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Riemannian approximation scheme in sub-Riemannian Heisenberg space $\mathbb{H}^1$ and rotation surfaces of constant Gaussian curvature

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*Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday*

Abstract. We verify if Gausssian curvature of surfaces and geodesic curvature of curves in surfaces introduced by Diniz-Veloso [3] and by Balogh-Tyson-Vecchi [1] to prove Gauss-Bonnet theorems in Heisenberg space $\mathbb{H}^1$ are equal. Curvatures do not coincide. For comparison sake we apply the same formalism of [3] to get the curvatures of [1]. With the obtained formulas, it is possible to prove the Gauss-Bonnet theorem in [1] as a straightforward application of Stokes theorem. To exemplify the Gaussian curvature of [1], we calculate the rotation surfaces of constant curvature in $\mathbb{H}^1$, which are only of three types.

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1 Introduction

In [3] Gaussian curvature for non-horizontal surfaces in sub-Riemannian Heisenberg space $\mathbb{H}^1$ was defined and a Gauss-Bonnet Theorem was proved. The definition was analogous to Gauss curvature of surfaces in $\mathbb{R}^3$ with particular normal to surface and Hausdorff measure of area. The image of Gauss map was in the cylinder of radius one. The curvature in [3] corresponds to the curvature of covariant derivative which is null on left invariant vector fields of $\mathbb{H}^1$. The proof of Gauss-Bonnet theorem follows as the classical one. In [1], the authors utilize a limit of Gaussian and geodesic curvatures defined in the Riemannian approximations scheme $(\mathbb{R}^3, g_L)$ introduced by Gromov [5] to study sub-Riemannian spaces. They show that these limits exist (unlike the limit of Riemannian surface area form or length form), and they obtain Gauss-Bonnet theorem in $\mathbb{H}^1$ as limit of Gauss-Bonnet theorems in $(\mathbb{R}^3, g_L)$ when L goes to infinity. This construction was extended by Wang-Wei in [6] to the affine group and the group of rigid motions of the Minkowski plane.

In this paper we compare curvatures defined in [1] and [3], and show they are not coincident. Furthermore, we show that the limit K^∞ and k^∞ of Riemannian Gaussian curvatures K^L and geodesic curvatures k^L depend only on one of the two functions that define the geometry of the surface. Also we obtain that $K^\infty d\sigma = d(k^\infty ds)$ and so to get Gauss-Bonnet theorem of [1] applying Stokes theorem without taking limit.

The space $\mathbb{H}^1$ is a Lie group. We define a distribution D generated by the left invariant vector fields $e_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}$ and $e_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}$ on $\mathbb{H}^1$. Introduce a scalar product in D such that e_1, e_2 are orthonormal. We complete e_1, e_2 to a basis of left invariant vector fields in $\mathbb{H}^1$ by introducing $e_3 = [e_1, e_2] = \frac{\partial}{\partial z}$. In [1], [2] they consider the family of g^L metrics such that e_1, e_2, $e_3^L = e_3/\sqrt{L}$ is an orthonormal basis, and the Levi-Civitta connection $\overline{\nabla}^L$ on $(\mathbb{H}^1, g^L)$.

At points of a surface S, where the distribution D does not coincide with TS, the intersection $D \cap TS$ has dimension 1, and we obtain a direction that we call *characteristic* at this point of S. We suppose that all points of surface S have this property. The vector field *normal horizontal* f_1 is a unitary vector field in D orthogonal to the characteristic direction which we suppose to be globally defined. We define f^1 by $f^1(f_1) = 1$ and $f^1(TS) = 0$. We denote by f_2 a unitary vector field in $D \cap TS$ and complete a basis of TS taking $f_3 = e_3 + Af_1$. If α is the angle between e_1 and f_1, then $f_1 = \cos\alpha\, e_1 + \sin\alpha\, e_2$ and $f_2 = -\sin\alpha\, e_1 + \cos\alpha\, e_2$. An orthonormal basis of TS in $(\mathbb{R}^3, g^L)$ is given by $X_2^L = f_2$, $X_3^L = \frac{1}{\sqrt{L+A^2}} f_3$. The normal vector in g^L to S is

$$X_1^L = \frac{\sqrt{L}}{\sqrt{L+A^2}} f_1 - \frac{A}{\sqrt{L+A^2}} e_3^L.$$

The curvature of S in the metric g^L is

$$K^L = \frac{L}{(L+A^2)^2} d\alpha \wedge dA(f_3, f_2) - \frac{L^2}{(L+A^2)^2} dA(f_2) - \frac{L}{L+A^2} A^2,$$

and

$$K^\infty = \lim_{L\to\infty} K^L = -dA(f_2) - A^2.$$

This formula shows K^∞ depends only on A.

We briefly introduce the curvature K of S as in [3]. We consider the dual horizontal normal $f^1 = \cos\alpha e^1 + \sin\alpha e^2 - Ae^3$ as a Gauss map

$$g : p \in S \to (\cos\alpha, \sin\alpha, -A) \in C,$$

in the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1\}$. If U is an open set of S, the area of U is defined as $\int_U i(f_1) dV$, where $dV = f^1 \wedge f^2 \wedge f^3 = e^1 \wedge e^2 \wedge e^3$ is the element of volume in $\mathbb{H}^1$. If $\tilde{f}_1$ is the horizontal normal to the cylinder C then the area of $g(U)$ is given by $\int_{g(U)} i(\tilde{f}_1) dV$. If $p \in S$, consider a sequence of open sets with $p \in U$ and that converges to p. We define the Gaussian curvature of surface S at point p by

$$K(p) = \lim_{U\to\{p\}} \frac{\int_{g(U)} i(\tilde{f}_1) dV}{\int_U i(f_1) dV}.$$

A simple calculation shows that

$$K = d\alpha \wedge dA(f_3, f_2).$$

Observe that the curvature K appears in the expression of K^L as a term that goes to 0 when $L \to \infty$.

We finish this work by calculating the surfaces invariant by rotations around z-axis with K^∞ constant. A surface invariant by rotations and without points p where $T_pM = D_p$ is foliated by curves which are tangents to f_2. When S is invariant by rotations, these curves are unique up to rotations. We write the ordinary differential equations which give these surfaces in terms of horizontal curves. If the rotation surface is obtained by rotating the curve $(r(t), 0, c(t))$ around the z-axis, then by a parameterization satisfying

$$\theta' = \frac{\sqrt{1-(r')^2}}{r} \tag{1.1}$$

we obtain the other equations

$$K^\infty + \frac{d^2}{dt^2}(\ln r^2) + \frac{d}{dt}(\ln r^2) = 0, \tag{1.2}$$

$$c' = \frac{1}{2} r\sqrt{1-(r')^2}. \tag{1.3}$$

In case of constant Gaussian curvature K^∞, we integrate these equations and give the graphics of these surfaces which are solutions of (1.1), (1.2), (1.3). There exists only three classes of rotation surfaces with K^∞ constant, which is different from the numerous rotation surfaces of constant curvature in Euclidean space $\mathbb{R}^3$ or those in $\mathbb{H}^1$ with K constant. For all the surface graphics, we show the horizontal curves inside them. For more details on these topics see [2], [4].

2 The Heisenberg group

We denote by $\mathbb{H}^1$ the Heisenberg nilpotent Lie group whose manifold is $\mathbb{R}^3$, with Lie algebra $H^1 = V_1 \oplus V_2$, $\dim V_1 = 2$, $\dim V_2 = 1$ and

$[V_1, V_1] = V_2$, $[V_1, V_2] = 0$. Since $\mathbb{H}^1$ is nilpotent, the exponential map $\exp : H^1 \to \mathbb{H}^1$ is a diffeomorphism. Let e_1, e_2 be a basis of V_1 and $e_3 = [e_1, e_2] \in V_2$. Applying the Baker-Campbell-Hausdorff formula we have

$$\exp^{-1}(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y],$$

where $X = x_1e_1 + y_1e_2 + z_1e_3$ and $Y = x_2e_1 + y_2e_2 + z_2e_3$. As $[e_1, e_2] = e_3$ we get

$$X+Y+\frac{1}{2}[X,Y] = (x_1+x_2)e_1+(y_1+y_2)e_2+(z_1+z_2+\frac{1}{2}(x_1y_2-x_2y_1))e_3.$$

We identify $\mathbb{H}^1$ with $\mathbb{R}^3$ by identifying (x, y, z) with $\exp(xe_1 + ye_2 + ze_3)$, and this identification is known as canonical coordinates of first kind or exponential coordinates. In these coordinates the group operation is

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1)),$$

the exponential is

$$\exp(xe_1 + ye_2 + ze_3) = (x, y, z),$$

and the left invariant vector fields e_1, e_2, e_3 are given by

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \\ e_2 &= \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \\ e_3 &= \frac{\partial}{\partial z}, \end{aligned}$$

with brackets $[e_1, e_2] = e_3$, and $[e_3, e_1] = [e_3, e_2] = 0$. The dual basis is

$$\begin{aligned} e^1 &= dx, \\ e^2 &= dy, \\ e^3 &= dz + \tfrac{1}{2}(ydx - xdy), \end{aligned}$$

with $de^3 = -e^1 \wedge e^2$, $de^1 = de^2 = 0$.

Let $D \subset T\mathbb{H}^1$ be the two dimensional distribution generated by the vector fields e_1, e_2, so that D is the null space of e^3. On D we define a scalar product $<, >$ such that $\{e_1, e_2\}$ is an orthonormal basis of D.

The *element of volume* dV in $\mathbb{H}^1$ is $dV = e^1 \wedge e^2 \wedge e^3 = dx \wedge dy \wedge dz$.

3 The approximation by scalar products

Consider the g_L metrics where e_1, e_2, $e_3^L = e_3/\sqrt{L}$ is an orthonormal basis. The dual basis is e^1, e^2, $e_L^3 = \sqrt{L}e^3$. Then the Carnot-Caratheodory metric space $\mathbb{H}^1$ is the limit in the sense of Gromov-Hausdorff of Riemannian metric spaces (R^3, d_L), when $L \to \infty$. We consider the Levi-Civitta connection $\overline{\nabla}^L$ in $(\mathbb{H}^1, g_L)$. A straightforward calculation shows that (see [2]) :

$$\begin{array}{rcl} \overline{\nabla}^L e_1 & = & \frac{\sqrt{L}}{2}(-e^2 \otimes e_3^L - e_L^3 \otimes e_2) \\ \overline{\nabla}^L e_2 & = & \frac{\sqrt{L}}{2}(e^1 \otimes e_3^L + e_L^3 \otimes e_1) \\ \overline{\nabla}^L e_3^L & = & \frac{\sqrt{L}}{2}(-e^1 \otimes e_2 + e^2 \otimes e_1). \end{array} \tag{3.1}$$

The curvatute tensor $\overline{R}_{L,ijkl} = \overline{R}_L(e_i, e_j, e_k, e_l)$ is given by

$$\overline{R}_{L,1212} = \overline{R}_{L,2121} = -\overline{R}_{L,1221} = -\overline{R}_{L,2112} = \frac{3L}{4},$$

$$\overline{R}_{L,1313} = \overline{R}_{L,3131} = \overline{R}_{L,2323} = \overline{R}_{L,3232} = \frac{-L}{4},$$

$$\overline{R}_{L,1331} = \overline{R}_{L,3113} = \overline{R}_{L,2332} = \overline{R}_{L,3223} = \frac{L}{4}$$

and $\overline{R}_{L,ijkl} = 0$ otherwise.

4 Surfaces

Suppose that S is an oriented differentiable two dimensional manifold in $\mathbb{H}^1$. We get that $\dim(D \cap TS) \geq 1$, and as $de^3 = e^1 \wedge e^2$, the set where $\dim(D \cap TS) = 2$ has empty interior. We denote by $\Sigma = \{x \in S : \dim(D_x \cap T_xS) = 2\}$ and by $S' = S - \Sigma$. The set S' is open in S. In what follows we will suppose $\Sigma = \emptyset$, so $S = S'$. With this hypothesis the one dimensional vector subbundle $D \cap TS$ is well defined on S. Suppose $U \subset S$ is an open set such that we can define a unitary vector field f_2 with values in $D \cap TS$, so that $< f_2, f_2 >= 1$. Suppose $f_2 = xe_1 + ye_2$.

Definition 4.1. The unitary vector field $f_1 \in \underline{D}$ defined by

$$f_1 = ye_1 - xe_2$$

is the *horizontal normal* to S.

Then we can define $f^1 \in \underline{(T\mathbb{H}^1)^*|_S}$ by $f^1(f_1) = 1$ and $f^1(TS) = 0$. We call f^1 the *horizontal conormal* to S. If

$$f_3 = e_3 - f^1(e_3)f_1,$$

then $\{f_2, f_3\}$ is a *special* basis of TS on the open set U. If we write

$$f_1 = \cos\alpha\, e_1 + \sin\alpha\, e_2,$$

for some real function α on U, reducing U if necessary, then

$$f_2 = -\sin\alpha\, e_1 + \cos\alpha\, e_2,$$

and if we denote by $A = -f^1(e_3)$,

$$f_3 = e_3 + Af_1.$$

The dual basis of $(T\mathbb{H}^1)^*$ on S is

$$\begin{aligned} f^1 &= \cos\alpha\, e^1 + \sin\alpha\, e^2 - Ae^3, \\ f^2 &= -\sin\alpha\, e^1 + \cos\alpha\, e^2, \\ f^3 &= e^3. \end{aligned}$$

The inverse relations are

$$\begin{aligned} e^3 &= f^3, \\ e^1 &= \cos\alpha\, f^1 - \sin\alpha\, f^2 + A\cos\alpha\, f^3, \\ e^2 &= \sin\alpha\, f^1 + \cos\alpha\, f^2 + A\sin\alpha\, f^3. \end{aligned}$$

As $f^1 = 0$ on S, we get

$$\begin{aligned} df^2 &= -A\, d\alpha \wedge f^3, \\ df^3 &= Af^2 \wedge f^3, \\ 0 &= (d\alpha + A^2 f^3) \wedge f^2 - dA \wedge f^3. \end{aligned}$$

From this last relation we obtain

$$d\alpha(f_3) = -(dA(f_2) + A^2).$$

We have on TS

$$\begin{array}{rcl} d(Af^3) & = & dA \wedge f^3 + Ade^3 \\ & = & i(f_2)dAf^2 \wedge f^3 - Ae^1 \wedge e^2 \\ & = & i(f_2)dAf^2 \wedge f^3 \\ & & -A(-\sin\alpha f^2 + A\cos\alpha f^3) \wedge (\cos\alpha f^2 + A\sin\alpha f^3) \\ & = & (i(f_2)dA + A^2)f^2 \wedge f^3 \end{array} \tag{4.1}$$

5 The orthonormal basis

An orthonormal basis of TS in $(\mathbb{R}^3, g_L)$ is given by

$$X_2^L = f_2 \text{ and } X_3^L = \frac{1}{\sqrt{L+A^2}} f_3. \tag{5.1}$$

The normal vector in g_L to S is

$$X_1^L = \frac{\sqrt{L}}{\sqrt{L+A^2}} f_1 - \frac{A}{\sqrt{L+A^2}} e_3^L = \frac{\sqrt{L+A^2}}{\sqrt{L}} f_1 - \frac{A}{\sqrt{L+A^2}\sqrt{L}} f_3. \tag{5.2}$$

If we write $\cos\beta = \frac{\sqrt{L}}{\sqrt{L+A^2}}$ and $\sin\beta = \frac{A}{\sqrt{L+A^2}}$, then our orthonormal basis is

$$\begin{array}{rcl} X_1^L & = & \cos\beta\cos\alpha e_1 + \cos\beta\sin\alpha e_2 - \sin\beta e_3^L \\ X_2^L & = & -\sin\alpha e_1 + \cos\alpha e_2 \\ X_3^L & = & \sin\beta\cos\alpha e_1 + \sin\beta\sin\alpha e_2 + \cos\beta e_3^L \end{array}$$

As $d\sin\beta = \cos\beta d\beta$, we get

$$d\beta = \frac{\sqrt{L}}{L+A^2} dA.$$

6 The projection of covariant derivatives

The connection ∇^L on TS is defined by

$$\nabla_X^L Y = \overline{\nabla}_X^L Y - < \overline{\nabla}_X^L Y, X_1^L > X_1^L,$$

for X, Y sections of TS. We have

$$\nabla^L X_2^L = < \overline{\nabla}^L X_2^L, X_3^L > X_3^L.$$

Taking into account (3.1) we get

$$\overline{\nabla}^L X_2^L = (-d\alpha + \frac{\sqrt{L}}{2} e_L^3) \otimes (\cos\alpha e_1 + \sin\alpha e_2) + \frac{\sqrt{L}}{2}(\cos\alpha e^1 + \sin\alpha e^2) \otimes e_3^L$$

and from

$$\begin{array}{rcl} e^1 & = & \cos\beta\cos\alpha X_L^1 - \sin\alpha X_L^2 + \sin\beta\cos\alpha X_L^3 \\ e^2 & = & \cos\beta\sin\alpha X_L^1 + \cos\alpha X_L^2 + \sin\beta\sin\alpha X_L^3 \\ e_L^3 & = & -\sin\beta X_L^1 + \cos\beta X_L^3 \end{array}$$

we get

$$\begin{array}{rcl} < \overline{\nabla}^L X_2^L, X_3^L > & = & (-d\alpha + \frac{\sqrt{L}}{2} e_L^3)\sin\beta + \frac{\sqrt{L}}{2}(\cos\alpha e^1 + \sin\alpha e^2)\cos\beta \\ & = & -d\alpha\sin\beta + \frac{\sqrt{L}}{2}\cos(2\beta) X_L^1 + \frac{\sqrt{L}}{2}\sin(2\beta) X_L^3. \end{array}$$

As X_L^1 is null on TS we get

$$\nabla^L X_2^L = (-d\alpha\sin\beta + \frac{\sqrt{L}}{2}\sin(2\beta) X_L^3) \otimes X_3^L.$$

In the same way as $< \overline{\nabla}^L X_3^L, X_2^L >= - < \overline{\nabla}^L X_2^L, X_3^L >$, we obtain

$$\nabla^L X_3^L = (d\alpha\sin\beta - \frac{\sqrt{L}}{2}\sin(2\beta) X_L^3) \otimes X_2^L.$$

Then

$$\begin{array}{rcl} \lim_{L\to\infty} \nabla^L X_2^L & = & \lim_{L\to\infty}(-d\alpha\sin\beta + \frac{\sqrt{L}}{2}\sin(2\beta) X_L^3) \otimes X_3^L \\ & = & \lim_{L\to\infty}(-d\alpha\frac{A}{\sqrt{L+A^2}} + \sqrt{L}\frac{A}{\sqrt{L+A^2}}\frac{\sqrt{L}}{\sqrt{L+A^2}}\sqrt{L+A^2} f^3) \\ & & \otimes \frac{1}{\sqrt{L+A^2}} f_3 \\ & = & A f^3 \otimes f_3. \end{array}$$

In the same way

$$\nabla^L X_3^L = \nabla^L \frac{1}{\sqrt{L+A^2}} f_3 = \frac{-AdA}{\sqrt{L+A^2}^3} \otimes f_3 + \frac{1}{\sqrt{L+A^2}} \nabla^L f_3$$

or

$$\begin{array}{rcl} \nabla^L f_3 & = & \sqrt{L+A^2}\frac{AdA}{\sqrt{L+A^2}^3} \otimes f_3 \\ & & + \sqrt{L+A^2}(d\alpha\frac{A}{\sqrt{L+A^2}} - \sqrt{L}\frac{A}{\sqrt{L+A^2}}\frac{\sqrt{L}}{\sqrt{L+A^2}}\sqrt{L+A^2} f^3) \otimes f_2 \end{array}$$

therefore $\lim_{L\to\infty}\nabla^L f_3$ does not exist. The limit of covariant derivatives ∇^L is not well defined.

Also

$$\begin{aligned} X_L^3([X_2^L, X_3^L]) &= \tfrac{1}{\cos\beta} e_L^3([X_2^L, X_3^L]) \\ &= \tfrac{-1}{\cos\beta}(\sin\beta\sqrt{L} - \sin\beta\sin\alpha d\beta(e_1) + \sin\beta\cos\alpha d\beta(e_2)) \\ &= \tfrac{\sin\beta}{\cos\beta}(-\sqrt{L} - d\beta(X_2^L)) \end{aligned}$$

7 The limit of curvatures

Now we will calculate the Gaussian curvature

$$K^L =< R^L(X_2^L, X_3^L)X_3^L, X_2^L >,$$

where $R^L(X,Y)Z = \nabla_X^L\nabla_Y^L Z - \nabla_Y^L\nabla_X^L Z - \nabla_{[X,Y]}^L Z$. Therefore

$$< \nabla^L_{X_2^L}\nabla^L_{X_3^L} X_3^L, X_2^L >= X_2^L(d\alpha(X_3^L)\sin\beta - \frac{\sqrt{L}}{2}\sin(2\beta))$$

$$< \nabla^L_{X_3^L}\nabla^L_{X_2^L} X_3^L, X_2^L >= X_3^L(d\alpha(X_2^L)\sin\beta)$$

$$\begin{aligned} < \nabla^L_{[X_2^L, X_3^L]} X_3^L, X_2^L > &= d\alpha([X_2^L, X_3^L])\sin\beta \\ &\quad -\tfrac{\sqrt{L}}{2}\sin(2\beta)X_L^3([X_2^L, X_3^L]) \\ &= d\alpha([X_2^L, X_3^L])\sin\beta \\ &\quad -\tfrac{\sqrt{L}}{2}\sin(2\beta)(\tfrac{\sin\beta}{\cos\beta}(-\sqrt{L} - d\beta(X_2^L))) \end{aligned}$$

Then

$$\begin{aligned} K^L &= X_2^L(d\alpha(X_3^L)\sin\beta - \tfrac{\sqrt{L}}{2}\sin(2\beta)) - X_3^L(d\alpha(X_2^L)\sin\beta) \\ &\quad -d\alpha([X_2^L, X_3^L])\sin\beta + \tfrac{\sqrt{L}}{2}\sin(2\beta)(\tfrac{\sin\beta}{\cos\beta}(-\sqrt{L} - d\beta(X_2^L))) \\ &= d\alpha(X_3^L)\cos\beta d\beta(X_2^L) - \tfrac{\sqrt{L}}{2}2\cos(2\beta)d\beta(X_2^L) \\ &\quad -d\alpha(X_2^L)\cos\beta d\beta(X_3^L) + \sqrt{L}\sin^2(\beta)(-\sqrt{L} - d\beta(X_2^L)) \\ &= \cos\beta d\alpha\wedge d\beta(X_3^L, X_2^L) - \sqrt{L}\cos^2\beta d\beta(X_2^L) - L\sin^2\beta \\ &= \tfrac{\sqrt{L}}{\sqrt{L+A^2}} d\alpha\wedge \tfrac{\sqrt{L}}{L+A^2} dA(\tfrac{1}{\sqrt{L+A^2}} f_3, f_2) \\ &\quad -\sqrt{L}(\tfrac{\sqrt{L}}{\sqrt{L+A^2}})^2 \tfrac{\sqrt{L}}{L+A^2} dA(f_2) - L(\tfrac{A}{\sqrt{L+A^2}})^2 \\ &= \tfrac{L}{(L+A^2)^2} d\alpha\wedge dA(f_3, f_2) - \tfrac{L^2}{(L+A^2)^2} dA(f_2) - \tfrac{L}{L+A^2}A^2. \end{aligned}$$

Therefore

$$K^\infty = \lim_{L\to\infty} K^L = -dA(f_2) - A^2. \qquad (7.1)$$

8 The limit of Riemannian area elements

It follows from (5.2) and (5.1) that

$$X_L^1 = \frac{\sqrt{L}}{\sqrt{L+A^2}} f^1,\ X_L^2 = f^2 \text{ and } X_L^3 = \sqrt{L+A^2} f^3 + \frac{A}{\sqrt{L+A^2}} f^1.$$

Therefore on S we get

$$d\sigma_L = X_L^2 \wedge X_L^3 = \sqrt{L+A^2} f^2 \wedge f^3$$

since that f^1 is null on TS. We can see that $\lim_{L\to\infty} K^L d\sigma_L$ does not exist. In [1], to get an area form on S it was necessary to multiply $d\sigma_L$ by $\frac{1}{\sqrt{L}}$ and take the limit as L goes to infinity to obtain a surface form, i.e.,

$$d\sigma = \lim_{L\to\infty} \frac{1}{\sqrt{L}} d\sigma_L = f^2 \wedge f^3,$$

which is the Hausdorff measure on S. Therefore

$$\lim_{L\to\infty} \frac{1}{\sqrt{L}} K^L d\sigma_L = K^\infty f^2 \wedge f^3 = K^\infty d\sigma.$$

9 The limit of geodesic curvatures of transverse curves

Suppose $\gamma(t)$ is a curve in S such that $\gamma'(t) = a(t) f_2(\gamma(t)) + b(t) f_3(\gamma(t))$, where $b(t) \neq 0$ for every t. Then $\gamma'(t) = a(t) X_2^L + b(t)\sqrt{L+A^2} X_3^L$ and the unitary tangent vector in the metric g_L is

$$T^L(t) = \frac{1}{\sqrt{a^2 + b^2(L+A^2)}} (a X_2^L + b\sqrt{L+A^2} X_3^L).$$

Let's write $a^L = \frac{a}{\sqrt{a^2+b^2(L+A^2)}}$ and $b^L = \frac{b\sqrt{L+A^2}}{\sqrt{a^2+b^2(L+A^2)}}$, so that $T^L = a^L X_2^L + b^L X_3^L$. Then

$$\begin{aligned} \nabla^L_{T^L} T^L &= \tfrac{d}{dt} a^L X_2^L + \tfrac{d}{dt} b^L X_3^L + a^L \nabla^L_{T^L} X_2^L + b^L \nabla^L_{T^L} X_3^L \\ &= \tfrac{d}{dt} a^L X_2^L + \tfrac{d}{dt} b^L X_3^L \\ &\quad + (-\sin\beta\, d\alpha(T^L) + \tfrac{\sqrt{L}}{2} \sin(2\beta) X_L^3(T^L))(a_L X_3^L - b^L X_2^L) \end{aligned}$$

Let be $N^L = -b^L X_2^L + a^L X_3^L$. Then $k^L =< \nabla^L_{T^L} T^L, N^L >$ is the geodesic curvature of γ, so

$$\begin{aligned}
k^L &= -b_L \tfrac{d}{dt} a^L + a_L \tfrac{d}{dt} b^L + (-\sin\beta d\alpha(T^L) + \tfrac{\sqrt{L}}{2}\sin(2\beta) X_L^3(T^L)) \\
&= \tfrac{1}{\sqrt{L+A^2}(a^2+b^2(L+A^2))}(abA\tfrac{d}{dt}A + (a\tfrac{d}{dt}b - b\tfrac{d}{dt}a)(L+A^2)) \\
&\quad - \sin\beta(a_L d\alpha(X_2^L) + b_L d\alpha(X_3^L)) + \sqrt{L}\sin\beta\cos\beta b_L \\
&= \tfrac{1}{\sqrt{L+A^2}(a^2+b^2(L+A^2))}(abA\tfrac{d}{dt}A + (a\tfrac{d}{dt}b - b\tfrac{d}{dt}a)(L+A^2)) \\
&\quad - \tfrac{A}{\sqrt{L+A^2}} \tfrac{a}{\sqrt{a^2+b^2(L+A^2)}} d\alpha(f_2) \\
&\quad - \tfrac{A}{\sqrt{L+A^2}} \tfrac{b\sqrt{L+A^2}}{\sqrt{a^2+b^2(L+A^2)}} d\alpha(\tfrac{1}{\sqrt{L+A^2}} f_3) \\
&\quad + \sqrt{L}\tfrac{A}{\sqrt{L+A^2}} \tfrac{\sqrt{L}}{\sqrt{L+A^2}} \tfrac{b\sqrt{L+A^2}}{\sqrt{a^2+b^2(L+A^2)}}
\end{aligned}$$

It follows from this formula that

$$k^\infty = \lim_{L\to\infty} k^L = A\frac{b}{|b|}$$

We can see, from expressions $K^\infty = -dA(f_2) - A^2$ and $k^\infty = A\frac{b}{|b|}$, that both formulas depend only on A. This means that the "horizontal" geometry of the surface S disappears as L goes to infinity. This will became clear as we proceed to find the rotation surfaces with K^∞ constant, which are composed only of three families.

10 The limit of length elements

The length element in the metric g^L on γ is

$$ds_L = a^L X_L^2 + b^L X_L^3.$$

As $\lim_{L\to\infty} a^L = 0$ and $\lim_{L\to\infty} b^L = \frac{b}{|b|}$, we get

$$\lim_{L\to\infty} ds_L = \lim_{L\to\infty} (a^L f^2 + b^L \sqrt{L+A^2} f^3) = \frac{b}{|b|} f^3 \lim_{L\to\infty} \sqrt{L+A^2}$$

that does not exist. But as in [1] and section 8, if we multiply by $\frac{1}{\sqrt{L}}$ we obtain

$$ds = \lim_{L\to\infty} \frac{1}{\sqrt{L}} ds_L = \frac{b}{|b|} f^3,$$

which is Hausdorff measure for transversal curves. It follows that

$$\lim_{L\to\infty} \frac{1}{\sqrt{L}} k^L ds_L = k^\infty ds = Af^3.$$

11 The Gauss-Bonnet theorem

The proof of Gauss-Bonnet theorem in [1] was done taking limits of Gauss-Bonnet formulas in (R^3, g_L) as L goes to infinity:

$$\int_S \frac{1}{\sqrt{L}} K^L d\sigma_L + \int_{\partial S} \frac{1}{\sqrt{L}} k^L ds_L = \frac{1}{\sqrt{L}} 2\pi\chi(S)$$

to obtain $\int_S K^\infty d\sigma + \int_{\partial S} k^\infty ds = 0$.

We will give below a straightforward proof due to expressions of K^∞ and k^∞ obtained in sections 7 and 9. We will restrict our theorem to regions where points are non singular and the boundary is constituted by transverse curves.

Let be $R \subset S$ a fundamental set, and c a fundamental 2-chain such that $|c| = R$. The oriented curve $\gamma = \partial c$ is the bounding curve of R. The curve γ is piecewise differentiable, and composed by differentiable curves $\gamma_j : [s_j, s_{j+1}] \to S$, $j = 1, \dots, r$, with $\gamma_1(s_1) = \gamma_r(s_{r+1})$ and $\gamma_j(s_{j+1}) = \gamma_{j+1}(s_{j+1})$, for $j = 1, \dots, r-1$.

Theorem 11.1. *(Gauss-Bonnet formula) Let R be contained in a coordinate domain U of S, let the bounding curve γ of R be a simple closed transverse curve. Then*

$$\int_\gamma k^\infty + \int_R K^\infty = 0,$$

where $k^\infty = \lim_{L\to\infty} k^L$ on γ and $K^\infty = \lim_{L\to\infty} K^L$ on R.

Proof. From Stokes theorem and using (4.1) we get

$$\begin{aligned} \textstyle\int_R K^\infty &= \textstyle\int_c K^\infty f^2 \wedge f^3 = \int_c (-i(f_2)dA - A^2) f^2 \wedge f^3 \\ &= \textstyle\int_c d(-Af^2) = -\int_{\partial c} Af^2 = -\int_\gamma k^\infty. \end{aligned}$$

□

12 The limit of curvatures for surfaces invariant by rotations

A curve γ in $\mathbb{H}^1$ is *horizontal* if $\gamma'(t)$ is contained in D for every t. If $(a(t), b(t))$ is a curve in $\mathbb{R}^2$ and $(a(t), b(t), c(t))$ is a horizontal curve in $\mathbb{H}^1$, then

$$c(t) = \frac{1}{2}\int_0^t (a(s)b'(s) - b(s)a'(s))ds. \tag{12.1}$$

Given a surface S in $\mathbb{H}^1$ such that $\dim(TS \cap D) = 1$, then the surface S is foliated by the integral curves of distribution $TM \cap D$. The distribution D is invariant by rotations (around the z-axis). If S is invariant by rotations, then S is foliated by horizontal curves tangent to f_2. These curves are unique up to rotations. Therefore we can obtain all surfaces invariant by rotations, with $\Sigma = \emptyset$, rotating horizontal curves around the z-axis. Below we write the equations that define these surfaces.

Rotating the curve (12.1) we obtain the surface

$$f(u, v) = (a(v)\cos u - b(v)\sin u, b(v)\cos u + a(v)\sin u, c(v)).$$

The coordinate vector fields are

$$\begin{aligned} f_u &= f_*(\tfrac{\partial}{\partial u}) = (-a(v)\sin u - b(v)\cos u)e_1 \\ &\quad +(-b(v)\sin u + a(v)\cos u)e_2 - \tfrac{1}{2}(a(v)^2 + b(v)^2)e_3 \end{aligned}$$

and

$$f_v = f_*(\frac{\partial}{\partial v}) = (a'(v)\cos u - b'(v)\sin u)e_1 + (a'(v)\sin u + b'(v)\cos u)e_2.$$

Then $f_v \in TS \cap D$, and as $< f_v, f_v >= a'(v)^2 + b'(v)^2$ we obtain

$$f_2(u, v) = \frac{1}{\sqrt{a'(v)^2 + b'(v)^2}} f_v(u, v). \tag{12.2}$$

Therefore

$$f_2(u, v) = \frac{a'(v)\cos u - b'(v)\sin u}{\sqrt{a'(v)^2 + b'(v)^2}} e_1 + \frac{a'(v)\sin u + b'(v)\cos u}{\sqrt{a'(v)^2 + b'(v)^2}} e_2,$$

and it follows

$$\cos\alpha(u,v) = \frac{a'(v)\sin u + b'(v)\cos u}{\sqrt{a'(v)^2 + b'(v)^2}},$$

$$\sin\alpha(u,v) = \frac{-a'(v)\cos u + b'(v)\sin u}{\sqrt{a'(v)^2 + b'(v)^2}}.$$

Then

$$f_1(u,v) = \frac{a'(v)\sin u + b'(v)\cos u}{\sqrt{a'(v)^2 + b'(v)^2}} e_1 - \frac{a'(v)\cos u - b'(v)\sin u}{\sqrt{a'(v)^2 + b'(v)^2}} e_2.$$

As $< f_u(u,v), f_v(u,v) >= a(v)b'(v) - b(v)a'(v)$, we get

$$f_u - \frac{a(v)b'(v) - b(v)a'(v)}{\sqrt{a'(v)^2 + b'(v)^2}} f_2 = -\frac{a(v)a'(v) + b(v)b'(v)}{\sqrt{a'(v)^2 + b'(v)^2}} f_1 - \frac{a(v)^2 + b(v)^2}{2} e_3$$

so

$$f_u - \frac{ab' - ba'}{\sqrt{(a')^2 + (b')^2}} f_2 = -\frac{a^2 + b^2}{2}\left(e_3 + 2\frac{aa' + bb'}{\sqrt{(a')^2 + (b')^2}(a^2 + b^2)} f_1\right).$$

It follows that

$$f_u = \frac{ab' - ba'}{\sqrt{(a')^2 + (b')^2}} f_2 - \frac{a^2 + b^2}{2} f_3$$

where $f_3 = e_3 + Af_1$ and

$$A = 2\frac{aa' + bb'}{\sqrt{(a')^2 + (b')^2}(a^2 + b^2)}.$$

From equation $d\sin\alpha = \cos\alpha\, d\alpha$ we get

$$d\sin\alpha = \frac{a'\sin u + b'\cos u}{\sqrt{(a')^2 + (b')^2}}(du + \frac{a''b' - b''a'}{(a')^2 + (b')^2} dv)$$

so

$$d\alpha = du + \frac{a''b' - b''a'}{(a')^2 + (b')^2} dv.$$

Introducing polar coordinates

$$a(t) = r(t)\cos\theta(t), \ b(t) = r(t)\sin\theta(t)$$

we get $a^2+b^2=r^2$, $ab'-ba'=r^2\theta'$, $aa'+bb'=rr'$, $(a')^2+(b')^2=(r')^2+r^2(\theta')^2$, $a'b''-b'a''=(r^2(\theta')^2+2(r')^2-rr'')\theta'+rr'\theta''$. Taking a parameterization such that

$$(a')^2+(b')^2=1,$$

then

$$A=2\frac{aa'+bb'}{(a^2+b^2)}=\frac{2rr'}{r^2}=\frac{d}{dv}\ln r^2 \tag{12.3}$$

Also from (12.2)

$$f_2(u,v)=f_v(u,v).$$

and (7.1)

$$K^\infty=-dA(f_v)-A^2=-\frac{dA}{dv}-A^2 \tag{12.4}$$

13 Surfaces of constant limit curvatures invariant by rotations

Suppose that K^∞ is constant on S. Solving equation (12.4) we obtain

$$A(v)=-\sqrt{K^\infty}\tan(\sqrt{K^\infty}v+c_1),$$

for $K^\infty>0$;

$$A(v)=\sqrt{-K^\infty}\tanh(\sqrt{-K^\infty}v+c_1),$$

for $K^\infty<0$;

$$A(v)=\frac{1}{v+c_1}$$

for $K^\infty=0$. From (12.3) we get:

$$r(v)=c_2\sqrt{\cos(\sqrt{K^\infty}v+c_1)}$$

for $K^\infty>0$;

$$r(v)=c_2\sqrt{\cosh(\sqrt{-K^\infty}v+c_1)}$$

for $K^\infty<0$;

$$r(v)=c_2\sqrt{v+c_1}$$

for $K^\infty = 0$. In any case

$$\begin{aligned} \theta(v) &= \int_0^v \frac{\sqrt{1-r'(t)^2}}{r(t)} dt \\ c(v) &= \frac{1}{2}\int_0^v r(t)\sqrt{1-r'(t)^2} dt. \end{aligned} \tag{13.1}$$

13.1 Positive case

We can write

$$r(v) = r_0\sqrt{\cos(\sqrt{K^\infty}v)},$$

and $c(v)$, $\theta(v)$ as in (13.1). The surface is defined only for

$$|v| < \frac{1}{\sqrt{K^\infty}}\cos^{-1}\left(-\frac{2}{r_0^2K^\infty} + \sqrt{\frac{4}{r_0^4(K^\infty)^2}+1}\right)$$

and, in figure 13.1 below, we can see the graphics of S with $r_0 = 1$ and $K^\infty = 1$, indicating the horizontal curve.

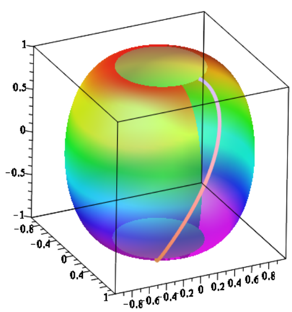

Figure 13.1: Surface with $r_0 = 1$ and $K = 1$

13.2 Null case

We can write

$$r(v) = r_0\sqrt{v},$$

and $c(v)$, $\theta(v)$ as in (13.1). The curve is defined only for $v > \frac{R^2}{4}$ and, in figure 13.2 below, we can see the graphics of S with $r_0 = 1$ and $K^\infty = 0$, showing the horizontal curve.

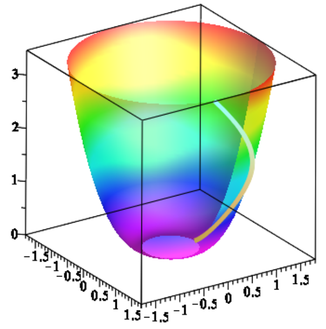

Figure 13.2: Surface with $r_0 = 1$ and $K^\infty = 0$

13.3 Negative case

We can write

$$r(v) = r_0\sqrt{\cosh(\sqrt{-K^\infty}v)},$$

and $c(v)$, $\theta(v)$ as in (13.1). The surface is defined only for

$$|v| < \frac{1}{\sqrt{-K^\infty}}\cosh^{-1}\left(\frac{-2}{r_0^2K^\infty} + \sqrt{1 + \frac{4}{r_0^4(K^\infty)^2}}\right)$$

and, in figure 13.3 below, we can see the graphics of S with $r_0 = 1$ and $K^\infty = -1$, showing the horizontal curve.

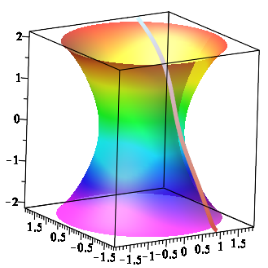

Figure 13.3: Surface with $r_0 = 1$ and $K^\infty = -1$

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