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Exponential attractor for a class of non local evolution equations

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Abstract. In this work we study the existence of exponential attractor for a non local evolution equation, which generalizes the model that describes the neuronal activity. Our results extend results obtained in [21] and in [11], where the studied models are configured as particular cases of the model presented here. Furthermore, by the existence of the exponential attractor, we conclude that the global attractor given in [20] has finite fractal dimension.

Keywords: Exponential attractor, global attractor, finite fractal dimension.

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1 Introduction

We consider the non local evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = -u(x,t) + g\big(\beta K(f \circ u)(x,t) + \beta h\big), \ x \in \Omega, t \in [0,\infty) \\ u(x,t) = 0, \ x \in \mathbb{R}^n \backslash \Omega, \ t \in [0,\infty) \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u(x,t)$ is a real function on $\mathbb{R}^N \times [0,\infty[$, Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$; h and β are nonnegative constants; K is an integral operator with symmetric kernel, given by

$$Kv(x) := \int\limits_{\mathbb{R}^N} J(x,y)v(y)dy, \tag{1.2}$$

where J is a non negative symmetric function of class $\mathcal{C}^1$, with

$$\int\limits_{\mathbb{R}^N} J(x,y)dy = \int\limits_{\mathbb{R}^N} J(x,y)dx = 1$$

and $f, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions satisfying the growth conditions

$$|g(x)| \leq k_1|x| + k_2, \ \forall\ x \in \mathbb{R}, \tag{1.3}$$

$$|f(x)| \leq c_1|x| + c_2, \ \forall\ x \in \mathbb{R}, \tag{1.4}$$

for some non-negative constants k_1, k_2, c_1, c_2. In addition, we will also assume that:

1. The function $g \in C^1(\mathbb{R})$ with

$$|g'(x)| \leq k_3|x| + k_4, \ \forall\ x \in \mathbb{R}, \tag{1.5}$$

   for some non-negative constants $k_3, k_4 > 0$ and that g' is Lipschitz on bounded sets.

2. The function f satisfy

$$|f(x) - f(y)| \leq c_0(1 + |x|^{p-1} + |y|^{p-1})|x-y|, \ \forall\ (x,y) \in \mathbb{R} \times \mathbb{R}, \tag{1.6}$$

   where c_0 is some non-negative constant.

The dynamics of non local evolution Equations like in (1.1) has attracted the attention of many researchers in the last years; see for instance [14, 15, 5, 16, 17, 6, 7, 20] and [8]. The formulation given in (1.1) is more complete and was recently introduced in the literature (see [20]). However in [20] only the existence of the global attractor and the gradient property have been analyzed. Here we follow a natural continuation of the results of [20], because we show the existence of an exponential attractor and conclude that the global attractor given in [20] has finite fractal dimension.

The approache considered here was motivated by similar approaches in [5, 4, 21] and [20], whose basic idea is to find an abstract way to impose Dirichlet boundary conditions in non local evolution equations.

This paper is organized as follows. In Section 2, we review the dynamical system generated by model (1.1), including the main result given in [20]. In Section 3, using the growth conditions assumed for the functions g and f and the techniques of [1] and [12] we prove that (1.1) has a exponential attractor in the phase space isometric to $L^p(\Omega)$. Furthermore, according to our assumptions, the results presented in this section are also extensions of Theorem 2.2, Theorem 3.2, Theorem 3.5, Theorem 3.6 and Corolary 3.7 proved in [21]; and Lemma 3.6, Theorem 3.10, Theorem 3.11, Lemma 3,12 and Corollary 3.13 obtained in [11].

2 Review on dynamical system generated by model

The map given by (1.2) is well defined as a bounded linear operator in various function spaces, depending on the properties assumed for function J. In particular, with the hypotheses considered in the introduction, we have the estimates of the lemma below, whose proof has been given in [3].

Lemma 2.1. *Let K be the map given by (1.2) and*

$$\|J\|_r := \sup_{x\in\Omega} \|J(x,.)\|_{L^r(\Omega)},\ 1 \leq r \leq \infty.$$

If $u \in L^p(\Omega)$, $1 < p < \infty$, then $Ku \in L^\infty(\Omega)$ with

$$|Ku(x)| \leq \|J\|_q \|u\|_{L^p(\Omega)},\ \forall\ x \in \Omega, \tag{2.1}$$

where $1 < q < \infty$ is he conjugate exponent of p, and

$$\|Ku\|_{L^p(\Omega)} \leq \|J\|_1 \|u\|_{L^p(\Omega)}. \tag{2.2}$$

Furthermore, if $u \in L^1(\Omega)$, *then* $Ku \in L^p(\Omega)$, $1 \leq p \leq \infty$, *and*

$$\|Ku\|_{L^p(\Omega)} \leq \|J\|_p \|u\|_{L^1(\Omega)}. \tag{2.3}$$

Proceeding as in [20], using Lemma 2.1 and the hypotheses on f and g, it easy to see that the function defined by the right side of the first equation in (1.1) is Lipschitz continuous on bounded set of the phase space

$$X = \{u \in L^p(\mathbb{R}^n) : u(x) = 0, \text{ if } x \in \mathbb{R}^n \backslash \Omega\},$$

with the induced norm of $L^p(\Omega)$, which is isometric to $L^p(\Omega)$ and we usually identify the two spaces, without further comment.

In [20], it has also been proven that the problem (1.1) generates a $\mathcal{C}^1$ flow in X which is given, by the variation of constant formula,

$$u(x,t) = \begin{cases} e^{-(t-t_0)}u_0(x) + \displaystyle\int_{t_0}^{t} e^{-(t-s)} g\big(\beta K f(u(x,s)) + \beta h\big) ds, & x \in \Omega, \\ 0, \ x \in \mathbb{R}^n \backslash \Omega. \end{cases} \tag{2.4}$$

Furthermore, assuming that $k_1 \beta c_1 < 1$ and using classic results of [22], the following results are proven:

Lemma 2.2. *([20]) For any positive number* σ, *the ball of radius*

$$R = (1+\sigma)\left[\frac{(k_1\beta c_2 + k_1 \beta h + k_2)|\Omega|^{\frac{1}{p}}}{1 - k_1 \beta c_1}\right].$$

is an absorbing set for the flow $S(t)$ *generated by* (1.1).

We denote by $\omega(C)$ the omega-limit set of a set C.

Theorem 2.3. *([20]) The set* $\mathcal{A} = \omega(B(0, R+\sigma))$ *is a global attractor for the flow* $S(t)$ *generated by* (1.1) *in* $M \equiv L^p(\Omega)$, *wich is contained in the ball of radius,* R.

3 Exponential Attractor for the discrete semigroup

In this section we prove the existence of Exponential Attractor, for the flow $\{S(t)\}$ generated by (1.1). For constructions of the exponential attractor, we follow the techniques used in [1] and [12] which is based on verification of the following conditions:

(C1) The semigroup $\{S(t)\}$ has a absorbing set $\mathcal{B}_1 \subset W^{1,p}(\Omega)$, which is bounded in $W^{1,p}(\Omega)$.

(C2) The set $\mathcal{B}_1$ is positively invariant for the flow $\{S(t)\}$, in $L^p(\Omega)$.

(C3) There exists $t^* \geq t_1$ such that, for any $u_1, u_2 \in \mathcal{B}_1$,

$$S(t^*)u_1 - S(t^*)u_2 = L(u_1, u_2) + K(u_1, u_2),$$

where $L : \mathcal{B}_1 \times \mathcal{B}_1 \to L^p(\Omega)$ satisfy,

$$\|L(u_1, u_2)\|_{L^p(\Omega)} \leq \alpha \|u_1 - u_2\|_{L^p(\Omega)}, \forall\, u_1, u_2 \in \mathcal{B}_1, \tag{3.1}$$

for some α satisfying $0 < \alpha < \dfrac{1}{2}$; and $K : \mathcal{B}_1 \times \mathcal{B}_1 \to W^{1,p}(\Omega)$ satisfy for any $u_1, u_2 \in \mathcal{B}_1$,

$$\|K(u_1, u_2)\|_{W^{1,p}(\Omega)} \leq C \|u_1 - u_2\|_{L^p(\Omega)}, \tag{3.2}$$

for some constant $C > 0$.

(C4) The map

$$\begin{array}{rrcl} F: & [0, t^*] \times \mathcal{B}_1 & \to & \mathcal{B}_1 \\ & (t, u) & \mapsto & F(t, u) := S(t)u \end{array}$$

is Lipschitz on $\mathcal{B}_1$ in the topology of $L^p(\Omega)$, uniformly in t, that is, there exists $\gamma_0 > 0$ such that, for any $u_1, u_2 \in \mathcal{B}_1$,

$$\|S(t)u_1 - S(t)u_2\|_{L^p(\Omega)} \leq \gamma_0 \|u_1 - u_2\|_{L^p(\Omega)},$$

for any $t \in [0, t^*]$.

(**C5**) The map

$$\begin{array}{rcl} F: \quad [0,t^*]\times\mathcal{B}_1 & \to & \mathcal{B}_1 \\ (t,u) & \mapsto & F(t,u) := S(t)u \end{array}$$

is Lipschitz on $[0,t^*]$, uniformly in $\mathcal{B}_1$, that is, there is $C_0>0$ such that, for any $u\in\mathcal{B}_1$,

$$\|S(t)u - S(t)u\|_2 \leq C_0|t_1-t_2|,$$

for any $t_1,t_2\in[0,t^*]$.

The conditions **(C1)** and **(C3)** allow to obtain a set $\mathcal{M}^*$ with all the properties of an exponential attractor for the discrete semigroup $\{S^n(t^*)\}_{n\in\mathbb{N}}$, except the compactness property. But precisely, we have the following result given in [12].

Theorem 3.1. *Let $(H,\|\cdot\|_H)$ and $(H_1,\|\cdot\|_{H_1})$ be Banach spaces with $H_1\subset H$, such that the immersion $H_1\hookrightarrow H$ is compact. Let $X\subset H$ be a bounded subset of H. Consider a non-linear map $L: X\to X$ such that L can be decomposed into a sum of two applications,*

$$L = L_0 + G_0,$$

where $L_0 : X\to H$ is a contraction with

$$\|L_0(x_1)-L_0(x_2)\|_H \leq \alpha\|x_1-x_2\|_H,\ \forall\ x_1,x_2\in X, \tag{3.3}$$

and $0<\alpha<\dfrac{1}{2}$; and $G_0 : X\to H$ is such that $G_0(X)\subset H_1$ and

$$\|G_0(x_1)-G_0(x_2)\|_{H_1} \leq C\|x_1-x_2\|_H,\ \forall\ x_1,x_2\in X, \tag{3.4}$$

for any positive constant C. Then the map L (or even, the discrete semigroup generated by L) admits a set $\mathcal{M}\subset\overline{X}^H$ satisfying the following properties:

(*i*) $L^k(\mathcal{M})\subset\mathcal{M},\ \forall\ k\in\mathbb{N}.$

(*ii*) $\dim_F(\mathcal{M},H)<\infty.$

(iii) There exist $\alpha, \omega > 0$ *such that*

$$dist_H(L^k(X), \mathcal{M}) \leq \alpha e^{-\omega k}, \ \forall k \in \mathbb{N}.$$

Furthermore, $\mathcal{M}$ *is closed in* H*.*

The conditions **(C2)**, **(C4)** and **(C5)** allow that, from $\mathcal{M}^*$, get a set $\mathcal{M}$ that is the candidate for the desired exponential attractor. However, to ensure that $\mathcal{M}$ exponentially attracts any bounded subset of $L^p(\Omega)$, it will be necessary to assume an additional assumption about f and g. For now, we continue with the assumptions initially assumed.

3.1 Relevant Estimates

We will now present some results that support the verification of **(C1)**-**(C5)** conditions. The next Lemma is an immediate consequence of the conditions imposed on f.

Lemma 3.2. *Let* $1 \leq p < \infty$ *and* $u \in L^p(\Omega)$ *be. Then* $f \circ u \in L^p(\Omega)$ *and*

$$\|f \circ u\|_{L^p(\Omega)} \leq c_1 \|u\|_{L^p(\Omega)} + c_2 |\Omega|^{\frac{1}{p}}, \tag{3.5}$$

where $|\Omega|$ *denotes the (Lebesgue) measure of* Ω*. If* $u \in L^\infty(\Omega)$ *then* $f \circ u \in L^\infty(\Omega)$ *and*

$$\|f \circ u\|_{L^\infty(\Omega)} \leq c_1 \|u\|_{L^\infty(\Omega)} + c_2. \tag{3.6}$$

Analogously to Lema 2.1 it is easy to check the following result:

Lemma 3.3. *If* $u \in L^p(\Omega)$, $1 < p < \infty$ *then* $\frac{\partial}{\partial x} K u \in L^\infty(\Omega)$ *and*

$$\left| \frac{\partial}{\partial x} K u(x) \right| \leq \|J_x\|_q \|u\|_{L^p(\Omega)}, \tag{3.7}$$

where $1 < q < \infty$ *is the conjugate exponent of* p*. Furthermore,*

$$\left\| \frac{\partial}{\partial x} K u \right\|_{L^p(\Omega)} \leq \|J_x\|_1 \|u\|_{L^p(\Omega)}. \tag{3.8}$$

If $u \in L^1(\Omega)$ *then* $\dfrac{\partial}{\partial x} Ku \in L^p(\Omega)$, $1 < p < \infty$ *and*

$$\left\| \frac{\partial}{\partial x} Ku \right\|_{L^p(\Omega)} \leq \|J_x\|_p \|u\|_{L^1(\Omega)}. \tag{3.9}$$

Using Lemma 3.3 and Lemma 3.2 we have the following result.

Lemma 3.4. *Let* $1 < p < \infty$, $a_1 := c_1 \|J_x\|_q$ *and* $a_2 := c_2 \|J_x\|_q |\Omega|^{\frac{1}{p}}$ *be. For any* $u \in L^p(\Omega)$,

$$\left| \frac{\partial}{\partial x} K\big(f \circ u\big)(x) \right| \leq a_1 \|u\|_{L^p(\Omega)} + a_2. \tag{3.10}$$

Proof. Just combine the estimates (3.7) and (3.5). □

Proposition 3.5. *For each bounded subset* C *in* $L^p(\Omega)$, *there exists* $R(C) > 0$ *such that, for each* $u_0 \in C$ *and* $t \geq 0$,

$$\|S(t)u_0\|_{L^p(\Omega)} \leq R(C).$$

Proof. From Lemma 2.2 there exists $t_0 = t_0(C) \geq 0$ such that

$$t \geq t_0 \Rightarrow S(t)C \subset \mathcal{B},$$

where $\mathcal{B}$ denotes the ball with center at the origin of $L^p(\Omega)$ and radius

$$R = (1+\sigma)\left[\frac{(k_1\beta c_2 + k_1\beta h + k_2)|\Omega|^{\frac{1}{p}}}{1 - k_1\beta c_1}\right].$$

Then, given $u_0 \in C$,

$$t \geq t_0 \Rightarrow \|S(t)u_0\|_{L^p(\Omega)} \leq R.$$

On the other hand, using Lemma 4.2 in [20], when t is such that

$$\|S(t)u_0\|_{L^p(\Omega)} \geq R,$$

it follows that,

$$\|S(t)u_0\|^p_{L^p(\Omega)} \leq e^{-\frac{\sigma p(1-k_1\beta c_1)}{1+\sigma}} \|S(0)u_0\|_{L^p(\Omega)},$$

and consequently

$$\|S(t)u_0\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)}.$$

Since C is bounded, there exists $\rho > 0$ such that $C \subset B(0,\rho)$. Thus, considering $R(C) := \max\{\rho, R\}$, we get the result. □

3.2 Existence of bounded absorbing set in $W^{1,p}(\Omega)$

In this subsection we show the existence of a bounded absorbing set in $W^{1,p}(\Omega)$.

Proposition 3.6. *The space $W^{1,p}(\Omega)$ is positively invariant under semi-group $\{S(t)\}$ generated by (1.1).*

Proof. Let $u_0 \in W^{1,p}(\Omega)$ and $t \geq 0$ be. By Constant Variation Formula (2.4), for any $x \in \Omega$, we can write,

$$S(t)u_0(x) = S_1(t)u_0(x) + S_2(t)u_0(x)$$

where

$$S_1(t)u_0(x) = e^{-t}u_0(x)$$

and

$$S_2(t)u_0(x) = \int_{t_0}^{t} e^{-(t-s)} g\Big(\beta K\big((f \circ S(s)u_0)(x)\big) + \beta h\Big) ds.$$

Clearly $S_1(t)u_0 \in W^{1,p}(\Omega)$. On the other hand, deriving $S_2(t)u_0(x)$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial x} S_2(t)u_0(x)\\ &= \int_{0}^{t} e^{-(t-s)} g'\Big(\beta K\big((f \circ S(s)u_0)(x)\big) + \beta h\Big) \frac{\partial}{\partial x}[\beta K\big((f \circ S(s)u_0)(x)\big)]\; ds.\end{aligned}$$

Using (1.5), (2.1) and (3.5), we have, for any $s \geq 0$,

$$\left| g'\Big(\beta K\big((f \circ S(s)u_0)(x)\big) + \beta h\Big) \right| \leq a_3 \|S(s)u_0\|_{L^p(\Omega)} + a_4,$$

where $a_3 = k_3 \beta \|J\|_q c_1$ and $a_4 := k_3 \beta \|J\|_q c_2 |\Omega|^{\frac{1}{p}} + k_3 \beta h + k_4$. Then, using Lemma 3.4,

$$\begin{aligned}&\left| \frac{\partial}{\partial x} S_2(t)u_0(x) \right|\\ &\leq \beta \int_{0}^{t} e^{-(t-s)} (a_3 \|S(s)u_0\|_{L^p(\Omega)} + a_4)(a_1 \|S(s)u_0\|_{L^p(\Omega)} + a_2)\; ds.\end{aligned}$$

Now, from Proposition 3.5, there exists $r_0 > 0$ such that $\|S(t)u_0\|_{L^p(\Omega)} \leq r_0$, for any $t \geq 0$. Thus,

$$\left|\frac{\partial}{\partial x} S_2(t)u_0(x)\right| \leq \beta(a_3 r_0 + a_4)(a_1 r_0 + a_2)$$

and so, we can to conclud that $S_2(t)u_0 \in W^{1,p}(\Omega)$ for any $t \geq 0$, what completes the proof. □

Lemma 3.7. *The semigroup $\{S(t)\}$ generated by the solutions of (1.1) admits a bounded absorbing set in $W^{1,p}(\Omega)$.*

Proof. Let $B \subset W^{1,p}(\Omega)$ be a bounded set. From Lemma 2.2, there exists $t_0 \geq 0$ such that

$$t \geq t_0 \Rightarrow S(t)B \subset \mathcal{B}, \tag{3.11}$$

where $\mathcal{B}$ is the ball of center at origin in $L^p(\Omega)$ and radius R (where R is given in the Lemma 3.5). Given $t \geq 0$, let $u_0 \in B$ and $u(\cdot,t) = S(t)u_0$ be. Then,

$$\begin{aligned}\frac{d}{dt}\|u_x(\cdot,t)\|^p_{L^p(\Omega)} &= \frac{d}{dt}\int_\Omega |u_x(x,t)|^p dx \\ &= \int_\Omega p|u_x(x,t)|^{p-1}\text{sgn}[u_x(x,t)]u_{tx}(x,t)dx.\end{aligned}$$

From (1.1) we get

$$u_{tx}(x,t) = -u_x(x,t) + g'(\beta K(f\circ u)(x,t) + \beta h)\beta\frac{\partial}{\partial x}K(f\circ u)(x,t).$$

Hence,

$$\begin{aligned}&\int_\Omega p|u_x(x,t)|^{p-1}\text{sgn}[u_x(x,t)]u_{tx}(x,t)dx = \\ &= \int_\Omega p|u_x(x,t)|^{p-1}\text{sgn}[u_x(x,t)](-u_x(x,t))dx \\ &+ \int_\Omega p|u_x(x,t)|^{p-1}\text{sgn}[u_x(x,t)]g'(\beta K(f\circ u)(x,t)+\beta h)\beta\frac{\partial}{\partial x}K(f\circ u)(x,t)dx \\ &\leq -p\int_\Omega |u_x(x,t)|^p dx \\ &+ \int_\Omega p\beta|u_x(x,t)|^{p-1}\left|g'(\beta K(f\circ u)(x,t)+\beta h)\right|\cdot\left|\frac{\partial}{\partial x}K(f\circ u)(x,t)\right|dx.\end{aligned}$$

Proceeding as in the Proposition 3.6 we have

$$\left| g'\Big(\beta K\big((f\circ u)(x,t)\big) + \beta h\Big)\right| \le a_3 \|u(\cdot,t)\|_{L^p(\Omega)} + a_4$$

and so, from (3.11), we obtain, for $t \ge t_0$,

$$\left| g'\Big(\beta K\big((f\circ u)(x,t)\big) + \beta h\Big)\right| \le a_3 R + a_4.$$

Furthermore, using Lemma 3.4 and (3.11) it follows that

$$\left| \frac{\partial}{\partial x} K(f\circ u)(x,t)\right| \le a_1 R + a_2$$

for $t \ge t_0$. Thus,

$$\begin{aligned}&\frac{d}{dt}\|u_x(\cdot,t)\|^p_{L^p(\Omega)}\\ &\le -p\int_\Omega |u_x(x,t)|^p dx + p\beta(a_1R+a_2)(a_3R+a_4)\int_\Omega |u_x(x,t)|^{p-1}dx\end{aligned}$$

for $t \ge t_0$ and, using the Hölder inequality to estimate $\int_\Omega |u_x(x,t)|^{p-1}dx$, we obtain

$$\begin{aligned}&\frac{d}{dt}\|u_x(\cdot,t)\|^p_{L^p(\Omega)}\\ &\le -p\|u_x(\cdot,t)\|^p_{L^p(\Omega)} + p\beta(a_1R+a_2)(a_3R+a_4)\|u_x(\cdot,t)\|^{p-1}_{L^p(\Omega)}|\Omega|^{\frac{1}{p}}.\end{aligned}$$

Since $\|u_x(\cdot,t)\|_{L^p(\Omega)} \ne 0$ (we can assume that, without loss of generality), it implies

$$\frac{d}{dt}\|u_x(\cdot,t)\|^p_{L^p(\Omega)} \le p\|u_x(\cdot,t)\|^p_{L^p(\Omega)}\left[-1 + \frac{\beta(a_1R+a_2)(a_3R+a_4)|\Omega|^{\frac{1}{p}}}{\|u_x(\cdot,t)\|_{L^p(\Omega)}}\right].$$

So while $t \ge t_0$ is such that

$$\|u_x(\cdot,t)\|_{L^p(\Omega)} \ge p\beta(a_1R+a_2)(a_3R+a_4) \tag{3.12}$$

we have

$$\frac{\beta(a_1R+a_2)(a_3R+a_4)|\Omega|^{\frac{1}{p}}}{\|u_x(\cdot,t)\|_{L^p(\Omega)}} \le \frac{1}{p}.$$

Thus,

$$\frac{d}{dt}\|u_x(\cdot,t)\|^p_{L^p(\Omega)} \leq p\|u_x(\cdot,t)\|^p_{L^p(\Omega)}\left(-1+\frac{1}{p}\right) = (-p+1)\|u_x(\cdot,t)\|^p_{L^p(\Omega)}$$

or equivalently

$$\frac{\frac{d}{dt}\|u_x(\cdot,t)\|^p_{L^p(\Omega)}}{\|u_x(\cdot,t)\|^p_{L^p(\Omega)}} \leq -p+1$$

and consequently,

$$\|u_x(\cdot,t)\|_{L^p(\Omega)} \leq e^{\frac{(1-p)}{p}(t-t_0)}\|u_x(\cdot,t_0)\|_{L^p(\Omega)}. \tag{3.13}$$

Hence,

$$\mathcal{B}_0 := \left\{u \in W^{1,p}(\Omega);\ \|u(\cdot,t)\|_{W^{1,p}(\Omega)} \leq R + p\beta(a_1R+a_2)(a_3R+a_4)\right\}$$

is an absorbing set for the flow $S(t)$ in $W^{1,p}(\Omega)$, which concludes the proof. □

From now on, we denote by $r(\mathcal{B}_0)$ the radius of $\mathcal{B}_0$ in $W^{1,p}(\Omega)$, that is

$$r(\mathcal{B}_0) := R + p\beta(a_1R+a_2)(a_3R+a_4).$$

From $\mathcal{B}_0$ we can construct a bounded and positively invariant absorbing set under $\{S(t)\}$ in $W^{1,p}(\Omega)$, satisfying the conditions **(C1)** and **(C2)** .

As $\mathcal{B}_0$ is bounded in $W^{1,p}(\Omega)$, there exists $\overline{t_0} \geq 0$ such that

$$t \geq \overline{t_0} \Rightarrow S(t)\mathcal{B}_0 \subset \mathcal{B}_0.$$

Define

$$\mathcal{B}_0^* = \bigcup_{t\in[0,\overline{t_0}]} S(t)\mathcal{B}_0 \qquad \text{and} \qquad \mathcal{B}_1 = \mathcal{B}_0 \cup \mathcal{B}_0^*. \tag{3.14}$$

Theorem 3.8. *The set $\mathcal{B}_1 \subset W^{1,p}(\Omega)$ given in (3.14) is bounded and it is positively invariant under $\{S(t)\}$.*

Proof. Since $\mathcal{B}_0$ is bounded in $W^{1,p}(\Omega)$, to verify that $\mathcal{B}_1$ is also so, just check that $\mathcal{B}_0^*$ is also bounded.

Let $u_0 \in \mathcal{B}_0$ and $t \in \left[0, \overline{t_0}\right]$ be. Write

$$S(t)u_0 = S_1(t)u_0 + S_2(t)u_0 \tag{3.15}$$

where

$$S_1(t)u_0 = e^{-t}u_0 \quad \text{and} \quad S_2(t)u_0 = \int_{t_0}^{t} e^{-(t-s)} g\Big(\beta K(f \circ S(s)u_0) + \beta h\Big) ds.$$

Note that

$$\|S_1(t)u_0\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)} \qquad \text{and} \qquad \left\|\frac{\partial}{\partial x} S_1(t)u_0\right\|_{L^p(\Omega)} \leq \left\|\frac{\partial u_0}{\partial x}\right\|_{L^p(\Omega)}$$

which implies $\|S_1(t)u_0\|_{W^{1,p}(\Omega)} \leq \|u_0\|_{W^{1,p}(\Omega)}$. Consequently

$$\|S_1(t)u_0\|_{W^{1,p}(\Omega)} \leq r(\mathcal{B}_0). \tag{3.16}$$

Now, by (3.15) we have $S_2(t)u_0 = S(t)u_0 - S_1(t)u_0$. Then, from (3.16),

$$\|S_2(t)u_0\|_{L^p(\Omega)} \leq \|S(t)u_0\|_{L^p(\Omega)} + r(\mathcal{B}_0).$$

Using Proposition 3.5, it follows that there exists $R(\mathcal{B}_0) > 0$ such that,

$$\|S_2(t)u_0\|_{L^p(\Omega)} \leq R(\mathcal{B}_0) + r(\mathcal{B}_0).$$

On the other hand, proceeding once more in a similar way to the proof of the Proposition 3.6, we obtain for all $x \in \Omega$,

$$\left|\frac{\partial}{\partial x} S_2(t)u_0(x)\right| \leq \beta \left(a_3 R(\mathcal{B}_0) + a_4\right)\left(a_1 R(\mathcal{B}_0) + a_2\right),$$

so that

$$\left\|\frac{\partial}{\partial x} S_2(t)u_0\right\|_{L^p(\Omega)} \leq \beta \left(a_3 R(\mathcal{B}_0) + a_4\right)\left(a_1 R(\mathcal{B}_0) + a_2\right) |\Omega|^{\frac{1}{p}}.$$

Hence

$$\|S_2(t)u_0\|_{W^{1,p}(\Omega)} \leq R(\mathcal{B}_0) + r(\mathcal{B}_0) + \beta \left(a_3 R(\mathcal{B}_0) + a_4\right)\left(a_1 R(\mathcal{B}_0) + a_2\right) |\Omega|^{\frac{1}{p}}. \tag{3.17}$$

From (3.15), (3.16) and (3.17) it folows that

$$\|S(t)u_0\|_{W^{1,p}(\Omega)} \leq R(\mathcal{B}_0) + 2r(\mathcal{B}_0) + \beta\left(a_3 R(\mathcal{B}_0) + a_4\right)\left(a_1 R(\mathcal{B}_0) + a_2\right)|\Omega|^{\frac{1}{p}}.$$

As $t \in \left[0, \overline{t_0}\right]$ and $u_0 \in \mathcal{B}_0$ were taken arbitrarily, we have that $\mathcal{B}_0^*$ is bounded and, consequently, $\mathcal{B}_1$ is also bounded.

It follows from the inclusion $\mathcal{B}_0 \subset \mathcal{B}_1$ that $\mathcal{B}_1$ is an absorbing set for $S(t)$ in $W^{1,p}(\Omega)$.

To check the positive invariance of $\mathcal{B}_1$, let $t \geq 0$. If $t > \overline{t_0}$ then $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ and more, $t + r > \overline{t_0}$ so that $S(t+r)\mathcal{B}_0 \subset \mathcal{B}_0$, for each $r \in \left[0, \overline{t_0}\right]$. Thus,

$$S(t)\mathcal{B}_0^* = \left(\bigcup_{r\in\left[0,\overline{t_0}\right]} S(t+r)\mathcal{B}_0\right) \subset \mathcal{B}_0.$$

Consequently $S(t)\mathcal{B}_1 \subset \mathcal{B}_1$.

Now, suppose that $0 \leq t \leq \overline{t_0}$. In this case $S(t)\mathcal{B}_0 \subset \mathcal{B}_0^*$. Noting that

$$S(\xi)\mathcal{B}_0^* \subset \bigcup_{\xi\leq r\leq \xi+\overline{t_0}} S(r)\mathcal{B}_0, \ \forall\, \xi \geq 0, \tag{3.18}$$

we have,

$$S(t)\mathcal{B}_0^* \subset \bigcup_{t\leq r\leq t+\overline{t_0}} S(r)\mathcal{B}_0 = \left(\bigcup_{t\leq r\leq \overline{t_0}} S(r)\mathcal{B}_0\right) \cup \left(\bigcup_{\overline{t_0}\leq r\leq t+\overline{t_0}} S(r)\mathcal{B}_0\right). \tag{3.19}$$

Since $t \leq r \leq \overline{t_0}$ implica em $r \in \left[0, \overline{t_0}\right]$ it follows immediately

$$\bigcup_{t\leq r\leq \overline{t_0}} S(r)\mathcal{B}_0 \subset \mathcal{B}_0^*. \tag{3.20}$$

On the other hand $r \geq \overline{t_0}$ implica rm $S(r)\mathcal{B}_0 \subset \mathcal{B}_0$ and so,

$$\bigcup_{\overline{t_0}\leq r\leq t+\overline{t_0}} S(r)\mathcal{B}_0 \subset \mathcal{B}_0. \tag{3.21}$$

Using (3.19), (3.20) and (3.21) it folows that $S(t)\mathcal{B}_0^* \subset \mathcal{B}_1$. Since $S(t)\mathcal{B}_0 \subset \mathcal{B}_1$, it folows that $S(t)\mathcal{B}_1 \subset \mathcal{B}_1$, which concludes the proof. □

Corollary 3.9. *The absorbing set $\mathcal{B}_1$ is positively invariant under $\{S(t)\}$ in $L^p(\Omega)$.*

Proof. In fact, for any $t \geq 0$ we have $S(t)\mathcal{B}_1 \subset \mathcal{B}_1$ in $W^{1,p}(\Omega)$. Considering the map $I : W^{1,p}(\Omega) \to L^p(\Omega)$, it follows that $I\big(S(t)\mathcal{B}_1\big) \subset I(\mathcal{B}_1)$ in $L^p(\Omega)$, for any $t \geq 0$, completing the result. □

3.3 Verification of the Conditions (C4) and (C5)

In this section, $B \subset L^p(\Omega)$ will always denote a bounded set. From Proposition 3.5 there exists $R(B) > 0$ such that

$$\|S(t)u\|_{L^p(\Omega)} \leq R(B), \tag{3.22}$$

for any $t \geq 0$ and $u \in B$. Using (3.22) and Lemmas 2.1 and 3.2 we obtain

$$|K(f \circ S(t)u)| \leq \|J\|_q \left(c_1 R(B) + c_2 |\Omega|^{\frac{1}{p}}\right).$$

Thus, writing $\tau(B) := \beta \left[\|J\|_q \left(c_1 R(B) + c_2 |\Omega|^{\frac{1}{p}}\right) + h\right]$, it follows that

$$|\beta K(f \circ S(t)u) + \beta h| \leq \tau(B).$$

Since g is of class C^1, it follows that g' is bounded in $[-\tau(B), \tau(B)]$. Hence g is Lipschitz in $[-\tau(B), \tau(B)]$.

Lemma 3.10. *Let $u_0, v_0 \in B$, $s \geq 0$, $u(\cdot, s) = S(s)u_0$ and $v(\cdot, s) = S(s)v_0$ be. Then,*

$$\|f \circ u(\cdot, s) - f \circ v(\cdot, s)\|_{L^1(\Omega)} \leq \Gamma(B) \|u(\cdot, s) - v(\cdot, s)\|_{L^p(\Omega)},$$

where $\Gamma(B) := c_0 \left(|\Omega|^{\frac{1}{q}} + 2[R(B)]^{\frac{p}{q}}\right)$.

Proof. From (1.6) we have

$$\begin{aligned}
&|f(u(x, s)) - f(v(x, s))| \\
&\leq c_0 (1 + |u(x, s)|^{p-1} + |v(x, s)|^{p-1}) |u(x, s) - v(x, s)|
\end{aligned}$$

which implies

$$\begin{aligned}&\|(f\circ u)(\cdot,s)-(f\circ v)(\cdot,s)\|_{L^1(\Omega)}\\&\leq\int_\Omega c_0(1+|u(x,s)|^{p-1}+|v(x,s)|^{p-1})|u(x,s)-v(x,s)|dx.\end{aligned}$$

Using Hölder inequality, it follows that

$$\begin{aligned}&\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}\\&\leq c_0\|1+u^{p-1}(\cdot,s)+v^{p-1}(\cdot,s)\|_{L^q(\Omega)}\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}\end{aligned}$$

and by the triangular inequality, we have

$$\begin{aligned}&\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}\leq\\&\leq c_0\left(|\Omega|^{\frac{1}{q}}+\|u^{p-1}(\cdot,s)\|_{L^q(\Omega)}+\|v^{p-1}(\cdot,s)\|_{L^q(\Omega)}\right)\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}\\&=c_0\left(|\Omega|^{\frac{1}{q}}+\|u(\cdot,s)\|^{\frac{p}{q}}_{L^p(\Omega)}+\|v(\cdot,s)\|^{\frac{p}{q}}_{L^p(\Omega)}\right)\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}.\end{aligned}$$

From (3.22) we have $\|u(\cdot,s)\|_{L^p(\Omega)}\leq R(B)$ and $\|v(\cdot,s)\|_{L^p(\Omega)}\leq R(B)$. Thus,

$$\begin{aligned}&\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}\\&\leq c_0\left(|\Omega|^{\frac{1}{q}}+2[R(B)]^{\frac{p}{q}}\right)\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}\end{aligned}$$

which concludes the proof. □

Proposition 3.11. *For any $u_0,v_0\in B$ and $t\geq 0$*

$$\|S(t)u_0-S(t)v_0\|_{L^p(\Omega)}\leq e^{[N(B)\Gamma(B)\beta\|J\|_p-1]t}\|u_0-v_0\|_{L^p(\Omega)},$$

where $N(B)$ is the Lipschitz constant of g in $[-\tau(B),\tau(B)]$.

Proof. Let $u_0,v_0\in B$, $u(\cdot,t)=S(t)u_0$ and $v(\cdot,t)=S(t)v_0$ be. By the variation of constants formula, we have

$$\begin{aligned}&\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)}\leq e^{-t}\|u_0-v_0\|_{L^p(\Omega)}\\&+\int_0^t e^{-(t-s)}\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)}ds.\end{aligned}\tag{3.23}$$

Using the fact that g is lipschitz and the linearity of K, we obtain

$$\begin{aligned}&\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)}\\&\leq N(B)\beta\|K(f\circ u-f\circ v)(\cdot,s)\|_{L^p(\Omega)}.\end{aligned}$$

From Lemmas 2.1 and 3.10 it follows that

$$\begin{aligned}&\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)}\\&\leq N(B)\beta\|J\|_p\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}\\&\leq N(B)\Gamma(B)\beta\|J\|_p\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}.\end{aligned}\tag{3.24}$$

By (3.23) it results

$$\begin{aligned}&\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)}\leq e^{-t}\|u_0-v_0\|_{L^p(\Omega)}\\&+\int_0^t e^{-(t-s)}N(B)\Gamma(B)\beta\|J\|_p\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}ds,\end{aligned}$$

or equivalently,

$$\begin{aligned}&e^t\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)}\\&\leq\|u_0-v_0\|_{L^p(\Omega)}+\int_0^t N(B)\Gamma(B)\beta\|J\|_p e^s\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}ds.\end{aligned}$$

Using Gronwall inequality, we have

$$e^t\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)}\leq e^{N(B)\Gamma(B)\beta\|J\|_p t}\|u_0-v_0\|_{L^p(\Omega)}$$

and multiplying both members by e^{-t}, the result follows. □

Corollary 3.12. *If $I\subset\mathbb{R}$ is an bounded interval, then the map*

$$\begin{aligned}F_1:\ &I\times\mathcal{B}_1\to\mathcal{B}_1\\&(t,u)\mapsto F(t,u):=S(t)u\end{aligned}$$

is Lipschitz in $\mathcal{B}_1$ in the topology of $L^p(\Omega)$, uniformly in I.

Proposition 3.13. *There exists a constant C_0 such that*

$$\left\|\frac{d}{dt}S(t)u\right\|_{L^p(\Omega)}\leq C_0$$

for any $u\in\mathcal{B}_1$ and $t\geq 0$.

Proof. Let $u \in \mathcal{B}_1$ and $t \geq 0$ be. Since

$$\frac{d}{dt}S(t)u = -S(t)u + g\big(\beta K(f \circ S(t)u) + \beta h\big),$$

then,

$$\left\|\frac{d}{dt}S(t)u\right\|_{L^p(\Omega)} \leq \|S(t)u\|_{L^p(\Omega)} + \big\|g\big(\beta K(f \circ S(t)u) + \beta h\big)\big\|_{L^p(\Omega)}.$$

Using (1.3), Lemma 2.1 and Lemma 3.2 it follows that, for $x \in \Omega$,

$$\big|g\big(\beta K(f \circ S(t)u)(x) + \beta h\big)\big| \leq \alpha_1 \|S(t)u\|_{L_p(\Omega)} + \alpha_2,$$

where $\alpha_1 = k_1\beta c_1\|J\|_q$ and $\alpha_2 = k_1\beta c_2\|J\|_q|\Omega|^{\frac{1}{p}} + k_1\beta h + k_2$. Writing $R_0 := R(\mathcal{B}_1)$, from Proposition 3.5 we have

$$\big|g\big(\beta K(f \circ S(t)u)(x) + \beta h\big)\big| \leq \alpha_1 R_0 + \alpha_2.$$

Thus,

$$\big\|g\big(\beta K(f \circ S(t)u) + \beta h\big)\big\|_{L^p(\Omega)} \leq (\alpha_1 R_0 + \alpha_2)|\Omega|^{\frac{1}{p}}.$$

Hence,

$$\left\|\frac{d}{dt}S(t)u\right\|_{L^p(\Omega)} \leq R_0 + (\alpha_1 R_0 + \alpha_2)|\Omega|^{\frac{1}{p}},$$

this completes the result with $C_0 = R_0 + (\alpha_1 R_0 + \alpha_2)|\Omega|^{\frac{1}{p}}$. □

The corollary below has immediate proof.

Corollary 3.14. *The map*

$$\begin{array}{rrcl} F_2: & \mathbb{R}_+ \times \mathcal{B}_1 & \to & \mathcal{B}_1 \\ & (t,u) & \mapsto & F(t,u) := S(t)u \end{array}$$

is Lipschitz in $\mathbb{R}_+$, uniformly in $\mathcal{B}_1$ with constant of Lipschitz C_0.

Corollary 3.15. *If $I \subset \mathbb{R}$ is a bounded interval, then the map*

$$\begin{array}{rrcl} F_1: & I \times \mathcal{B}_1 & \to & \mathcal{B}_1 \\ & (t,u) & \mapsto & F(t,u) := S(t)u \end{array}$$

is Lipschitz in $I \times \mathcal{B}_1$.

Proof. Let $\gamma_0 := \sup_{t\in I} e^{(N\beta\Gamma\|J\|_p - 1)t}$ and $\overline{C} := \max\{\gamma_0, C_0\}$ be. Given $(t_1,u_1),(t_2,u_2) \in [t^*, 2t^*] \times \mathcal{B}_1$ we have

$$\begin{aligned} &\|S(t_1)u_1 - S(t_2)u_2\|_{L^p(\Omega)} \leq \\ &\|S(t_1)u_1 - S(t_1)u_2\|_{L^p(\Omega)} + \|S(t_1)u_2 - S(t_2)u_2\|_{L^p(\Omega)}. \end{aligned}$$

Using Proposition 3.11 and Corolary 3.14 it follows that

$$\begin{aligned} &\|S(t_1)u_1 - S(t_2)u_2\|_{L^p(\Omega)} \\ &\leq \gamma_0\|u_1 - u_2\|_{L^p(\Omega)} + C_0|t_1 - t_2| \leq \overline{C}(|t_1 - t_2| + \|u_1 - u_2\|_{L^p(\Omega)}). \end{aligned}$$

Hence,

$$\|S(t_1)u_1 - S(t_2)u_2\|_{L^p(\Omega)} \leq \overline{C}\|(t_1 - t_2, u_2 - u_2)\|_{\mathbb{R}\times L^p(\Omega)}.$$

Therefore

$$\|F(t_1,u_1) - F(t_2,u_2)\|_{L^p(\Omega)} \leq \overline{C}\|(t_1,u_1) - (t_2,u_2)\|_{\mathbb{R}\times L^p(\Omega)}.$$

□

3.4 Estimate for difference of two solutions

Let $u_0, v_0 \in L^p(\Omega)$ be. Given $t \geq 0$, consider the solutions $u(\cdot,t) = S(t)u_0$ and $v(\cdot,t) = S(t)v_0$. Define $z(t) := u(\cdot,t) - v(\cdot,t)$ and $z_0 = u_0 - z_0$ and note that z is the solution of Cauchy's problem

$$\begin{cases} \dfrac{dy}{dt} + y = g(\beta K(f \circ S(t)u_0) + \beta h) - g(\beta K f \circ S(t)v_0 + \beta h) \\ y(0) = z_0 \end{cases} \tag{3.25}$$

in $L^p(\Omega)$. Furthermore, the Cauchy-Lipschitz-Picard Theorem (see [10]) gives uniqueness of this solution. Now, consider the problems of Cauchy

$$\begin{cases} \dfrac{d\phi}{dt} + \phi = 0 \\ \phi(0) = z_0 \end{cases} \tag{3.26}$$

and

$$\begin{cases} \dfrac{d\theta}{dt} + \theta = g(\beta K(f \circ u) + \beta h) - g(\beta K(f \circ v) + \beta h) \\ \theta(0) = 0 \end{cases} \tag{3.27}$$

By Cauchy-Lipschitz-Picard Theorem, we have the existence and uniqueness of solutions for the two problems, both belonging to $C^1\big([0,\infty), L^p(\Omega)\big)$. We will denote by $\phi(\cdot,t,u_0,v_0)$ and $\theta(\cdot,t,u_0,v_0)$ the solutions, in $L^p(\Omega)$, of problems (3.26) and (3.27), respectively. Defining $\psi(t) = \phi(\cdot,t,u_0,v_0) + \theta(\cdot,t,u_0,v_0)$ we have that ψ is solution of (3.25) and, by uniqueness of the solution, it follows that $z = \psi$, that is,

$$u(\cdot,t) - v(\cdot,t) = \phi(\cdot,t,u_0,v_0) + \theta(\cdot,t,u_0,v_0), \ \forall\, t \geq 0. \tag{3.28}$$

Lemma 3.16. *For any $u_0, v_0 \in L^p(\Omega)$ and any $t > \ln 3$ we have*

$$\|\phi(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} < \frac{1}{3}\|u_0 - v_0\|_{L^p(\Omega)}.$$

Proof. For simplicity, we write $\phi(\cdot,t,u_0,v_0) = \phi(\cdot,t)$, for any $t \geq 0$. Then

$$\frac{d}{dt}\|\phi(\cdot,t)\|^p_{L^p(\Omega)} + p\|\phi(\cdot,t)\|^p_{L^p(\Omega)} = \frac{d}{dt}\int_\Omega |\phi(x,t)|^p dx + p\int_\Omega |\phi(x,t)|^p dx.$$

Note that $\phi(t,x) = z_0(x)e^{-t}$, for any $x \in \Omega$ and $t \geq 0$. Thus,

$$\begin{aligned} \left|\frac{\partial}{\partial t}|\phi(x,t)|^p\right| &= \left|p|\phi(x,t)|^{p-1}\mathrm{sgn}[\phi(x,t)]\frac{\partial}{\partial t}\phi(x,t)\right| = p|\phi(x,t)|^{p-1}|-\phi(x,t)| \\ &= p|\phi(x,t)|^p = p\left|z_0(x)e^{-t}\right|^p \leq p|z_0(x)|^p. \end{aligned}$$

Since $p|z_0|^p \in L^1(\Omega)$, from Dominated Convergence Theorem (see [9], Theorem 2.27), we can derive under the integration signal, getting

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\phi(x,t)|^p dx &= \int_\Omega p|\phi(x,t)|^{p-1}\mathrm{sgn}[\phi(x,t)]\phi_t(x,t)dx \\ &= \int_\Omega p|\phi(x,t)|^{p-1}\mathrm{sgn}[\phi(x,t)]\big(-\phi(x,t)\big)dx \\ &= -p\int_\Omega |\phi(x,t)|^p dx.\end{aligned}$$

Thus,

$$\frac{d}{dt}\|\phi(\cdot,t)\|^p_{L^p(\Omega)} + p\|\phi(\cdot,t)\|^p_{L^p(\Omega)} = 0$$

and consequently

$$\frac{\frac{d}{dt}\|\phi(\cdot,t)\|^p_{L^p(\Omega)}}{\|\phi(\cdot,t)\|^p_{L^p(\Omega)}} = -p.$$

Thus, we obtain

$$\|\phi(\cdot,t)\|_{L^p(\Omega)} = e^{-t}\|\phi(\cdot,0)\|_{L^p(\Omega)} = e^{-t}\|z_0\|_{L^p(\Omega)}.$$

For $t > \ln 3$ we have $e^{-t} < \frac{1}{3}$. Therefore,

$$\|\phi(\cdot,t)\|_{L^p(\Omega)} < \frac{1}{3}\|z_0\|_{L^p(\Omega)} = \frac{1}{3}\|u_0 - v_0\|_{L^p(\Omega)},$$

which concludes the result. □

Fron now on, to simplify, we will write τ instead of $\tau(\mathcal{B}_1)$.

Lemma 3.17. *Fixed $t^* > \ln 3$, there exists $C > 0$ such that*

$$\|\theta(\cdot,t,u_0,v_0)\|_{W^{1,p}(\Omega)} \leq C\|u_0 - v_0\|_{L^p(\Omega)}$$

for any $u_0, v_0 \in \mathcal{B}_1$ and t in $[0,t^*]$.

Proof. Let $u_0, v_0 \in \mathcal{B}_1$, $t \in [0, t^*]$, $u(\cdot, t) = S(t)u_0$ and $v(\cdot, t) = S(t)v_0$ be. For the problem (3.27), the variation of constant formula is given by

$$\theta(x, t, u_0, v_0) = \int_0^t e^{-(t-s)} \big[g(\beta K(f \circ u)(x, s) + \beta h) \tag{3.29}$$

$$- g(\beta K(f \circ v)(x, s) + \beta h) \big] ds, \tag{3.30}$$

for any $x \in \mathbb{R}^n$ and $t \geq 0$. Derivating with respect to x we get

$$\theta_x(x, t, u_0, v_0) = \beta \int_0^t e^{-(t-s)} \Big[g'(\beta K(f \circ u)(x, s) + \beta h) \frac{\partial}{\partial x} K(f \circ u)(x, s) - g'(\beta K(f \circ v)(x, s) + \beta h) \frac{\partial}{\partial x} K(f \circ v)(x, s) \Big] ds. \tag{3.31}$$

Let $M = \|g'\|_{L^\infty([-\tau, \tau])}$ be. Then, we have

$$|\theta_x(x, t, u_0, v_0)| \leq \beta M \int_0^t e^{-(t-s)} \left| \frac{\partial}{\partial x} K(f \circ u)(x, s) \right| ds + \beta M \int_0^t e^{-(t-s)} \left| \frac{\partial}{\partial x} K(f \circ v)(x, s) \right| ds.$$

Using Lemma 3.3, Lemma 3.2 and as $S(t)\mathcal{B}_1 \subset B(0, R_0)$, we obtain

$$|\theta_x(x, t, u_0, v_0)| \leq 2\beta M \|J_x\|_q (c_1 R_0 + c_2 |\Omega|^{\frac{1}{p}}).$$

Therefore $\theta_x(\cdot, t, u_0, v_0) \in L^p(\Omega)$. Now, adding and subtracting the term $g'(\beta K(f \circ u)(\cdot, s) + \beta h) \frac{\partial}{\partial x} K(f \circ v)(\cdot, s)$ in the integration in (3.31) and rearranging the terms, we have

$$\theta_x(\cdot, t, u_0, v_0) = \beta \int_0^t e^{-(t-s)} \left\{ g'(\beta K(f \circ u)(\cdot, s) + \beta h) \left[\frac{\partial}{\partial x} K(f \circ u)(\cdot, s) - \frac{\partial}{\partial x} K(f \circ v)(\cdot, s) \right] + \left[g'(\beta K(f \circ u)(\cdot, s) + \beta h) - g'(\beta K(f \circ v)(\cdot, s) + \beta h) \right] \frac{\partial}{\partial x} K(f \circ v)(\cdot, s) \right\} ds.$$

Since $|g'(x)| \leq M$ for all $x \in [-\tau, \tau]$, it follows that

$$\left\| g'(\beta K(f \circ u)(\cdot, s) + \beta h) \left[\frac{\partial}{\partial x} K(f \circ u)(\cdot, s) - \frac{\partial}{\partial x} K(f \circ v)(\cdot, s) \right] \right\|_{L^p(\Omega)} \leq M \left\| \frac{\partial}{\partial x} K(f \circ u(\cdot, s) - f \circ v(\cdot, s)) \right\|_{L^p(\Omega)}$$

and, from Lemma (3.3) results

$$\left\| g'(\beta K(f\circ u)(\cdot,s)+\beta h)\left[\frac{\partial}{\partial x}K(f\circ u)(\cdot,s)-\frac{\partial}{\partial x}K(f\circ v)(\cdot,s)\right]\right\|_{L^p(\Omega)}$$
$$\leq M\|J_x\|_p\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}.$$

On the other hand, let L be the Lipschitz constant of g' in $[-\tau,\tau]$. Using (3.7), (3.5) and Lemma 2.1 we obtain,

$$\begin{aligned}&\Big\|\Big[g'(\beta K(f\circ u)(\cdot,s)+\beta h)\\ &-g'(\beta K(f\circ v)(\cdot,s)+\beta h)\Big]\frac{\partial}{\partial x}K(f\circ v)(\cdot,s)\Big\|_{L^p(\Omega)}\\ &\leq L\beta\|J_x\|_q\|f\circ v(\cdot,s)\|_{L^p(\Omega)}\|K(f\circ u-f\circ v)(\cdot,s)\|_{L^p(\Omega)}\\ &\leq L\beta\|J_x\|_q\|J\|_p(c_1R_0+c_2|\Omega|^{\frac{1}{p}})\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}.\end{aligned}$$

Therefore,

$$\|\theta_x(\cdot,t,u_0,v_0)\|_{L^p(\Omega)}\leq\beta\int_0^t e^{-(t-s)}\Big\{M\|J_x\|_p\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}$$
$$+L\beta\|J_x\|_q\|J\|_p(c_1R_0+c_2|\Omega|^{\frac{1}{p}})\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}\Big\}ds.$$

Now, defining $\gamma:=\beta\big[M\|J_x\|_p+L\|J_x\|_q\|J\|_p(c_1R_0+c_2|\Omega|^{\frac{1}{p}})\big]$ we can rewrite the inequality above as follows

$$\|\theta_x(\cdot,t,u_0,v_0)\|_{L^p(\Omega)}\leq\gamma\int_0^t e^{-(t-s)}\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^1(\Omega)}ds.$$

From Lemma (3.10) it follows that

$$\|\theta_x(\cdot,t,u_0,v_0)\|_{L^p(\Omega)}\leq\gamma\int_0^t e^{-(t-s)}\mathit{\Gamma}\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}ds,$$

where $\mathit{\Gamma}:=c_0\left(|\Omega|^{\frac{1}{q}}+2R_0^{\frac{p}{q}}\right)$. Using Proposition 3.11 and rearranging terms with exponentials, we have

$$\|\theta_x(\cdot,t,u_0,v_0)\|_{L^p(\Omega)}\leq\gamma e^{-t}\|u_0-v_0\|_{L^p(\Omega)}\int_0^t e^{N\beta\mathit{\Gamma}\|J\|_p s}ds,$$

where N denotes the Lipschitz constante of g in $[-\tau,\tau]$. Solving the integral on the right and disregarding the negative terms, we see that

$$\|\theta_x(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq \frac{\gamma}{N\beta\Gamma\|J\|_p} e^{(N\beta\Gamma\|J\|_p-1)t}\|u_0-v_0\|_{L^p(\Omega)}$$

and so,

$$\|\theta_x(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq \frac{\gamma}{N\beta\Gamma\|J\|_p} e^{|N\beta\Gamma\|J\|_p-1|t^*}\|u_0-v_0\|_{L^p(\Omega)}. \quad (3.32)$$

Now, by (3.29), we obtain

$$\|\theta(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq \int_0^t e^{-(t-s)}\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)}ds.$$

Proceeding as done to get (3.24), we conclude that

$$\begin{aligned}\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)}\\ \leq N\beta\Gamma\|J\|_p\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}\end{aligned}$$

and, from Proposition 3.11 we obtain

$$\begin{aligned}\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)}\\ \leq N\beta\Gamma\|J\|_p e^{(N\beta\Gamma\|J\|_p-1)s}\|u_0-v_0\|_{L^p(\Omega)}.\end{aligned}$$

Therefore,

$$\|\theta(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq \int_0^t e^{-(t-s)}N\beta\Gamma\|J\|_p e^{(N\beta\Gamma\|J\|_p-1)s}\|u_0-v_0\|_{L^p(\Omega)}ds,$$

wich implies

$$\|\theta(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq \|u_0-v_0\|_{L^p(\Omega)}e^{-t}\int_0^t N\beta\Gamma\|J\|_p e^{N\beta\Gamma\|J\|_p s}ds,$$

that is,

$$\|\theta(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq \|u_0-v_0\|_{L^p(\Omega)}e^{-t}\int_0^t \frac{d}{ds}\left(e^{N\beta\Gamma\|J\|_p s}\right)ds.$$

Consequently, by Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}\|\theta(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} &\leq \|u_0 - v_0\|_{L^p(\Omega)} e^{-t} e^{N\beta\Gamma\|J\|_p t} ds \\ &= e^{(N\beta\Gamma\|J\|_p - 1)t}\|u_0 - v_0\|_{L^p(\Omega)}.\end{aligned}$$

Hence,

$$\|\theta(\cdot,t,u_0,v_0)\|_{L^p(\Omega)} \leq e^{|N\beta\Gamma\|J\|_p - 1|t^*}\|u_0 - v_0\|_{L^p(\Omega)}. \tag{3.33}$$

Using (3.33) and (3.32) it follows that

$$\|\theta(\cdot,t,u_0,v_0)\|_{W^{1,p}(\Omega)} \leq \left(1 + \frac{\gamma}{N\beta\Gamma\|J\|_p}\right) e^{|N\beta\Gamma\|J\|_p - 1|t^*}\|u_0 - v_0\|_{L^p(\Omega)}.$$

Then the result is obtained with $C := \left(1 + \frac{\gamma}{N\beta\Gamma\|J\|_p}\right) e^{|N\beta\Gamma\|J\|_p - 1|t^*}$.

□

Proposition 3.18. *For any $t \geq 0$ and $u_1, u_2 \in L^p(\Omega)$,*

$$\phi(\cdot,t,u_1 - u_2, 0) = \phi(\cdot,t,u_1,0) - \phi(\cdot,t,u_2,0).$$

Proof. Denote by $x = \phi(\cdot,t,u_1,0)$, the solution of the problem

$$\begin{cases} \phi_t + \phi = 0 \\ \phi(0) = u_1 \end{cases}$$

and by $y = \phi(\cdot,t,u_2,0)$ is solution of the problem

$$\begin{cases} \phi_t + \phi = 0 \\ \phi(0) = u_2, \end{cases}.$$

Defining $\psi = x - y$, it follows that ψ is a solution to the Cauchy Problem

$$\begin{cases} \phi_t + \phi = 0 \\ \phi(0) = u_1 - u_2 \end{cases}$$

and, by uniqueness of solution, it results that $\psi = \phi(\cdot,t,u_1 - u_2,0)$, concluding the result. □

Proposition 3.19. *For any* $t \geq 0$ *and* $u_1, u_2 \in L^p(\Omega)$,

$$\theta(\cdot, t, u_1, 0) - \theta(\cdot, t, u_2, 0) = \theta(\cdot, t, u_1, u_2).$$

Proof. Let $x = \theta(\cdot, t, u_1, 0)$ and $y = \theta(\cdot, t, u_2, 0)$ be the solution of the problems

$$\begin{cases} \theta_t + \theta = g(\beta K(f \circ S(t)u_1) + \beta h) - g(\beta K(f \circ S(t) \cdot 0) + \beta h) \\ \theta(0) = 0 \end{cases}$$

and

$$\begin{cases} \theta_t + \theta = g(\beta K(f \circ S(t)u_2) + \beta h) - g(\beta K(f \circ S(t) \cdot 0) + \beta h) \\ \theta(0) = 0 \end{cases}$$

respectively. Define $\psi := x - y$, it follows that ψ is a solution to the Cauchy Problem

$$\begin{cases} \theta_t + \theta = g(\beta K(f \circ S(t)u_1) + \beta h) - g(\beta K(f \circ S(t)u_2) + \beta h) \\ \theta(0) = 0 \end{cases}$$

and, by uniqueness of solution, it results that $\psi = \theta(\cdot, t, u_1, u_2)$, concluding the result. $\square$

Define the operators $L, G : \mathcal{B}_1 \times \mathcal{B}_1 \to L^p(\Omega)$ given by

$$L(u_0, v_0) := \phi(\cdot, t^*, u_0, v_0) \tag{3.34}$$

and

$$G(u_0, v_0) = \theta(\cdot, t^*, u_0, v_0). \tag{3.35}$$

Now define $L_0 : \mathcal{B}_1 \to L^p(\Omega)$ and $G_0 : \mathcal{B}_1 \to L^p(\Omega)$ por

$$L_0(u) = L(u, 0) + S(t^*) \cdot 0 \quad \text{and} \quad G_0(u) = G(u, 0). \tag{3.36}$$

From Proposition 3.18 and Lemma 3.16 we get

$$\|L_0(u_1) - L_0(u_2)\|_{L^p(\Omega)} < \frac{1}{3}\|u_1 - u_2\|_{L^p(\Omega)}, \tag{3.37}$$

for all $u_1, u_2 \in \mathcal{B}_1$. Analogously, using Proposition 3.19 and Lemma 3.17 we obtain, for all $u_1, u_2 \in \mathcal{B}_1$,

$$\|G_0(u_1) - G_0(u_2)\|_{W^{1,p}(\Omega)} < C\|u_1 - u_2\|_{L^p(\Omega)}. \tag{3.38}$$

Furthemore, by (3.28) we have

$$S(t^*)u_0 - S(t^*)v_0 = L(u_0, v_0) + G(u_0, v_0),$$

for any $u_0, v_0 \in \mathcal{B}_1$. Therefore

$$S(t^*)u = L_0(u) + G_0(u), \ \forall\ u \in \mathcal{B}_1. \tag{3.39}$$

3.5 Existence of exponential attractor for the discrete semigroup $S^n(t^*)$

Since $W^{1,p}(\Omega)$ is compactly immersed in $L^p(\Omega)$ (see [10], Rellich- Kondrachov Theorem), $\mathcal{B}_1$ is bounded and the equations (3.37), (3.38) and (3.39) hold, the map $S(t^*)$ satisfy the conditions of Theorem 3.1. Consequently, we have the existence of a set $\mathcal{M}^* \subset \overline{\mathcal{B}_1}^{L^p(\Omega)}$ satisfying the following properties:

(i) $S^n(t^*)(\mathcal{M}^*) \subset \mathcal{M}^*$, for any $n \in \mathbb{N}$;

(ii) $\dim_F(\mathcal{M}^*, L^p(\Omega)) < \infty$;

(iii) There exist $\alpha^*, \omega^* > 0$ such that

$$\operatorname{dist}_{L^p(\Omega)}(S^n(t^*)(\mathcal{B}_1, \mathcal{M}^*) \le \alpha^* e^{-\omega^* n}, \ \forall\ n \in \mathbb{N}; \tag{3.40}$$

(iv) $\mathcal{M}^*$ is closed in $L^p(\Omega)$.

Remark 3.20. (i) The notation $S^n(t^*)$ indicates the composition of $S(t^*)$ with itself n times, that is,

$$S^n(t^*) = \underbrace{S(t^*) \circ S(t^*) \circ \cdots S(t^*)}_{n \text{ times}}.$$

(ii) From semigroup properties it follows that

$$S^n(t^*) = S(nt^*),\ \forall\, n \in \mathbb{N}.$$

(iii) The set $\mathcal{M}^*$ is compact in $L^p(\Omega)$. In fact, since $\mathcal{B}_1$ is bounded in $W^{1,p}(\Omega)$, the compact immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ implies that $\overline{\mathcal{B}_1}^{L^p(\Omega)}$ is compact in $L^p(\Omega)$. Since $\mathcal{M}^*$ is closed in $L^p(\Omega)$, it follows that $\mathcal{M}^*$ is compact.

4 Exponential attractor for continuous semigroup

In this section, using the same tecniques of [2], from $\mathcal{M}^*$, we build an exponential attractor for the semigroup generated by the solutions of (1.1) in X. For this, we started with the following Proposition for fractal dimension properties, which can be proved by proceeding as in [2].

Proposition 4.1. *Let H_1 and H_2 be Banach spaces.*

(i) *If K_1 and K_2 are compact sets in H_1, then*

$$K_1 \subset K_2 \Rightarrow \dim_F(K_1, H_1) \leq \dim_F(K_2, H_1).$$

(ii) *If K_1 is compact in H_1 and K_2 is compact in H_2, then*

$$\dim_F(K_1 \times K_2, H_1 \times H_2) \leq \dim_F(K_1, H_1) + \dim_F(K_2, H_2).$$

(iii) *If $K \subset H_1$ is a compact subset and $f : K \to H_2$ is Lipschitz, then we have*

$$\dim_F \big(f(K), H_2\big) \leq \dim_F \big(K, H_1\big).$$

(iv) *$I \subset \mathbb{R}$ is a closed and bounded, then* $\dim_F(I, \mathbb{R}) = 1$.

Using Propositin 4.1 and the results from the previous section, we obtain the following result:

Theorem 4.2. *The flow $\{S(t)\}_{t\geq 0}$ has a compact set $\mathcal{M} \subset L^p(\Omega)$ with the following properties:*

(*i*) *The set* $\mathcal{M}$ *is positively invariant under semigroup* $\{S(t)\}$*, that is,* $S(t)\mathcal{M} \subset \mathcal{M}$ *for any* $t \geq 0$.

(*ii*) *The set* $\mathcal{M}$ *has finite fractal dimension, that is,* $\dim_F(\mathcal{M}, L^p(\Omega)) < \infty$.

(*iii*) *The set* $\mathcal{M}$ *attracts exponentially* $\mathcal{B}_1$*, that is, there exist* $\alpha \geq 0$ *and* $\omega > 0$ *such that, for any* $t \geq 0$,

$$dist_{L^p(\Omega)}\big(S(t)\mathcal{B}_1, \mathcal{M}\big) < \alpha e^{-\omega t}.$$

Proof. Define

$$\mathcal{M} := \bigcup_{t\in[0,t^*]} S(t)\mathcal{M}^*.$$

Note that $\mathcal{M} = F\left([0,t^*] \times \mathcal{M}^*\right)$, where

$$\begin{array}{rrcl} F: & [0,\infty) \times L^p(\Omega) & \rightarrow & L^p(\Omega) \\ & (t,u) & \mapsto & F(t,u) := S(t)u. \end{array}$$

From Corolary 3.15, it follows that F is continuous in $[0,t^*] \times \mathcal{M}^*$. Since $[0,t^*] \times \mathcal{M}^*$ is compact in $[0,\infty) \times L^p(\Omega)$, from continuity of F, it follows that $\mathcal{M}$ is compact in $L^p(\Omega)$.

We will show that $\mathcal{M}$ satisfies the conditions (i), (ii) and (iii) stated in the Theorem 4.2.

(*i*) <u>The set $\mathcal{M}$ is positively invariant under the semigroup $\{S(t)\}$.</u>
First note that

$$S(t)\mathcal{M} \subset \bigcup_{t\leq r\leq t+t^*} S(r)\mathcal{M}^*, \forall\, t \geq 0. \tag{4.1}$$

Let $t \geq 0$ be. Suppose first $0 \leq t \leq t^*$. From (4.1) we have

$$S(t)\mathcal{M} \subset \bigcup_{t\leq r\leq t+t^*} S(r)\mathcal{M}^* = \left(\bigcup_{t\leq r\leq t^*} S(r)\mathcal{M}^*\right) \cup \left(\bigcup_{t^*\leq r\leq t+t^*} S(r)\mathcal{M}^*\right). \tag{4.2}$$

Clearly

$$\bigcup_{t\le r\le t^*} S(r)\mathcal{M}^* \subset \mathcal{M}. \tag{4.3}$$

Now, $r \in [t^*, t^* + t]$ if and only if $r = t^* + s$ with $s \in [0, t]$. Hence,

$$\bigcup_{t^*\le r\le t+t^*} S(r)\mathcal{M}^* = \bigcup_{s\in[0,t]} S(s+t^*)\mathcal{M}^* = \bigcup_{s\in[0,t]} S(s)S(t^*)\mathcal{M}^*.$$

Using the invariance of $\mathcal{M}^*$ under $S(t^*)$ and that $t \le t^*$, it follows that

$$\bigcup_{t^*\le r\le t+t^*} S(r)\mathcal{M}^* \subset \bigcup_{s\in[0,t]} S(s)\mathcal{M}^* \subset \bigcup_{s\in[0,t^*]} S(s)\mathcal{M}^* = \mathcal{M}.$$

Hence,

$$\bigcup_{t^*\le r\le t+t^*} S(r)\mathcal{M}^* \subset \mathcal{M}. \tag{4.4}$$

Using (4.2), (4.3) and (4.4) it follows that $S(t)\mathcal{M} \subset \mathcal{M}$.

Now, suppose that $t > t^*$ and write $t = nt^* + \sigma$ with $n \in \mathbb{N}$ and $\sigma \in [0, t^*)$. Then,

$$S(t)\mathcal{M} = S(nt^*)S(\sigma)\mathcal{M}.$$

From the first part of the proof $S(\sigma)\mathcal{M} \subset \mathcal{M}$. Thus, $S(t)\mathcal{M} \subset S(nt^*)\mathcal{M}$. Using (4.1) we obtain

$$S(t)\mathcal{M} \subset \bigcup_{nt^*\le r\le nt^*+t^*} S(r)\mathcal{M}^* = \bigcup_{nt^*\le r\le (n+1)t^*} S(r)\mathcal{M}^*.$$

Given $r \in [nt^*, (n+1)t^*]$ we can to write $r = nt^* + s$, with $s \in [0, t^*]$. Then, from invariance of $\mathcal{M}^*$ under $S(nt^*)$ it follows that

$$S(r)\mathcal{M}^* = S(s)S(nt^*)\mathcal{M}^* \subset S(s)\mathcal{M}^*.$$

Therefore

$$\bigcup_{nt^*\le r\le (n+1)t^*} S(r)\mathcal{M}^* \subset \bigcup_{s\in[0,t^*]} S(s)\mathcal{M}^* = \mathcal{M}$$

and consequently $S(t)\mathcal{M} \subset \mathcal{M}$. This completes the proof of (i).

(ii) The finite fractal dimension of $\mathcal{M}$

Since F is Lipschitz in $[0,t^*]\times\mathcal{B}_1$ (Corollary 3.15) it follows that F is Lipschitz in $[0,t^*]\times\overline{\mathcal{B}_1}^{L^p(\Omega)}$. Consequently F is Lipschitz in $[0,t^*]\times\mathcal{M}^*$. As $[0,t^*]\times\mathcal{M}^*$ is compact in $\mathbb{R}\times L^p(\Omega)$ and $\mathcal{M}=F([0,t^*]\times\mathcal{M}^*)$, from Proposition 4.1 we have

$$\begin{aligned}\dim_F(\mathcal{M},L^p(\Omega)) &\leq \dim_F([0,t^*],\mathbb{R})+\dim_F(\mathcal{M}^*,L^p(\Omega)).\\ &\leq 1+\dim_F(\mathcal{M}^*,L^p(\Omega)).\end{aligned}$$

Since $\dim_F(\mathcal{M}^*,L^p(\Omega))<\infty$, it follows that $\dim_F(\mathcal{M},L^p(\Omega))<\infty$.

(iii) The set $\mathcal{M}$ attracts exponentially $\mathcal{B}_1$

Let $t\geq 0$ be. Suppose initially $t\in[0,t^*]$. Given $x\in S(t)\mathcal{B}_1$ and $z^*\in\mathcal{M}^*$ we have $x=S(t)b$ with $b\in\mathcal{B}_1$ and $S(t)z^*\in\mathcal{M}$. Then, from Proposition 3.11 extended to $\overline{\mathcal{B}_1}^{L^p(\Omega)}$, as b, $z^*\in\overline{\mathcal{B}_1}^{L^p(\Omega)}$ we have,

$$\begin{aligned}\inf_{y\in\mathcal{M}}\|x-y\|_{L^p(\Omega)} \leq \|x-S(t)z^*\|_{L^p(\Omega)} &= \|S(t)b-S(t)z^*\|_{L^p(\Omega)}\\ &\leq e^{N\beta\Gamma\|J\|_p t}e^{-t}\|b-z^*\|_{L^p(\Omega)}.\end{aligned}$$

Defining $\alpha_1:=\operatorname{diam}\left(\overline{\mathcal{B}_1}^{L^p(\Omega)}\right)e^{N\beta\Gamma\|J\|_p t^*}$ and $\omega_1:=1$ we obtain

$$\operatorname{dist}_{L^p(\Omega)}(S(t)\mathcal{B}_1,\mathcal{M})\leq\alpha_1 e^{-\omega_1 t}. \tag{4.5}$$

Suppose now that $t>t^*$. In this case, we can write $t=mt^*+\sigma$, where $m\in\mathbb{N}$ and $\sigma\in[0,t^*]$. Let $x\in S(t)\mathcal{B}_1$ be, then $x=S(t)b$ with $b\in\mathcal{B}_1$. Given $z^*\in\mathcal{M}^*$, define $\overline{y}=S(\sigma)z^*$. Note que $\overline{y}\in\mathcal{M}$ and

$$\begin{aligned}\inf_{y\in\mathcal{M}}\|x-y\|_{L^p(\Omega)} &\leq \|S(\sigma)S(mt^*)b-S(\sigma)z^*\|_{L^p(\Omega)}\\ &\leq L^*\|S(mt^*)b-z^*\|_{L^p(\Omega)},\end{aligned}$$

where L^* is the Lipschitz constant of $S(t)$ on $\mathcal{B}_1$, for $t\in[0,t^*]$. Using that $z^*\in\mathcal{M}^*$ is arbitrary, it follows that

$$\begin{aligned}\inf_{y\in\mathcal{M}}\|x-y\|_{L^p(\Omega)} &\leq L^*\sup_{z\in S(mt^*)\mathcal{B}_1}\inf_{z^*\in\mathcal{M}^*}\|z-z^*\|_{L^p(\Omega)}\\ &= L^*\operatorname{dist}_{L^p(\Omega)}\left(S(mt^*)\mathcal{B}_1,M^*\right).\end{aligned}$$

From (3.40) it follows that

$$\inf_{y\in\mathcal{M}} \|x-y\|_{L^p(\Omega)} \leq L^*\alpha^* e^{-\omega^* m}.$$

Since $x \in S(t)\mathcal{B}_1$ is also arbitrary, it follows that

$$\operatorname{dist}_{L^p(\Omega)}(S(t)\mathcal{B}_1, \mathcal{M}) \leq L^*\alpha^* e^{-\omega^* m}.$$

Noting that $-m\omega^* \leq -\frac{\omega^*}{t^*}t+\omega^*$ and defining $\alpha_2 := L^*\alpha^* e^{\omega^*}$ and $\omega_2 := \dfrac{\omega^*}{t^*}$, we have

$$\operatorname{dist}_{L^p(\Omega)}(S(t)\mathcal{B}_1, \mathcal{M}) \leq \alpha_2 e^{-\omega_2 t}.$$

Finally, considering $\alpha := \max\{\alpha_1, \alpha_2\}+1$ and $\omega := \min\{\omega_1, \omega_2\}$ we obtain, for any $t \geq 0$

$$\operatorname{dist}_{L^p(\Omega)}(S(t)\mathcal{B}_1, \mathcal{M}) < \alpha e^{-\omega t}.$$

□

4.1 Transitivity of the exponential attraction

Initially we know that the set $\mathcal{M}$ obtained from the Theorem 4.2 attracts exponentially only $\mathcal{B}_1$. Under these conditions $\mathcal{M}$ is an exponential attractor for $\{S(t)\}_{t\geq 0}$ restricted to $\mathcal{B}_1$, in the sense defined in [1].

However, we want to extend exponential attraction to any bounded subset of $L^p(\Omega)$, obtaining an exponential attractor in the sense of [12]. For this, we will need to assume the additional hypothesis that f and g are Lipschitz, and thus be able to apply the following theorem, proved in [13] (see Teorema 5.1).

Theorem 4.3. *Let (H, d) be a metric space and $\{S(t)\}$ a semigroup acting on H such that, for any $t \geq 0$,*

$$d\big(S(t)u_1, S(t)u_2\big) \leq Ce^{-Kt}d(u_1, u_2), \ \forall\ u_1, u_2 \in H, \tag{4.6}$$

where $C \geq 0$ *and* $K > 0$ *are constants. Suppose there are subsets* $M_1, M_2, M_3 \subset H$ *and constants* $C_1, C_2, \alpha_1, \alpha_2 > 0$ *such that*

$$dist_H\big(S(t)M_1, M_2\big) < C_1 e^{-\alpha_1 t},\ \forall\ t \geq 0 \tag{4.7}$$

and

$$dist_H\big(S(t)M_2, M_3\big) < C_2 e^{-\alpha_2 t},\ \forall\ t \geq 0. \tag{4.8}$$

Then,

$$dist_H(S(t)M_1, M_3) \leq C' e^{-\alpha' t},\ \forall\ t \geq 0$$

where

$$C' = CC_1 + C_2 \quad \textit{and} \quad \alpha' = \frac{\alpha_1\alpha_2}{K + \alpha_1 + \alpha_2}.$$

Before applying the above theorem we need some additional results, which complete the argument that our problem is in accordance with the hypotheses required in it.

We will start by reformulating Proposition 5, under the assumption that f and g are Lipschitzian. We will denote by L_g and L_f, the Lipschitz constants of g and f, respectively. Note that under these hypotheses, we can consider $k_1 = L_g$ and $c_1 = L_f$. Thus, since we are assuming $k_1\beta c_1 < 1$, it follows that $L_g\beta L_f < 1$. Furthermore, from the assumptions about $\int_{\mathbb{R}^n} J(x, y)\ dy$, we can conclude that $\|J\|_1 \leq 1$. Therefore

$$L_g\beta L_f\|J\|_1 < 1.$$

Proposition 4.4. *For any* $u_0, v_0 \in L^p(\Omega)$ *and* $t \geq 0$

$$\|S(t)u_0 - S(t)v_0\|_{L^p(\Omega)} \leq e^{\left[L_g\beta L_f\|J\|_1 - 1\right]t}\|u_0 - v_0\|_{L^p(\Omega)}.$$

Proof. Let $u_0, v_0 \in L^p(\Omega)$, $u(\cdot, t) = S(t)u_0$ and $v(\cdot, t) = S(t)v_0$ be. By the formula of variation of the constants, we have

$$\begin{aligned} u(\cdot, t) - v(\cdot, t) = e^{-t}(u_0 - v_0) + \int_0^t e^{-(t-s)}\big[&g(\beta K(f \circ u)(\cdot, s) + \beta h) \\ &- g(\beta K(f \circ v)(\cdot, s) + \beta h)\big]ds. \end{aligned}$$

Since g is Lipschitz we have

$$\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)} \\ \leq L_g\beta\|K(f\circ u-f\circ v)(\cdot,s)\|_{L^p(\Omega)}.$$

Using the Lemma 2.1 follows that

$$\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)} \\ \leq L_g\beta\|J\|_1\|f\circ u(\cdot,s)-f\circ v(\cdot,s)\|_{L^p(\Omega)}.$$

and as f is Lipschitz, we get

$$\|g(\beta K(f\circ u)(\cdot,s)+\beta h)-g(\beta K(f\circ v)(\cdot,s)+\beta h)\|_{L^p(\Omega)} \\ \leq L_g\beta L_f\|J\|_1\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}.$$

Therefore,

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)}\leq e^{-t}\|u_0-v_0\|_{L^p(\Omega)} \\ +\int_0^t e^{-(t-s)}L_g\beta L_f\|J\|_1\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}ds,$$

Thus,

$$e^t\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)} \\ \leq \|u_0-v_0\|_{L^p(\Omega)}+\int_0^t L_g\beta L_f\|J\|_1 e^s\|u(\cdot,s)-v(\cdot,s)\|_{L^p(\Omega)}ds.$$

Using Gronwall inequality, we obtain

$$e^t\|u(\cdot,t)-v(\cdot,t)\|_{L^p(\Omega)}\leq e^{L_g\beta L_f\|J\|_1 t}\|u_0-v_0\|_{L^p(\Omega)}.$$

and multiplying both members by e^{-t}, the result follows. □

Proposition 4.5. *For any bounded set B in $L^p(\Omega)$, there exists a constant $C_1(B)\geq 0$ depending only on B, such that for any $t\geq 0$,*

$$dist_{L^p(\Omega)}\left(S(t)B,\mathcal{B}_1\right)\leq C_1(B)e^{[L_g\beta L_f\|J\|_1-1]t}.$$

Proof. Let B be a bounded set in $L^p(\Omega)$. Fix $t \geq 0$ and lets $b \in S(t)B$, that is, $b = S(t)x$ with $x \in B$. Then,

$$\inf_{y \in \mathcal{B}_1} \|b - y\|_{L^p(\Omega)} = \inf_{y \in \mathcal{B}_1} \|S(t)x - y\|_{L^p(\Omega)}.$$

Fixed $z \in \mathcal{B}_1$, for every $y \in \mathcal{B}_1$, from Proposition 4.4, we have

$$\begin{aligned}\|S(t)x - y\|_{L^p(\Omega)} &\leq \|S(t)x - S(t)z\|_{L^p(\Omega)} + \|S(t)z - y\|_{L^p(\Omega)} \\ &\leq e^{[L_g \beta L_f \|J\|_1 - 1]t} \|x - z\|_{L^p(\Omega)} + \|S(t)z - y\|_{L^p(\Omega)}.\end{aligned}$$

We denote by $\overline{C_1}(B)$ the radius of a ball containing B and by r_1 the radius of a ball containing $\mathcal{B}_1$. Then, writing $C_1(B) := \overline{C_1}(B) + r_1$ we have

$$\|S(t)x - y\|_{L^p(\Omega)} \leq C_1(B) e^{[L_g \beta L_f \|J\|_1 - 1]t} + \|S(t)z - y\|_{L^p(\Omega)},$$

for any $y \in \mathcal{B}_1$. Thus,

$$\inf_{y \in \mathcal{B}_1} \|S(t)x - y\|_{L^p(\Omega)} \leq C_1(B) e^{[L_g \beta L_f \|J\|_1 - 1]t} + \inf_{y \in \mathcal{B}_1} \|S(t)z - y\|_{L^p(\Omega)}.$$

Since $z \in \mathcal{B}_1$ and, from Corollary 3.9, $\mathcal{B}_1$ is positively invariant under $\{S(t)\}$, we have $S(t)z \in \mathcal{B}_1$. Hence,

$$\inf_{y \in \mathcal{B}_1} \|S(t)z - y\|_{L^p(\Omega)} = 0$$

and therefore,

$$\inf_{y \in \mathcal{B}_1} \|S(t)x - y\|_{L^p(\Omega)} \leq C_1(B) e^{[L_g \beta L_f \|J\|_1 - 1]t}.$$

Consequently

$$\operatorname{dist}_{L^p(\Omega)} (S(t)B, \mathcal{B}_1) \leq C_1(B) e^{[L_g \beta L_f \|J\|_1 - 1]t}.$$

□

Corollary 4.6. *The set $\mathcal{M}$ given in the Theorem 4.2 is an exponential attractor for the semigroup $\{S(t)\}$ generated by the solutions of the equation (1.1).*

Proof. From Proposition 4.4

$$d(S(t)u_1, S(t)u_2) \leq Ce^{\left[L_g\beta L_f\|J\|_1-1\right]t}d(u_1,u_2),\ \forall\ u_1,u_2 \in L^p(\Omega), \quad (4.9)$$

where $C = 1$. Given a bounded set $B \subset L^p(\Omega)$, by Proposition 4.5, there exists $C_1(B) \geq 0$ such that

$$\mathrm{dist}_{L^p(\Omega)}\left(S(t)B, \mathcal{B}_1\right) \leq C_1(B)e^{[L_g\beta L_f\|J\|_1-1]t},\ \forall\ t \geq 0. \quad (4.10)$$

From Theorem 4.2, there exist $\alpha \geq 0$ and $\omega > 0$ such that,

$$\mathrm{dist}_{L^p(\Omega)}(S(t)\mathcal{B}_1, \mathcal{M}) \leq \alpha e^{-\omega t},\ \forall\ t \geq 0. \quad (4.11)$$

Applying the Theorem 4.3 with $K = L_g\beta L_f\|J\|_1-1$, $\alpha_1 = 1-L_g\beta L_f\|J\|_1$, $C = 1$, $C_1 = C_1(B)$, $C_2 = \alpha$, $\alpha_2 = \omega$, $M_1 = B$, $M_2 = \mathcal{B}_1$ and $M_3 = \mathcal{M}$, we obtain

$$\mathrm{dist}_{L^p(\Omega)}\left(S(t)B, \mathcal{M}\right) \leq C'(B)e^{-\alpha' t},\ \forall\ t \geq 0,$$

where $C'(B) = C_1(B) + \alpha$ and $\alpha' = \alpha_1$, completing the result. □

Corollary 4.7. *The global attractor $\mathcal{A}$ given in the Theorem 2.3 has a finite fractal dimension*

Proof. We affirm that $\mathcal{A} \subset \mathcal{M}$. In fact, since $\mathcal{A}$ is bounded, there exists $C(\mathcal{A}) > 0$ such that

$$\mathrm{dist}_{L^p(\Omega)}(S(t)\mathcal{A}, \mathcal{M}) \leq C(\mathcal{A})e^{-\omega t},\ \forall t \geq 0.$$

From the invariance of $\mathcal{A}$ it follows that

$$\mathrm{dist}_{L^p(\Omega)}(\mathcal{A}, \mathcal{M}) \leq C(\mathcal{A})e^{-\omega t},\ \forall\ t \geq 0.$$

Thus, $\mathrm{dist}_{L^p(\Omega)}(\mathcal{A}, \mathcal{M}) \to 0$, as $t \to \infty$. Using semi-distance properties $\mathrm{dist}_{L^p(\Omega)}$, it follows that $\mathcal{A} \subset \overline{\mathcal{M}}$. As $\mathcal{M}$ is compact, we obtain $\mathcal{A} \subset \mathcal{M}$. Therefore, as both sets are compact (relative to the same phase space), $\dim_F \mathcal{A} \leq \dim_F \mathcal{M}$. □

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