \tag{5.34}$$

where:

$$\{A_{0\mu}, A_{1\mu}, B_{0\mu}, B_{1\mu}, C_{0\mu}, C_{1\mu}\}_{\mu\in\mathbb{N}} \subset (D(0,L))^6$$
$$\|\{A_{0\mu}, A_{1\mu}, B_{0\mu}, B_{1\mu}, C_{0\mu}, C_{1\mu}\} - \{A_0, A_1, B_0, B_1, C_0, C_1\}\|_* \to 0. \tag{5.35}$$

We note that the previous definition is independent of the sequence $\{A_{0\mu}, A_{1\mu}, B_{0\mu}, B_{1\mu}, C_{0\mu}, C_{1\mu}\}$ which approaches $\{A_0, A_1, B_0, B_1, C_0, C_1\}$.

We will prove that

$$\widetilde{\wedge}\{A_0, A_1, B_0, B_1, C_0, C_1\}$$
$$= \{\rho_1\Phi_t(.,0), -\rho_1\Phi(.,0), \rho_2\Psi_t(.,0), -\rho_2\Psi(.,0), \rho_1 W_t(.,0), -\rho_1 W(.,0)\} \tag{5.36}$$

where $\{\Phi, \Psi, W\}$ is solution by transposition of

$$\begin{cases} \rho_1\Phi_{tt} - k(\Phi_x + \Psi + lW)_x - k_0 l[W_x - l\phi] = 0 \\ \rho_2\Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi + lW) = 0 \\ \rho_1 W_{tt} - k_0[W_x - l\Phi]_x + kl(\Phi_x + \Psi + lW) = 0 \\ \Phi(0,t) = \Psi(0,t) = W(0,t) = 0, \quad t \in (0,T) \\ \Phi(L,t) = A_x(L), \quad \Psi(L,t) = B_x(L), \quad W(L,t) = C_x(L), \quad t \in (0,T) \\ \Phi(.,T) = \Phi_t(.,T) = \Psi(.,T) = \Psi_t(.,T) = W(.,T) = W_t(.,T) = 0, \\ in \ \ (0,L) \end{cases} \tag{5.37}$$

where $\{A, B, C\}$ is solution of

$$\begin{cases} \rho_1 A_{tt} - k(A_x + B + lC)_x - k_0 l[C_x - lA] = 0 \\ \rho_2 B_{tt} - bB_{xx} + k(A_x + B + lC) = 0 \\ \rho_1 C_{tt} - k_0[C_x - lA]_x + kl(A_x + B + lC) = 0 \\ A(0,t) = B(0,t) = C(0,t) = A(L,t) = B(L,t) = C(L,t) = 0 \\ A(.,0) = A_0, A_t(.,0) = A_1 \\ B(.,0) = B_0, B_t(.,0) = B_1 \\ C(.,0) = C_0, C_t(.,0) = C_1 \end{cases} \tag{5.38}$$

indeed,

let $\{A_0, A_1, B_0, B_1, C_0, C_1\} \in F$ and let us consider $\{A_{0\mu}, A_{1\mu}, B_{0\mu}, B_{1\mu}, C_{0\mu}, C_{1\mu}\} \subset D^6(0, L)$ such that

$$\left\{ \quad \{A_{0\mu}, A_{1\mu}, B_{0\mu}, B_{1\mu}, C_{0\mu}, C_{1\mu}\} \to \{A_0, A_1, B_0, B_1, C_0, C_1\} \quad in \quad F. \right. \tag{5.39}$$

We have to (5.8) and (5.34) that

$$\left\{ \begin{array}{l} \widetilde{\wedge}\{A_0, A_1, B_0, B_1, C_0, C_1\} = \lim\limits_{\mu\to+\infty} \wedge\{A_{0\mu}, A_{1\mu}, B_{0\mu}, B_{1\mu}, C_{0\mu}, C_{1\mu}\} \\ = \lim\limits_{\mu\to+\infty} \{\rho_1\Phi_{\mu t}(., 0), -\rho_1\Phi_\mu(., 0), \rho_2\Psi_{\mu t}(., 0), -\rho_2\Psi_\mu(., 0), \\ \rho_1 W_{\mu t}(., 0), -\rho_1 W_\mu(., 0)\} \end{array} \right. \tag{5.40}$$

where, for each $\mu \in \mathbb{N}$, $\{\Phi_\mu, \Psi_\mu, W_\mu\}$ is the unique solution by transposing the problem:

$$\left\{ \begin{array}{l} \rho_1\Phi_{\mu tt} - k(\Phi_{\mu x} + \Psi_\mu + lW_\mu)_x - k_0 l[W_{\mu x} - l\phi_\mu] = 0 \\ \rho_2\Psi_{\mu tt} - b\Psi_{\mu xx} + k(\Phi_{\mu x} + \Psi_\mu + lW_\mu) = 0 \\ \rho_1 W_{\mu tt} - k_0[W_{\mu x} - l\Phi_\mu]_x + kl(\Phi_{\mu x} + \Psi_\mu + lW_\mu) = 0 \\ \Phi_\mu(0, t) = \Psi_\mu(0, t) = W_\mu(0, t) = 0, \quad t \in (0, T) \\ \Phi_\mu(L, t) = A_{\mu x}(L), \quad \Psi_\mu(L, t) = B_{\mu x}(L), \\ W_\mu(L, t) = C_{\mu x}(L), \quad t \in (0, T) \\ \Phi_\mu(., T) = \Phi_{\mu t}(., T) = \Psi_\mu(., T) = \Psi_{\mu t}(., T) \\ = W_\mu(., T) = W_{\mu t}(., T) = 0, \ in \ (0, L) \end{array} \right. \tag{5.41}$$

where $\{A_\mu, B_\mu, C_\mu\}$ is solution of

$$\left\{ \begin{array}{l} \rho_1 A_{\mu tt} - k(A_{\mu x} + B_\mu + lC_\mu)_x - k_0 l[C_{\mu x} - lA_\mu] = 0 \\ \rho_2 B_{\mu tt} - bB_{\mu xx} + k(A_{\mu x} + B_\mu + lC_\mu) = 0 \\ \rho_1 C_{\mu tt} - k_0[c_{\mu x} - lA_\mu]_x + kl(A_{\mu x} + B_\mu + lC_\mu) = 0 \\ A_\mu(0, t) = B_\mu(0, t) = C_\mu(0, t) = A_\mu(L, t) = B_\mu(L, t) = C_\mu(L, t) = 0 \\ A_\mu(., 0) = A_{\mu 0}, A_{\mu t}(., 0) = A_{\mu 1} \\ B_\mu(., 0) = B_{\mu 0}, B_{\mu t}(., 0) = B_{\mu 1} \\ C_\mu(., 0) = C_{\mu 0}, C_{\mu t}(., 0) = C_{\mu 1}. \end{array} \right. \tag{5.42}$$

It follows that $\{\Phi_\mu-\Phi,\Psi_\mu-\Psi,W_\mu-W\}$ is the unique solution by tranposing

$$\left\{\begin{array}{l}
\rho_1(\Phi_\mu-\Phi)_{tt}-k((\Phi_\mu-\Phi)_x+(\Psi_\mu-\Psi)+l(W_\mu-W))_x\\
-k_0l[(W_\mu-W)_x-l(\Phi_\mu-\Phi)]=0\\
\rho_2(\Psi_\mu-\Psi)_{tt}-b(\Psi_\mu-\Psi)_{xx}\\
+k((\Phi_\mu-\Phi)_x+(\Psi_\mu-\Psi)+l(W_\mu-W))=0\\
\rho_1(W_\mu-W)_{tt}-k_0[(W_\mu-W)_x-l(\Phi_\mu-\Phi)]_x\\
+kl((\Phi_\mu-\Phi)_x+(\Psi_\mu-\Psi)+l(W_\mu-W))=0\\
\Phi_\mu(0,t)-\Phi(0,t)=\Psi_\mu(0,t)-\Psi(0,t)\\
=W_\mu(0,t)-W(0,t)=0,\quad t\in(0,T)\\
\Phi_\mu(L,t)-\Phi(L,t)=A_{\mu x}(L)-A_x(L),\\
\Psi_\mu(L,t)-\Psi(L,t)=B_{\mu x}(L)-B_x(L)\\
W_\mu(L,t)-W(L,t)=C_{\mu x}(L)-C_x(L),\\
\Phi_\mu(.,T)-\Phi_(.,T)=\Phi_{\mu t}(.,T)-\Phi_t(.,T)\\
=\Psi_\mu(.,T)-\Psi(.,T)=\Psi_{\mu t}(.,T)-\Psi_t(.,T)\\
=W_\mu(.,T)-W(.,T)=W_{\mu t}(.,T)-W_t(.,T)=0
\end{array}\right. \tag{5.43}$$

where $\{A_\mu-A,B_\mu-B,C_\mu-C\}$ is the unique solution of

$$\left\{\begin{array}{l}
\rho_1(A_\mu-A)_{tt}-k((A_\mu-A)_x+(B_\mu-B)+l(C_\mu-C))_x\\
-k_0l[(C_\mu-C)_x-l(A_\mu-A)]=0\\
\rho_2(B_\mu-B)_{tt}-b(B_\mu-B)_{xx}\\
+k((A_\mu-A)_x+(B_\mu-B)+l(C_\mu-C))=0\\
\rho_1(C_\mu-C)_{tt}-k_0[(C_\mu-C)_x-l(A_\mu-A)]_x\\
+kl((A_\mu-A)_x+(B_\mu-B)+l(C_\mu-C))=0\\
A_\mu(0,t)-A(0,t)=B_\mu(0,t)-B(0,t)=C_\mu(0,t)-C(0,t)\\
=A_\mu(L,t)-A(L,t)=B_\mu(L,t)-B(L,t)=C_\mu(L,t)-C(L,t)=0\\
A_\mu(.,0)-A(.,0)=A_{0\mu}-A_0,A_{\mu t}(.,0)-A_t(.,0)=A_{1\mu}-A_1\\
B_\mu(.,0)-B(.,0)=B_{0\mu}-B_0,B_{\mu t}(.,0)-B_t(.,0)=B_{1\mu}-B_1\\
C_\mu(.,0)-C(.,0)=C_{0\mu}-C_0,C_{\mu t}(.,0)-C(.,0)=C_{1\mu}-C_1.
\end{array}\right. \tag{5.44}$$

from (2.4) and (2.5)

$$
\begin{aligned}
&\|\Phi_\mu-\Phi\|_{C([0,T];L^2(0,L))}+\|\Psi_\mu-\Psi\|_{C([0,T];L^2(0,L))}\\
&+\|W_\mu-W\|_{C([0,T];L^2(0,L))}+\|\Phi_{\mu t}-\Phi_t\|_{C([0,T];H^{-1}(0,L))}\\
&+\|\Psi_{\mu t}-\Psi_t\|_{C([0,T];H^{-1}(0,L))}+\|W_{\mu t}-W_t\|_{C([0,T];H^{-1}(0,L))}\\
&\leq C\{k\|A_{\mu x}(L)-A_x(L)\|_{L^2(0,T)}+b\|B_{\mu x}(L)-B_x(L)\|_{L^2(0,T)}\\
&+k_0\|C_{\mu x}(L)-C_x(L)\|_{L^2(0,T)}\}=\\
&C(\|\{A_{0\mu}-A_0,A_{1\mu}-A_1,B_{0\mu}-B_0,B_{1\mu}-B_1,C_{0\mu}-C_0,C_{1\mu}-C_1\}\|_*)
\end{aligned}
\tag{5.45}
$$

where the last inequality stems from (5.26). Finally from (5.39) and (5.45) it turns out that:

$$
\begin{aligned}
&\Phi_\mu\to\Phi \quad in \;\; C([0,T];L^2(0,L))\\
&\Psi_\mu\to\Psi \quad in \;\; C([0,T];L^2(0,L))\\
&W_\mu\to W \quad in \;\; C([0,T];L^2(0,L))
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
&\Phi_\mu\to\Phi_t \quad in \;\; C([0,T];H^{-1}(0,L))\\
&\Psi_{\mu t}\to\Psi_t \quad in \;\; C([0,T];H^{-1}(0,L))\\
&W_{\mu t}\to W_t \quad in \;\; C([0,T];H^{-1}(0,L)).
\end{aligned}
\tag{5.47}
$$

From (5.40),(5.46) and (5.47) follow (5.36).

We defined:

$$
\begin{aligned}
&\mathcal{B}(\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\})\\
&\langle\overline{\wedge}\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}\rangle\\
&\forall\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}\in F
\end{aligned}
\tag{5.48}
$$

which is clearly a bilinear form.

We prove that $\mathcal{B}(.,.)$ is continuous and coercive. In fact, be $\{u_0,u_1,v_0,v_1,z_0,z_1\},\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}\in F$ and $(\{u_{0\mu},u_{1\mu},v_{0\mu},v_{1\mu},z_{0\mu},z_{1\mu}\}),(\{X_{0\mu},X_{1\mu},Y_{0\mu},Y_{1\mu},Z_{0\mu},Z_{1\mu}\})$ $\subset(D(0,L))^6$ such that
$\{u_{0\mu},u_{1\mu},v_{0\mu},v_{1\mu},z_{0\mu},z_{1\mu}\}\to\{u_0,u_1,v_0,v_1,z_0,z_1\}$ and
$\{X_{0\mu},X_{1\mu},Y_{0\mu},Y_{1\mu},Z_{0\mu},Z_{1\mu}\}\to\{X_0,X_1,Y_0,Y_1,Z_0,Z_1\}$ in F.

For each $\mu\in\mathbb{N}$ from (5.32) it comes:

$|\langle \wedge\{u_{0\mu}, u_{1\mu}, v_{0\mu}, v_{1\mu}, z_{0\mu}, z_{1\mu}\}, \{X_{0\mu}, X_{1\mu}, Y_{0\mu}, Y_{1\mu}, Z_{0\mu}, Z_{1\mu}\}\rangle_{F',F}|$
$\leq \|\{u_{0\mu}, u_{1\mu}, v_{0\mu}, v_{1\mu}, z_{0\mu}, z_{1\mu}\}\|_* \|\{X_{0\mu}, X_{1\mu}, Y_{0\mu}, Y_{1\mu}, Z_{0\mu}, Z_{1\mu}\}\|_*.$

Taking the limit in the previous inequality we have:

$$\begin{aligned}&|\langle \widetilde{\wedge}\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}\rangle| \\ &\leq \|\{u_0, u_1, v_0, v_1, z_0, z_1\}\|_* \|\{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}\|_*\end{aligned} \tag{5.49}$$

which proves the continuity of $\mathcal{B}(.,.)$.

To prove the continuity of the same note that of (5.20) and (5.26), for each $\mu \in \mathbb{N}$ we can write

$\langle \wedge\{u_{0\mu}, u_{1\mu}, v_{0\mu}, v_{1\mu}, z_{0\mu}, z_{1\mu}\}, \{u_{0\mu}, u_{1\mu}, v_{0\mu}, v_{1\mu}, z_{0\mu}, z_{1\mu}\}\rangle_{F',F}|$
$= \|\{u_{0\mu}, u_{1\mu}, v_{0\mu}, v_{1\mu}, z_{0\mu}, z_{1\mu}\}\|_*^2$

in the limit we get:

$|\langle \wedge\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{u_0, u_1, v_0, v_1, z_0, z_1\}\rangle_{F',F}|$
$= \|\{u_0, u_1, v_0, v_1, z_0, z_1\}\|_*^2$

which proves the coercivity of $\mathcal{B}(.,.)$.

So for Lax-milgran given $\{P_0, P_1, Q_0, Q_1, R_0, R_1\} \in F'$
$\exists!\ \{u_0, u_1, v_0, v_1, z_0, z_1\} \in F$ such that

$$\begin{aligned}&\langle\{P_0, P_1, Q_0, Q_1, R_0, R_1\}, \{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}\rangle = \\ &\mathcal{B}(\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}) \\ &= \langle \widetilde{\wedge}\{u_0, u_1, v_0, v_1, z_0, z_1\}, \{X_0, X_1, Y_0, Y_1, Z_0, Z_1\}\rangle \\ &\forall\{X_0, X_1, Y_0, Y_1, Z_0, Z_1\} \in F,\end{aligned} \tag{5.50}$$

which implies in function of the definition of $\mathcal{B}(.,.)$ given in (5.48) that:

$$\begin{aligned}&Given\ \ \{P_0, P_1, Q_0, Q_1, R_0, R_1\} \in F'\ \ \exists!\ \ \{u_0, u_1, v_0, v_1, z_0, z_1\} \in F \\ &such\, that \\ &\{P_0, P_1, Q_0, Q_1, R_0, R_1\} = \widetilde{\wedge}\{u_0, u_1, v_0, v_1, z_0, z_1\}\end{aligned} \tag{5.51}$$

or by virtue of (5.36) we conclude that:

$$
\begin{aligned}
&\textit{Given} \quad \{P_0, P_1, Q_0, Q_1, R_0, R_1\} \in F' \quad \exists! \ \{u_0, u_1, v_0, v_1, z_0, z_1\} \in F \\
&\textit{such that} \\
&P_0 = \rho_1 \Phi_t(.,0), P_1 = -\rho_1 \Phi_t(.,0), Q_0 = \rho_2 \Psi_t(.,0), Q_1 = -\rho_2 \Psi_t(.,0), \\
&R_0 = \rho_1 W_t(.,0), R_1 = -\rho_1 W_t(.,0)
\end{aligned}
\tag{5.52}
$$

where $\{\Phi, \Psi, W\}$ is the unique solution, by transposition of:

$$
\begin{cases}
\rho_1 \Phi_{tt} - k(\Phi_x + \Psi + lW)_x - k_0 l[W_x - l\Phi] = 0, \quad in \quad (0,L) \times (0,T) \\
\rho_2 \Psi_{tt} - b\Psi_{xx} + k(\Phi_x + \Psi + lW) = 0, \quad in \quad (0,L) \times (0,T) \\
\rho_1 W_{tt} - k_0[W_x - l\Phi]_x + kl(\Phi_x + \Psi + lW) = 0, \quad in \quad (0,L) \times (0,T) \\
\Phi(0,t) = \Psi(0,t) = W(0,t) = 0, \quad t \in (0,T) \\
\Phi(L,t) = u_x(L), \quad \Psi(L,t) = v_x(L), \quad W(L,t) = z_x(L), \quad t \in (0,T) \\
\Phi(.,T) = \Phi_t(.,T) = \Psi(.,T) = \Psi_t(.,T) = W(.,T) = W_t(.,T) = 0, \\
in \quad (0,L).
\end{cases}
\tag{5.53}
$$

and $\{u, v, z\}$ is solution of

$$
\begin{cases}
\rho_1 u_{tt} - k(u_x + v + lz)_x - k_0 l[z_x - lu] = 0, \quad in \quad (0,L) \times (0,T) \\
\rho_2 v_{tt} - bv_{xx} + k(u_x + v + lz) = 0, \quad in \quad (0,L) \times (0,T) \\
\rho_1 z_{tt} - k_0[z_x - lu]_x + kl(u_x + v + lz) = 0, \quad in \quad (0,L) \times (0,T) \\
u(0,t) = v(0,t) = z(0,t) = 0, \quad t \in (0,T) \\
u(L,t) = 0 = v(L,t) = z(L,t) = 0, \ \ t \in (0,T) \\
u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad in \quad (0,L) \\
v(.,0) = v_0, \quad v_t(.,0) = v_1, \quad in \quad (0,L) \\
z(.,0) = z_0, \quad z_t(.,0) = z_1, \quad in \quad (0,L).
\end{cases}
\tag{5.54}
$$

Let's remember that

$F = (H_0^1(0,L) \times L^2(0,L))^3$ and $F' = (H^{-1}(0,L) \times L^2(0,L))^3$.

So choosing

$$
T_0 = 2(L - x_0) \quad e \quad H = (L^2(0,L) \times H^{-1}(0,L))^3 \tag{5.55}
$$

then given

$$\begin{aligned}&\{\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1\}\in H \quad it \quad has \quad to \quad be\\&\{P_0,P_1,Q_0,Q_1,R_0,R_1\}\\&=\{\rho_1\varphi_1,-\rho_1\varphi_0,\rho_2\psi_1,-\rho_2\psi_0,\rho_1\omega_1,-\rho_1\omega_0\}\in F'\end{aligned} \tag{5.56}$$

and the (5.52) is that $\exists!\ \{u_0,u_1,v_0,v_1,z_0,z_1\}\in F$ such that

$$\begin{aligned}&\Phi(.,0)=\varphi_0,\Phi_t(.,0)=\varphi_1\\&\Psi(.,0)=\psi_0,\Psi_t(.,0)=\psi_1\\&W(.,0)=\omega_0,W_t(.,0)=\omega_1\end{aligned} \tag{5.57}$$

where $\{\Phi,\Psi,W\}$ is the unique solution by transposition of (5.53) and $\{u,v,z\}$ is the unique weak solution of (5.54) with data u_0,u_1,v_0,v_1,z_0,z_1.

Considering

$$g_1=u_x(L),g_2=v_x(L),g_3=z_x(L) \tag{5.58}$$

in the problem (5.2) subject to the initial data as in (5.57)we have such that problem has an unique solution by transposition φ,ψ,ω.

We observe that from (5.53) and (5.57) it results that $\{\Phi,\Psi,W\}$ is also solution by transposition of problem (5.2).

Then by the uniqueness of solution comes that $\varphi=\Phi$, $\psi=\Psi$, $W=\omega$ and consequently of (5.53) we conclude that:

$$\varphi(.,T)=\varphi_t(.,T)=\psi(.,T)=\psi_t(.,T)=W(.,T)=W_t(.,T)=0. \tag{5.59}$$

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