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# On a variational inequality for a beam equation with internal damping and source terms

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**Abstract.** In this paper we investigate the unilateral problem for a extensible beam equation with internal damping and source terms

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t = |u|^{r-1}u$$

where $r > 1$ is a constant, $M(s)$ is a continuous function on $[0, +\infty)$. The global solutions are constructed by using the Faedo-Galerkin approximations, taking into account that the initial data is in appropriate set of stability created from the Nehari manifold.

**Keywords:** beam equations, internal damping.

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# 1 Introduction

In [22] the authors establish existence of global solution to the problem

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t = |u|^{r-1}u \tag{1.1}$$

$$u(.,0) = u_0,\ u_t(.,0) = u_1 \text{ in } \Omega, \tag{1.2}$$

$$u(.,t) = \frac{\partial u}{\partial \eta}(.,t) \text{ in } \partial\Omega, t \geq 0, \tag{1.3}$$

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where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ , $r > 1$ is a constant and $M(s)$ is a continuous function on $[0, +\infty)$, $u = 0$ is the homogeneous Dirichlet boundary condition and the normal derivative $\frac{\partial u}{\partial \eta} = 0$ is the homogeneous Neumann boundary condition, where $\eta$ unit outward normal on $\partial\Omega$. In this work they use the potential well theory which approach is different from those in [11, 28]. In [11], the existence of global solutions was proved by means of the Fixed point theorem and the asymptotic behavior was obtained by using the perturbed energy method. In [28], the global existence and the longtime dynamics of solutions were considered by using semigroup theory.

The physical meaning of the clamped boundary conditions (1.3) is that, with the natural boundary conditions, we imposed no a priori conditions on the function space and it turns out that a weak solution automatically satisfies the boundary conditions.

In 1955, Berger [8] established the equation

$$u_{tt} - \left(Q + \int_\Omega |\nabla u|^2 dx\right) \Delta u = p(u, t, x), \tag{1.4}$$

which is called the Berger plate model, where the parameter $Q$ describes in-plane forces applied to the plate and the function $p$ represents transverse loads which may depend on the displacement $u$ and the velocity $u_t$ . If $n = 2$, the equation (1.4) represents the "Berger approximation" of the Von Kárman equations, modelling the nonlinear vibrations of a plate (see [25], pg. 501-507). When n = 1 and p = 0, the corresponding equation had been introduced by Woinowsky-Krieger [26] as a model for the transverse motion of an extensible beam. It means that the equation (1.1) describes the transverse deflection of an extensible beam of the length $L$ whose ends are attached at a fixed distance is the following equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} + \left(\beta + \int_0^L u_\epsilon^2(\epsilon, t) d\epsilon\right)\left(-\frac{\partial^2 u}{\partial t^2}\right) = f,$$

where $\alpha$ is a positive constant, $\beta$ is a constant not necessarily positive and the nonlinear term represents the change in the tension of the beam due to its extensibility.

The physical origin of the problem here relates to the study of the dynamical buckling of the hinged extensible beam which is either stretched or compressed by an axial force. The readers could also see in Burgreen [9] and Eisley [17] for more physical justifications and the model background.

Cavalcanti et al [11] studied the equation

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u_t) + f(u) = 0 \tag{1.5}$$

with $g(s) = |s|^{\rho-1}s$ and $f(s) = |s|^{\gamma-1}s$, where $\rho$ and $\gamma$ are positives constants such that $1 < \rho, \gamma \leq n/(n-2)$ if $n \geq 3$; $\rho, \gamma > 1$ if $n = 1, 2$. The global existence and asymptotic stability were proved by means of the fixed point theorem and continuity arguments.

Zhijian [28] investigated the problem (1.5) more generally as follows

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u_t) + f(u) = h(x),$$

where the source terms $f, g \in \mathcal{C}^1(\mathbb{R}), |f'(s)| \leq C(1+|s|^{p-1})$ and $K_0|s|^{q-1} < g'(s) \leq C(1 + |s|^{q-1}), K_0, C > 0$ with $1 \leq p < \infty, 1 \leq q < \infty$ if $n \leq 4; 1 \leq p \leq p* = (n+4)/(n-4)$ and $p \leq q$ if $n \geq 5$. By Galerkin approximation combined with the monotone arguments, the author proved the existence of global solution.

In article [2], Andrade et al. establish existence of global solution to the problem

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t + f(u) = 0, \tag{1.6}$$

we observe that for viscoelastic plate equation, it is usual consider a memory of the form $\int_0^t g(t-s)\Delta^2 u(s)ds$ (e. g. [10, 21]). However, because the main dissipation of the system (1.1) is given by strong damping $-\Delta u_t$, here we consider a weaker memory, acting only on $\Delta u$. There is a large literature about stability in viscoelasticity. We refer the reader to, for example [13, 12].

The equation (1.1) without source terms was studied by several authors in different contexts. The problem without damping was considered in Dickey [16], Ball [4], Medeiros [20], Pereira [23] among others. When the damping term is considered, the problem was studied by, Brito [7], Biler [6], Ball [5] see also the references therein.

A nonlinear perturbation of problem (1.1) is given by

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t - |u|^{r-1}u \geq 0. \tag{1.7}$$

In the present work we investigated the unilateral problem associated with this perturbation, that is, a variational inequality given for (1.7) (see [19]). Making use of the penalty method, the potential well theory and Galerkin's approximations, we establish existence and uniqueness of global solutions.

Unilateral problem is very interesting too, because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problem on elasticity and finite element method see Kikuchi-Oden [18] and reference there in. For contact problems on viscoelastic materials see [21]. For contact problems on Klein-Gordon operator see [24]. For contact problems on Oldroyd Model of Viscoelastic fluids see [15]. For contact problems on Navier-stokes Operator with variable viscosity see [14]. For contact problems on viscoelastic plate equation (1.6) see [3].

This work is organized as follows: In the Section 2, we introduce some notations and the stability set created from the Nehari Manifold. In the Section 3, we present the main theorem. In Section 4 we prove the existence of strong solution through the Faedo-Galerkin method and finally in Section 5 we prove a simple result of uniqueness.

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## 2 Notations and The potential well

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with the boundary $\Gamma$ of class $C^2$. For $T > 0$, we denote by $Q$ the cylinder $\Omega \times (0, T)$, with lateral boundary

$\Sigma = \Gamma \times (0,T)$. By $\langle .,. \rangle$ we will represent the duality pairing between $X$ and $X'$, $X'$ being the topological dual of the space $X$, and by $C$ we denote various positive constants.

We propose the variational inequality

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t - |u|^{r-1}u \geq 0 \ \text{ in } \ Q \tag{2.1}$$

This inequality is satisfied in a certain sense, that is, we formulate the problem as follows.

Let $K = \{v \in H_0^1(\Omega); v \geq 0 \text{ a.e. in } \Omega\}$ a closed and convex subset of $H_0^1(\Omega)$, the variational problem is to find a solution $u(x,t)$ satisfying

$$\int_Q (u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t - |u|^{r-1}u)(v - u_t) \geq 0, \forall v \in K, \tag{2.2}$$

with $u_t(x,t) \in K$ a. e. on $[0,T]$ and talking the initial and boundary data

$$\begin{aligned} u &= 0, \ \ u_t = 0 \text{ on } \Sigma, \\ u &= \Delta u = 0 \text{ on } \Sigma, \\ u(.,0) &= u_0, \ \ u_t(.,0) = u_1 \text{ in } \Omega. \end{aligned} \tag{2.3}$$

In order to formulate problem (2.1) we need some notations about Sobolev spaces. We use standard notation of $L^2(\Omega)$, $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $C^p(\Omega)$ for functions that are defined on $\Omega$ and range in $\mathbb{R}$. To complete this recall on functional spaces, see for instance Lions [19]. Particularly we denote the inner product and norm in $H_0^1(\Omega)$ and $L^2(\Omega)$ , respectively, by

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\, dx, \quad \|u\|^2 = \sum_{i=1}^{n} \int_\Omega \left( \frac{\partial u}{\partial x_i}(x) \right)^2 dx \text{ and}$$

$$(u,v) = \int_\Omega u(x)v(x)\, dx, \quad |u|^2 = \int_\Omega |u(x)|^2\, dx.$$

It is well-known that the energy of a PDE system, in some sense, splits into the kinetic and the potential energy. By following the idea of Y. Ye [27], we are able to construct a set of stability. We will prove that there is a valley or a well of the depth d created in the potential energy. If d is strictly positive, then we find that, for solutions with the initial data in the good part of the potential well, the potential energy of the solution can never escape the potential well. In general, it is possible that the energy from the source term causes the blow-up in a finite time. However, in the good part of the potential well, it remains bounded. As a result, the total energy of the solution remains finite on any time interval [0, T), providing the global existence of the solution. We started by introducing the functional $J : H_0^1(\Omega) \cap H^2(\Omega) \to \mathbb{R}$ definid by

$$J(u) = \frac{1}{2}|\Delta u|^2 + \frac{m_0}{2}|\nabla u|^2 - \frac{1}{r+1}|u|_{r+1}^{r+1}.$$

For $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$J(\lambda u) = \frac{\lambda^2}{2}|\Delta u|^2 + \frac{m_0 \lambda^2}{2}|\nabla u|^2 - \frac{\lambda^{r+1}}{r+1}|u|_{r+1}^{r+1}, \lambda > 0$$

Associated with $J$, we have the well-known Nehari Manifold given by

$$\mathcal{N} \overset{def}{=} \left\{ u \in H_0^1(\Omega) \cap H^2(\Omega) \setminus \{0\}; \left[ \frac{d}{d\lambda} J(\lambda u) \right]_{\lambda=1} = 0 \right\}$$

equivalently,

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) \cap H^2(\Omega)) \setminus \{0\}; |\Delta u|^2 + m_0 |\nabla u|^2 = |u|_{r+1.}^{r+1} \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [1]

$$d \overset{def}{=} \inf_{u \in (H_0^1(\Omega) \cap H^2(\Omega) \setminus \{0\}} \sup J(\lambda u)_{\lambda \geq 0}.$$

It is well-known for $1 < r \leq 5$ that the depth of the well is a real constant strictly positive ([25], theorem 4.2) and $d = \inf_{u \in \mathcal{N}} J(u)$. We now introduce the potential well

$$W = \{u \in H_0^1(\Omega) \cap H^2(\Omega); J(u) < d\} \cup \{0\}$$

and partition it into two sets as follows

$$W_1 = \left\{ u \in H_0^1(\Omega) \cap H^2(\Omega); \frac{1}{2}|\Delta u|^2 + \frac{m_0}{2}|\nabla u|^2 > \frac{1}{r+1}|u|_{r+1.}^{r+1} \right\} \cup \{0\}$$

and

$$W_2 = \left\{ u \in H_0^1(\Omega) \cap H^2(\Omega); \frac{1}{2}|\Delta u|^2 + \frac{m_0}{2}|\nabla u|^2 < \frac{1}{r+1}|u|_{r+1}^{r+1} \right\}.$$

We will refer to $W_1$ as the "good" part of the potential well. Then we define by $W_1$ the set of stability for the problem (1.1)-(1.3).

To study the existence and uniqueness of the problem (2.2)-(2.3), let us consider the following hypothesis

$$(H_1) \quad M \in C^1([0,\infty)) \text{ with } M(\lambda) \geq m_0 > 0, \ \ \forall \lambda \geq 0.$$

$$(H_2) \qquad (r-1)n \leq rn \leq q = \frac{2n}{n-2}, n > 2.$$

Next we shall state the main results of this paper.

## 3 Main Results

**Theorem 3.1.** *Consider the spaces*

$$H_\Gamma^4(\Omega) = \{u \in H^4(\Omega) | u = \Delta u = 0 \ \text{on } \Gamma\} \text{ and similarly}$$

$$H_\Gamma^3(\Omega) = \{u \in H^3(\Omega) | u = \Delta u = 0 \ \text{on } \Gamma\}.$$

*If* $u_0 \in W_1 \cap H_\Gamma^4(\Omega), J(u_0) < d, u_1 \in H_0^1(\Omega) \cap H^2(\Omega), 1 < r \leq 5$ *and the hypothesis* $(H_1)$ *and* $(H_2)$ *holds, then there exists a function* $u : [0,T] \to L^2(\Omega)$ *in the class*

$$u \in L^\infty(0,T; (H_0^1(\Omega) \cap H^2(\Omega)) \cap H_\Gamma^3(\Omega) \cap L^{r+1}(\Omega)) \tag{3.1}$$

$$u_t \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \tag{3.2}$$

$$u_{tt} \in L^\infty(0,T; L^2(\Omega)) \tag{3.3}$$

$$u_t(t) \in K \ \ a.e. \ in \ [0,T], \tag{3.4}$$

*satisfying*

$$\int_Q (u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + u_t - |u|^{r-1}u)(v - u_t) \geq 0,$$
$$\forall v \in L^2(0,T; H_0^1(\Omega)), \ v(t) \in K \ \ a.e. \ \ in \ t \tag{3.5}$$

$$u(0) = u_0, \ \ u_t(0) = u_1$$

The proof of Theorem 3.1 is made in Section 4 by the penalization method. It consists in considering a perturbation of the problem (1.1) adding a singular term called penalty, depending on a parameter $\epsilon > 0$. We solve the mixed problem in $Q$ for the penalization operator and the estimates obtained for the local solution of the penalized equation, allow to pass to limits, when $\epsilon$ goes to zero, in order to obtain a function $u$ which is the solution of our problem.

First of all, let us consider the penalization operator

$$\beta : L^2(\Omega) \longrightarrow L^2(\Omega)$$

associated to the closed convex set $K$, cf. Lions [19], p. 370. The operator $\beta$ is monotonous, hemicontinuous and takes bounded sets of $H_0^1(\Omega)$ into bounded sets of $H^{-1}(\Omega)$, its kernel is $K$ and

$$\beta : L^2(0,T; L^2(\Omega)) \longrightarrow L^2(0,T; L^2(\Omega))$$

is equally monotone and hemicontinous. The penalized problem associated with the variational inequality (3.5), consists in given $0 < \epsilon < 1$, find $u^\epsilon$ satisfying

$$u^\epsilon_{tt} \quad + \quad \Delta^2 u^\epsilon + M(|\nabla u^\epsilon|^2)(-\Delta u^\epsilon) + u^\epsilon_t + \frac{1}{\epsilon}(\beta(u^\epsilon_t)) = |u^\epsilon|^{r-1} u^\epsilon \text{ in } Q \tag{3.6}$$

$$\begin{aligned} u^\epsilon &= 0 \text{ on } \Sigma \\ u^\epsilon_t &= 0 \text{ on } \Sigma, \\ u^\epsilon(x,0) = u^\epsilon_0(x), &\qquad u^\epsilon_t(x,0) = u^\epsilon_1(x) \text{ in } \Omega. \end{aligned} \tag{3.7}$$

**Definition 3.2.** A strong solution to the boundary value problem (3.6)-(3.7) is a functions $u^\epsilon$,

$$u^\epsilon \in L^\infty(0,T; (H_0^1(\Omega) \cap H^2(\Omega)) \cap L^{r+1}(\Omega)),$$

$$u^\epsilon_t \in L^\infty(0,T; L^2(\Omega)),$$

satisfying for all $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$

$$
\begin{aligned}
(u_{tt}^\epsilon, \varphi) &+ (\Delta^2 u^\epsilon, \varphi) + (M(|\nabla u^\epsilon|^2)(-\Delta u^\epsilon), \varphi) + (u_t^\epsilon, \varphi) \\
&+ \frac{1}{\epsilon}((\beta(u_t^\epsilon)), \varphi) = (|u^\epsilon|^{r-1} u^\epsilon, \varphi) \text{ in } \mathcal{D}'(0,T) \\
u^\epsilon(0) &= u_0^\epsilon,\ u_t^\epsilon(0) = u_1^\epsilon,
\end{aligned}
\tag{3.8}
$$

The solution of problem (3.6)-(3.7) is given by the following theorem:

**Theorem 3.3.** *Assume that hypotheses* $(H_1)$ *and* $(H_2)$ *holds,*

$$u_0^\epsilon \in W_1, J(u_0^\epsilon) < d, 1 < r \leq 5 \text{ and} \tag{3.9}$$

$$u_1^\epsilon \in L^2(\Omega). \tag{3.10}$$

*Then, for each* $0 < \epsilon < 1$, *there exists a function* $u^\epsilon$ *with*

$$u^\epsilon \in L^\infty(0,T; (H_0^1(\Omega) \cap H^2(\Omega)) \cap L^{r+1}(\Omega)),$$

$$u_t^\epsilon \in L^\infty(0,T; L^2(\Omega)),$$

*strong solution of (3.6)-(3.7).*

# 4 Proof of the Results

**Proof of Theorem 3.3.** In order to prove Theorem 3.1, we first prove the penalized Theorem 3.3. The existence of solutions will be given by using Faedo-Galerkin approximations.

Let $\{w_j\}$ be the Galerkin basis given by eigenfunctions of $\Delta^2$ and let $V_m \subset \mathcal{N}$ be the subspace spanned by the vectors $w_1, w_2, ..., w_m$. Given initial date $u_0^\epsilon \in W_1$, $J(u_0^\epsilon) < d$ and $u_1^\epsilon \in L^2(\Omega)$, we a search for a function

$$u^{\epsilon m}(t) = \sum_{j=1}^{m} g_{j_m}(t) w_j, j = 1, 2, ...m,\ \forall w_j \in V_m$$

solution of approximate problem

$$\begin{aligned}&(u_{tt}^{\epsilon m}(t),w)+(\Delta u^{\epsilon m}(t),\Delta w)+M(|\nabla u^{\epsilon m}(t)|^2)\langle -\Delta u^{\epsilon m}(t),w\rangle\\&+(u_t^{\epsilon m}(t),w)+\frac{1}{\epsilon}\langle\beta(u_t^{\epsilon m})(t),w\rangle-\langle|u^{\epsilon m}(t)|^{r-1}u^{\epsilon m}(t),w\rangle=0,\forall w\in V_m\end{aligned}\tag{4.1}$$

with initial conditions

$$u^{\epsilon m}(0)=u_0^{\epsilon m}\to u_0^{\epsilon}\in H_0^1(\Omega)\cap H^2(\Omega),\quad u_t^{\epsilon m}(0)=u_1^{\epsilon m}\to u_1^{\epsilon}\in L^2(\Omega)\tag{4.2}$$

The system of ordinary differential equation (4.1) has a solution on a interval $[0,t_m[,\ 0<t_m<T$. The first estimate permits us to extend this solution to the whole interval $[0,T]$.

**Remark 4.1.** In order to obtain a better notation, we omit the parameter $\epsilon$ in the approximate solutions.

**FIRST ESTIMATE.** Substituting $w=u_t^m(t)$ in (4.1) it follows

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left(|u_t^m(t)|^2+|\Delta u^m(t)|^2+\widehat{M}(|\nabla u^m(t)|^2)\right)-\frac{1}{r+1}|u^m(t)|_{r+1}^{r+1}\right]\\&\quad+|u_t^m(t)|^2\le 0\end{aligned}\tag{4.3}$$

because $(\beta(u_t^m(t)),u_t^m(t))\ge 0$ and $\widehat{M}(s)=\int_0^s M(\xi)d\xi$.
Integrating (4.3) from 0 to $t$ we obtain

$$\begin{aligned}\frac{1}{2}|u_t^m(t)|^2 &+ \frac{1}{2}|\Delta u^m(t)|^2+\frac{1}{2}\widehat{M}(|\nabla u^m(t)|^2)-\frac{1}{r+1}|u^m(t)|_{r+1}^{r+1}\\&+\int_0^t|u_t^m(t)|^2ds\le\frac{1}{2}|u_1^m|^2+\frac{1}{2}|\Delta u_0^m|^2\\&+\frac{1}{2}\widehat{M}(|\nabla u_0^m|^2)-\frac{1}{r+1}|u_0^m|_{r+1}^{r+1}\end{aligned}\tag{4.4}$$

Now,

$$\widehat{M}(|\nabla u_0^m|^2)\le m_1|\nabla u_0^m|^2,\tag{4.5}$$

where $m_1=\max_{0\le s\le|\nabla u_0^m|^2\le C_0}M(s)$, $C_0$ is a positive constant independent of $m$ and $t$ . Therefore, the approximate energy

$$E_m(t)=\frac{1}{2}|u_t^m(t)|^2+\frac{1}{2}|\Delta u^m(t)|^2+\frac{1}{2}\widehat{M}(|\nabla u^m(t)|^2)-\frac{1}{r+1}|u^m(t)|_{r+1}^{r+1}$$

satisfies

$$E_m(t) \leq E_m(0) \leq \frac{1}{2}|u_1^m|^2 + C_1 J(u_0^m),$$

where $C_1 > 0$ is a constant independent of $m$ and $t$. We have that $J(u_0^m) < d$ and by convergence $(4.2)_2$, there exists a constant $C_2 > 0$ independent of $m$ and $t$ such that $\frac{1}{2}|u_1^m|^2 \leq C_2$. So, there exists a constant $C_3 > 0$ such that

$$E_m(t) \leq E_m(0) < C_3.$$

By $(H_1)$ , $\widehat{M}(|\nabla u^m(t)|^2) \geq m_0 |\nabla u^m(t)|^2$, then

$$\frac{1}{2}|u_t^m(t)|^2 + \frac{1}{2}|\Delta u^m(t)|^2 + \frac{m_0}{2}|\nabla u^m(t)|^2 - \frac{1}{r+1}|u^m(t)|_{r+1}^{r+1} \leq E_m(t) < C_3 \quad (4.6)$$

Returning to the notation $u^{\epsilon m}$, the first estimate (4.6) implies that

$$\begin{aligned} &(u^{\epsilon m}) \text{ is bounded in } L^\infty(0,T;(H_0^1(\Omega)\cap H^2(\Omega))\cap L^{r+1}(\Omega)) \\ &(u_t^{\epsilon m}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \end{aligned} \quad (4.7)$$

From the estimates (4.7), there exists a subsequence of $(u^{\epsilon m})$, still denoted by $(u^{\epsilon m})$, such that

$$u^{\epsilon m} \to u^\epsilon \text{ weakly star in } L^\infty(0,T;H_0^1(\Omega)\cap H_0^2(\Omega)) \quad (4.8)$$

$$u^{\epsilon m} \to u^\epsilon \text{ weakly star in } L^\infty(0,T;L^{r+1}(\Omega)) \quad (4.9)$$

$$u_t^{\epsilon m} \to u_t^\epsilon \text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \quad (4.10)$$

As

$$(\beta(u_t^{\epsilon m})) \text{ is bounded in } L^2(0,T;L^2(\Omega))$$

we have

$$\beta(u_t^{\epsilon m}) \to \psi \text{ weakly in } L^2(0,T;L^2(\Omega)) \quad (4.11)$$

Follows from (4.8), (4.10) and Aubin-Lions Theorem, for any $T > 0$,

$$u^{\epsilon m} \to u^\epsilon \text{ in } L^2(0,T;H_0^1(\Omega)), \text{ strong and a.e. in } Q. \quad (4.12)$$

and since M is continous, it follows

$$M(|\nabla u^{\epsilon m}|^2) \to M(|\nabla u^\epsilon|^2) \text{ strongly in } L^2(0,T).$$

Therefore,

$$M(|\nabla u^{\epsilon m}|^2)(-\Delta u^{\epsilon m}) \to M(|\nabla u^{\epsilon}|^2)(-\Delta u^{\epsilon}) \text{ weakly in } L^2(0,T;L^2(\Omega)). \quad (4.13)$$

Now, observe that

$$\int_0^T (|u^{\epsilon m}(t)|^{r-1}u^{\epsilon m}(t))^{\frac{r+1}{r}}dt \leq \int_0^T |u^{\epsilon m}(t)|^{r+1}dt \leq C,$$

so

$$(|u^{\epsilon m}|^{r-1}u^{\epsilon m}) \text{ is bounded in } L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)) \quad (4.14)$$

From the convergence (4.12) and continuity of function $s \to |s|^{r-1}s$ it follows

$$|u^{\epsilon m}|^{r-1}u^{\epsilon m} \to |u^{\epsilon}|^{r-1}u^{\epsilon} \text{ a.e. in } Q.$$

Therefore, from [19] Lemma 1.3, we have

$$|u^{\epsilon m}|^{r-1}u^{\epsilon m} \rightharpoonup |u^{\epsilon}|^{r+1}u^{\epsilon} \text{ weakly in } L^{\frac{r+1}{r}}(0,T;L^{\frac{r+1}{r}}(\Omega)). \quad (4.15)$$

To prove that $\psi = \beta(u_t^{\epsilon})$ we can use the monotony of the operators $\beta$ (see Lions [19], chap. 2). With these convergences we can pass to limit in the terms of the approximate problem (4.1)-(4.2). Therefore, we have proved the Theorem 3.3.

To prove the Theorem 3.1 we will need another estimates

**SECOND ESTIMATE**

Let's continue omitting the parameter $\epsilon$. Let us consider data initial

$$\begin{aligned} &u_0 \in H^4_\Gamma(\Omega), u_1 \in H^1_0(\Omega) \cap H^2(\Omega) \\ &u_0^m \to u_0 \text{ in } H^4_\Gamma, u_1^m \to u_1 \text{ in } H^1_0(\Omega) \cap H^2(\Omega) \\ &u^m = \Delta u^m = 0 \text{ on } \Sigma \end{aligned} \quad (4.16)$$

Thanks to the choice of the base $\{w_j\}$, we can replace $w$ with $-\Delta u_t^m$.

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}|\nabla u_t^m(t)|^2 + \frac{1}{2}|\nabla\Delta u^m(t)|^2\right\} + M(|\nabla u^m(t)|^2)\frac{1}{2}\frac{d}{dt}|\Delta u^m(t)|^2 \quad (4.17) \\ + \quad &\|u_t^m(t)\|^2 + \frac{1}{\epsilon}(\beta(u_t^m(t)), -\Delta u_t^m(t)) = \langle |u^m(t)|^{r-1}u^m(t), -\Delta u^m(t)\rangle. \end{aligned}$$

We observe that

$$\begin{aligned} \frac{d}{dt} & \left\{M(|\nabla u^m(t)|^2)|\Delta u^m(t)|^2\right\} = M(|\nabla u^m(t)|^2)\frac{d}{dt}(|\Delta u^m(t)|^2) \\ + & \frac{d}{dt}(M(|\nabla u^m(t)|^2))|\Delta u^m(t)|^2 = M(|\nabla u^m(t)|^2)\frac{d}{dt}(|\Delta u^m(t)|^2) \\ + & 2M'(|\nabla u^m(t)|^2)((u^m(t), u_t^m(t)))|\Delta u^m(t)|^2. \end{aligned} \tag{4.18}$$

Substituting (4.18) in (4.17), we get

$$\begin{aligned} \frac{d}{dt} & \left\{\frac{1}{2}|\nabla u_t^m(t)|^2 + \frac{1}{2}|\nabla\Delta u^m(t)|^2 + \frac{1}{2}(M(|\nabla u^m(t)|^2)|\Delta u^m(t)|^2)\right\} \\ + & \|u_t^m(t)\|^2 \leq \langle |u^m(t)|^{r-1}u^m(t), -\Delta u^m(t)\rangle \\ + & 2M'(|\nabla u^m(t)|^2)\|u^m(t)\|\|u_t^m(t)\||\Delta u^m(t)|^2. \end{aligned} \tag{4.19}$$

because

$$(\beta(u_t^m(t)), -\Delta u_t^m(t)) \geq 0.$$

Since

$$(r-1)n \leq q = \frac{2n}{n-2}, n > 2,$$

we have

$$\frac{1}{2} + \frac{1}{q} + \frac{1}{n} = 1 \text{ and } H_0^1(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{(r-1)n},$$

this implies

$$\|u^m(t)^{r-1}\|_{L^n(\Omega)} = \left(\int_\Omega |u^m(x,t)|^{(r-1)n}dx\right)^{\frac{1}{n}} = \|u^m(t)\|^{r-1}_{L^{(r-1)n}(\Omega)} \leq C\|u^m(t)\|^{r-1} \leq C$$

also

$$\left\|\frac{\partial u^m(t)}{\partial x_i}\right\|_{L^q(\Omega)} \leq C\left\|\frac{\partial um(t)}{\partial x_i}\right\| \leq C$$

because $u^m \in L^\infty(0,T;H^2(\Omega))$. Therefore, using Holder inequality, we obtain

$$\begin{aligned}\langle |u^m(t)|^{r-1}u^m(t)\,,-\Delta u^m(t)\rangle &= \int_\Omega |u^m(x,t)|^{r-1}u^m(x,t)(-\Delta u_t^m(x,t))dx\\ &= \sum_{i=1}^n \int_\Omega \frac{\partial}{\partial x_i}(|u^m(x,t)|^{r-1}u^m(x,t))\frac{\partial u_t^m}{\partial x_i}dx\\ &= \sum_{i=1}^n \int_\Omega r|u^m(x,t)|^{r-1}\frac{\partial u^m(x,t)}{\partial x_i}\frac{\partial u_t^m(x,t)}{\partial x_i}dx \quad (4.20)\\ &\leq \sum_{i=1}^n \int_\Omega r|u^m(x,t)|^{r-1}\left|\frac{\partial u^m(x,t)}{\partial x_i}\right|\left|\frac{\partial u_t^m(x,t)}{\partial x_i}\right|dx\\ &\leq \sum_{i=1}^n r\|u^m(t)^{r-1}\|_{L^n(\Omega)}\left\|\frac{\partial u^m(t)}{\partial x_i}\right\|_{L^q(\Omega)}\left|\frac{\partial u_t^m(t)}{\partial x_i}\right| \quad (4.21)\\ &\leq C+\frac{1}{4}\|u_t^m(t)\|^2\end{aligned}$$

Note that (4.7) implies $\|u^m(t)\| \leq C$, therefore $\|u^m(t)\| \in [0, C]$, for aech $m$ and $t \in [0, t_m[$. Since $M \in C^1([0,\infty);\mathbb{R}$, this implies that

$$|M'(\|u^m(t)\|^2) \leq C,\ \forall m, t \in [0, t_m[. \tag{4.22}$$

Therefore, using (4.22) and (4.7) we can write

$$2M'\ (|\nabla u^m(t)|^2)\|u^m(t)\|\|u_t^m(t)\||\Delta u^m(t)|^2 \leq C+\frac{1}{4}\|u_t^m(t)\|^2 \tag{4.23}$$

We observe that

$$\frac{1}{2}|\nabla u_t^m(0)|^2+\frac{1}{2}|\nabla\Delta u^m(0)|^2+\frac{1}{2}(M(|\nabla u^m(0)|^2)|\Delta u^m(0)|^2) \leq C \tag{4.24}$$

Integrating (4.19) on $(0,t)$, follows from (4.20), (4.23), (4.24) and $(H_1)$

$$\frac{1}{2}\|u_t^m(t)\|^2+\frac{1}{2}|\nabla\Delta u^m(t)|^2+\frac{1}{2}\int_0^t \|u_t^m(t)\|^2 \leq C. \tag{4.25}$$

Returning to the notation $u^{\epsilon m}$, this implies that

$$u^{\epsilon m} \to u^\epsilon \text{ in } L^\infty(0,T;H^3_\Gamma(\Omega)), \quad \text{weakly star.} \tag{4.26}$$

$$u_t^{\epsilon m} \to u_t^\epsilon \text{ in } L^\infty(0,T;H^1_0(\Omega)), \quad \text{weakly star} \tag{4.27}$$

Taking into account $(H_2)$ we have

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^{rn},$$

this implies that

$$\begin{aligned} |u^{\epsilon m}(t)|^{r-1}u^{\epsilon m}(t)| &\leq C\|u^{\epsilon m}(t)|^{r-1}u^{\epsilon m}(t)\|_{L^n(\Omega)} = \left(\int_\Omega (|u^{\epsilon m}(t)|^r)^n\right)^{\frac{1}{n}} \\ &= \left(\left(\int_\Omega (|u^{\epsilon m}(t)|^{rn}\right)^{\frac{1}{rn}}\right)^r = \|u^{\epsilon m}(t)\|^r_{L^{rn}(\Omega)} \quad (4.28) \\ &\leq C\|u^{\epsilon m}(t)\|^r_{L^q(\Omega)} \leq C\|u^{\epsilon m}(t)\|^r \leq C \end{aligned}$$

because $u^{\epsilon m} \in L^\infty(0,T;H_0^1(\Omega))$, therefore

$$|u^{\epsilon m}|^{r-1}u^{\epsilon m} \to |u^\epsilon|^{r-1}u^\epsilon \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star} \quad (4.29)$$

**THIRD ESTIMATE**

Taking derivatives in the distribution sense in approximate problem (4.1), omitting parameter $\epsilon$ and considering $w = u_{tt}^m$, we obtain using an argument similar to that in (4.18)

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}|u_{tt}^m(t)|^2 + \frac{1}{2}|\Delta u_t^m(t)|^2 + M(|\nabla u^m|^2)\|u_t^m(t)\|^2\right] + |u_{tt}^m(t)|^2 \\ &+ \frac{1}{\epsilon}(\beta(u_t^m(t))', u_{tt}^m(t)) = \left(r|u^m(t)|^{r-1}u_t^m(t), u_{tt}^m(t)\right) \quad (4.30) \\ &- 2M'(|\nabla u^m(t)|^2)((u_t^m(t), u^m(t)))((u^m(t), u_{tt}^m(t))) \\ &+2M'(|\nabla u^m(t)|^2)((u_t^m(t), u^m(t)))\|u_t^m(t)\|^2. \end{aligned}$$

Using previous estimates, the terms of second, third and fourth line of (4.30) can be increased as follows:

$$\begin{aligned} &|(r|u^m(t)|^{r-1}u_t^m(t), u_{tt}^m(t))| \leq C\|u^m(t)\|^{r-1}_{L^n(\Omega)}\|u_t^m(t)\|_{L^q(\Omega)}|u_{tt}^m(t)| \\ &\leq C + \tfrac{1}{4}|u_{tt}^m(t)|^2 \end{aligned} \quad (4.31)$$

$$\begin{aligned} &|\,2M'(|\nabla u^m(t)|^2)((u_t^m(t), u^m(t)))((u^m(t), u_{tt}^m(t))) \quad (4.32) \\ \leq\; &|2M'(|\nabla u^m(t)|^2)\|u_t^m(t)\|\|u^m(t)\||u^m(t)||u_{tt}^m(t)| \leq C + \frac{1}{4}|u_{tt}^m(t)|^2 \end{aligned}$$

$$\begin{aligned}&|\,2M'(|\nabla u^m(t)|^2)((u_t^m(t),u^m(t)))\|u_t^m\|^2| \\ \leq\ & |2M'(|\nabla u^m(t)|^2)\|u_t^m(t)\|\|u^m(t)\|\|u_t^m\|^2\leq C\end{aligned} \tag{4.33}$$

also

$$(\beta(u_t^m(t))',u_{tt}^m(t))=\lim_{h\to 0}\left(\frac{\beta(u_t^m(t+h)-\beta(u_t^m(t))}{h},\frac{u_t^m(t+h)-u_t^m(t)}{h}\right)\geq 0 \tag{4.34}$$

Integrating (4.30) on $(0,t)$ and using $(H_1)$ and (4.31)-(4.34), we obtain

$$\frac{1}{2}\quad |u_{tt}^m(t)|^2+\frac{1}{2}|\Delta u_t^m(t)|^2+\frac{1}{2}\int_0^t |u_{tt}^m(t)|^2+\leq C. \tag{4.35}$$

because

$$\frac{1}{2}|u_{tt}^m(0)|^2+\frac{1}{2}|\Delta u_t^m(0)|^2+M(|\nabla u^m(0)|^2)\|u_t^m(0)\|^2\leq C$$

We note that (4.1) implies that $|u_{tt}^m(0)|\leq C$ because $u_0^m\to u_0$ in $H_\Gamma^4(\Omega)$ Returning to the notation $u^{\epsilon m}$ we have from (4.35)

$$u_t^{\epsilon m}\quad\to u_t^\epsilon \text{ in } L^\infty(0,T;H_0^1(\Omega)H^2(\Omega)) \text{ weakly star} \tag{4.36}$$

$$u_{tt}^{\epsilon m}\quad\to u_{tt}^\epsilon \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star.} \tag{4.37}$$

Follows from (4.27), (4.36), (4.37) and Aubin-Lions compactness Theorem that there exists a subsequence from $(u_t^{\epsilon m})$, still denoted by $(u_t^{\epsilon m})$, such that

$$u_t^{\epsilon m}\to u_t^\epsilon \text{ strongly in } L^2(0,T;H_0^1(\Omega)) \text{ and a.e. in } Q. \tag{4.38}$$

.

**Proof of Theorem 3.1.** Finally, we prove the main theorem of this work. Let $v\in L^2(0,T;H_0^1(\Omega))$ be $v(t)\in K$ a.e. for $t\in(0,T)$. From $(3.6)_1$ follows that

$$\begin{aligned}&\int_0^T(u_{tt}^\epsilon,v-u_t^\epsilon)dt+\int_0^T(\Delta^2u^\epsilon,v-u_t^\epsilon)dt\\ &+\int_0^T(M(|\nabla u^\epsilon|^2)(-\Delta u^\epsilon),v-u_t^\epsilon)+(u_t^\epsilon,v-u_t^\epsilon)\\ &-(|u^\epsilon|^{r-1}u^\epsilon,v-u_t^\epsilon)dt=-\frac{1}{\epsilon}(\beta(u_t^\epsilon),v-u_t^\epsilon)\geq 0\end{aligned} \tag{4.39}$$

because $v \in K$ ($\beta(v) = 0$) and $\beta$ is monotone.

From (4.26), (4.27), (4.37), (4.13), (4.29), (4.38) and the Bannach-Steinhauss Theorem, it follows that there exists a subnet $(u^\epsilon)_{0<\epsilon<1}$, such that it converge to $u$ as $\epsilon \to 0$, that is

$$u^\epsilon \to u \text{ in } L^\infty(0,T;H^3_\Gamma(\Omega)), \quad \text{weakly star,} \tag{4.40}$$

$$u^\epsilon_t \to u_t \text{ in } L^2(0,T;H^1_0(\Omega)), \text{ strong and a.e. in } Q. \tag{4.41}$$

$$\Delta u^\epsilon \to \Delta u \text{ in } L^\infty(0,T;L^2(\Omega)), \quad \text{weakly star.} \tag{4.42}$$

$$u^\epsilon_{tt} \to u_{tt} \text{ in } L^\infty(0,T;L^2(\Omega), \quad \text{weakly star} \tag{4.43}$$

$$|u^\epsilon|^{r-1}u^\epsilon \to |u|^{r-1}u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weakly star} \tag{4.44}$$

The convergences above are sufficient to pass to the limit in (4.39) with $\epsilon > 0$ to conclude that (3.5) is valid. To complete the proof of Theorem 3.1, it remains to show that $u_t(t) \in K$ a.e.

In the position, we observe that using convergences obtained in the estimates above, making $m \to \infty$ in (4.1), we can find $u^\epsilon$ such that

$$\begin{aligned} u^\epsilon_{tt} &+ \Delta^2 u^\epsilon + M(|\nabla u^\epsilon|^2)(-\Delta u^\epsilon) + u^\epsilon_t + \frac{1}{\epsilon}\beta(u^\epsilon_t) \\ &= |u^\epsilon|^{r-1}u^\epsilon \text{ in } L^2(0,T;L^2(\Omega)), \end{aligned} \tag{4.45}$$

therefore

$$\beta(u^\epsilon_t) = \epsilon[-u^\epsilon_{tt} - \Delta^2 u^\epsilon - M(|\nabla u^\epsilon|^2)(-\Delta u^\epsilon) - u^\epsilon_t + |u^\epsilon|^{r-1}u^\epsilon]. \tag{4.46}$$

Then

$$\beta(u^\epsilon_t) \to 0 \text{ in } \mathcal{D}'(0,T;L^2(\Omega)).$$

From (4.46) it follows that

$$\beta(u^\epsilon_t) \text{ is bounded in } L^2(0,T;L^2(\Omega)),$$

therefore

$$\beta(u_t^\epsilon) \rightarrow 0 \text{ weak in } L^2(0,T;L^2(\Omega)). \tag{4.47}$$

On the other hand we deduce from (4.46) that

$$0 \leq \int_0^T (\beta(u_t^\epsilon), u_t^\epsilon)\, dt \leq \epsilon\, C. \tag{4.48}$$

Thus

$$\int_0^T (\beta(u_t^\epsilon), u_t^\epsilon) dt \longrightarrow 0. \tag{4.49}$$

We have that

$$\int_0^T (\beta(u_t^\epsilon) - \beta(\varphi), u_t^\epsilon - \varphi)\, dt \geq 0, \ \ \forall \varphi \text{ in } L^2(0,T;H_0^1(\Omega)),$$

because $\beta$ is a monotonous operator. Thus,

$$\int_0^T (\beta(u_t^\epsilon), u_t^\epsilon)\, dt - \int_0^T (\beta(u_t^\epsilon), \varphi)\, dt - \int_0^T (\beta(\varphi), u_t^\epsilon - \varphi)\, dt \geq 0. \tag{4.50}$$

We have from (4.49), (4.47), (4.50) and (4.41) that

$$\int_0^T (\beta(\varphi), u_t(t) - \varphi)\, dt \leq 0. \tag{4.51}$$

Taking $\varphi = u_t - \lambda v$, with $v \in L^2(0,T;H_0^1(\Omega))$ and $\lambda > 0$, we obtain

$$\int_0^T (\beta(u_t - \lambda v), \lambda v)\, dt \leq 0. \tag{4.52}$$

Multiplying (5.1) by $\dfrac{1}{\lambda} > 0$ it follows that

$$\int_0^T (\beta(u_t - \lambda v), v)\, dt \leq 0. \tag{4.53}$$

Now,

$$\int_0^T (\beta(u_t - \lambda v), v) dt \rightarrow \int_0^T (\beta(u_t, v)\, dt \text{ a.e. in } [0,T],$$

because $\beta$ is hemicontinuous. Therefore

$$\int_0^T (\beta(u_t, v)\, dt \leq 0 \;\forall v \in L^2(0,T; H_0^1(\Omega)).$$

In particular

$$\int_0^T (\beta(u_t, v)\, dt \geq 0,$$

to $-v \in L^2(0,T; H_0^1(\Omega))$. These last two inequalities imply that

$$\int_0^T (\beta(u_t, v)\, dt = 0 \;\forall v \in L^2(0,T; H_0^1(\Omega)),$$

therefore

$$\beta(u_t(t)) = 0, \tag{4.54}$$

and this implies that $u_t(t) \in K$ a.e.

## 5 Uniqueness

Let $u_1, u_2$ two solutions of (3.5) and set $w = u_2 - u_1$ and $t \in (0,T)$. Because $u_t \in K$, we can talk $u_{1t}$ (resp. $u_{2t}$) in the inequality (3.5) relative to $v_2$ (resp. $v_1$) and adding up the results we obtain

$$\begin{aligned}
&-\int_0^t (w_{tt}, w_t)ds - \int_0^t (\Delta^2 w, w_t)ds + \int_0^t (M(|\nabla u_2|^2)\Delta u_2, w_t)ds \\
&-\int_0^t (M(|\nabla u_1|^2)\Delta u_1, w_t)ds - \int_0^t (w_t, w_t)ds + \int_0^t (|u_1|^{r-1}u_1, w_t)ds \\
&-\int_0^t (|u_2|^{r-1}u_2, w_t)ds \geq 0,
\end{aligned}$$

or equivalently

$$\begin{aligned}
&-\int_0^t (w_{tt}, w_t)ds - \int_0^t (\Delta^2 w, w_t)ds + \int_0^t (M(|\nabla u_2|^2)\Delta u_2, w_t)ds \\
&-\int_0^t (M(|\nabla u_2|^2)\Delta u_1, w_t)ds + \int_0^t (M(|\nabla u_2|^2)\Delta u_1, w_t)ds \\
&-\int_0^t (M(|\nabla u_1|^2)\Delta u_1, w_t)ds - \int_0^t (w_t, w_t)ds + \int_0^t (|u_1|^{r-1}u_1, w_t)ds \\
&-\int_0^t (|u_2|^{r-1}u_2, w_t)ds = -\int_0^t (w_{tt}, w_t)ds - \int_0^t (\Delta^2 w, w_t)ds \\
&+\int_0^t (M(|\nabla u_2|^2)\Delta w, w_t)ds + \int_0^t ([M(|\nabla u_2|^2) - M(|\nabla u_1|^2)]\Delta u_1, w_t)ds \\
&-\int_0^t (w_t, w_t)ds + \int_0^t (|u_1|^{r-1}u_1, w_t)ds - \int_0^t (|u_2|^{r-1}u_2, w_t)ds \geq 0,
\end{aligned}$$

By hypotesis $(H_1)$, we can use the Mean Value Theorem to write

$$\begin{aligned}
-\int_0^t (w_{tt}, w_t)ds \quad &- \quad \int_0^t (\Delta^2 w, w_t)ds + \int_0^t (M(|\nabla u_2|^2)\Delta w, w_t)ds \\
&+ \quad \int_0^t (M'(\psi)[|\nabla u_2|^2 - |\nabla u_1|^2]\Delta u_1, w_t)ds \qquad (5.1) \\
&+ \quad \int_0^t (|u_1|^{r-1}u_1, w_t)ds - \int_0^t (|u_2|^{r-1}u_2, w_t)ds \geq 0,
\end{aligned}$$

where

$$|\nabla u_1|^2 \leq \psi \leq |\nabla u_2|^2.$$

From (5.1) it follows

$$\begin{aligned}
&\frac{1}{2} \quad \int_0^t \frac{d}{dt}\left(|w_t(s)|^2 + |\Delta w(s)|^2\right) ds + \int_0^t M(|\nabla u_2(s)|^2)\frac{d}{dt}\|w(s)\|^2 ds \\
&\leq \quad \int_0^t (|u_1(s)|^{r-1}u_1(s) - |u_2(s)|^{r-1}u_2(s), w_t(s))ds \qquad (5.2) \\
&+ \quad 2\int_0^t |M'(\psi)[(|\nabla u_2(s)| - |\nabla u_1(s)|)(|\nabla u_2(s)| + |\nabla u_1(s)|)]||\Delta u_1(s)||w_t(s)|ds.
\end{aligned}$$

Using an argument similar to that in (4.18), from (5.2) it follows

$$
\begin{aligned}
& \int_0^t \frac{d}{dt}\left\{|w_t(s)|^2+|\Delta w(s)|^2+2M(|\nabla u_2(s)|^2)\|w(s)\|ds\right\} \\
\leq\ & 2\int_0^t (|u_1(s)|^{r-1}u_1(s)-|u_2(s)|^{r-1}u_2(s),w_t(s))ds \qquad (5.3)\\
+\ & 4\int_0^t |M'(\psi)[(|\nabla u_2(s)|-|\nabla u_1(s)|)(|\nabla u_2(s)|+|\nabla u_1(s)|)]|\Delta u_1(s)||w_t(s)|ds \\
+\ & 4\int_0^t |M'(|\nabla u_2(s)|^2)|\|u_2(s)\|\|u_{2t}(s)\|\|w(s)\|^2 ds.
\end{aligned}
$$

The second line of (5.3) can be estimated in the following form:

$$
\begin{aligned}
& \int_0^t (|u_1(s)|^{r-1}u_1(s)-|u_2(s)|^{r-1}u_2(s),w_t(s))ds \\
\leq\ & \int_0^t\int_\Omega ||u_1(s)|^{r-1}u_1(s)-|u_2(s)|^{r-1}u_2(s)||w_t(s)|dxds \\
\leq\ & \int_0^t\int_\Omega (|u_1(s)|^{r-1}+|u_2(s)|^{r-1})|w(s)||w_t(s)|dxds \qquad (5.4)\\
\leq\ & C\int_0^t |u_1(s)|^{r-1}_{L^n(\Omega)}|w(s)|_{L^q(\Omega)}|w_t(s)|+|u_2(s)|^{r-1}_{L^n(\Omega)}\|w(s)\|_{L^q(\Omega)}|w_t(s)|ds \\
\leq\ & C\int_0^t (|w_t(s)|^2+\|w(s)\|^2)ds
\end{aligned}
$$

because

$$
\||u|^{r-1}\|_{L^n(\Omega)} \leq C\|u\|^{r-1} \text{ and } \||u|\|_{L^q(\Omega)} \leq C\|u\|.
$$

Also, third and fourth lines of (5.3) can be estimated by:

$$\begin{aligned}
&4\int_0^t |M'(\psi)[(|\nabla u_2(s)| - |\nabla u_1|(s))(|\nabla u_2(s)| + |\nabla u_1(s)|)]|\Delta u_1(s)||w_t(s)|ds \\
+ \quad & 4\int_0^t |M'(|\nabla u_2(s)|^2)|\|u_2(s)\|\|u_{2t}(s)\|\|w(s)\|^2 ds \\
\leq \quad & 4\int_0^t |M'(\psi)[(|\nabla u_2(s) - \nabla u_1(s)|)(|\nabla u_2(s)| + |\nabla u_1(s)|)]|\Delta u_1(s)||w_t(s)|ds \\
+ \quad & 4\int_0^t |M'(|\nabla u_2(s)|^2)|\|u_2(s)\|\|u_{2t}(s)\|\|w(s)\|^2 ds \qquad (5.5) \\
\leq \quad & 4\int_0^t |M'(\psi)[\|w(s)\|(|\nabla u_2(s)| + |\nabla u_1(s)|)]|\Delta u_1(s)||w_t(s)|ds \\
+ \quad & 4\int_0^t |M'(|\nabla u_2(s)|^2)|\|u_2(s)\|\|u_{2t}(s)\|\|w(s)\|^2 ds \\
\leq \quad & C\int_0^t (|w_t(s)|^2 + \|w(s)\|^2)ds.
\end{aligned}$$

Therefore, it follows from (5.3)-(5.5) and $H_1$

$$|w_t(s)|^2 + \|w(s)\|^2 \leq C\int_0^t (|w_t(s)|^2 + \|w(s)\|^2)ds \qquad (5.6)$$

because

$$w_t(0) = w(0) = \Delta w(0) = 0$$

Using Gronwall's inequality in (5.6), we conclude that $w(t) = 0$ therefore $u_1 = u_2$.

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