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The Spectrum of the Hodge - Laplacian

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Dedicated to Professor Renato Tribuzy on the occasion of his 75th birthday

Abstract. With this expository paper, we aim to revive the ideas and the tools used by Ikeda and Taniguchi for the explicit computation of the spectrum of the Hodge - Laplacian defined on the p-exterior bundles over $\mathbb{S}^n$ and $\mathbb{CP}^n$. Firstly, we briefly explain the general underlying philosophy of the latter computation. Secondly, starting almost from scratch, we try to develop as assiduously as possible all the interdisciplinary background necessary to understand in-depth the work of Ikeda and Taniguchi. Thirdly, we study the structure of the spectrum of the Hodge-Laplacian and explain how representation theory naturally comes into play. Fourthly, we give a broad-brush sketch of its actual computation in the case of spheres. Finally, we conclude by posing some questions which could conceivably be worth pondering.

Keywords: Hodge-Laplacian, Spectrum, Casimir operator, Freudenthal formula.

2020 Mathematics Subject Classification: 12A34, 67B89.

1 Introduction

The Laplacian operator, immediately recognisable by its symbol Δ, is a remarkable mathematical object. Miraculously, it pops up in almost all fundamental areas in both mathematics and exact sciences. Forsooth, there would hardly exist a person who read an exact science and not being familiar with one of the definitions of this intriguing operator. Either in Analysis or PDEs, in Quantum mechanics or Thermodynamics and rather remarkably in Riemannian geometry, it manifests itself as a fundamental concept without which the landscape of modern science would be impossible to grasp. Indeed, the Laplacian operator quietly lies in the intersection of many areas of the natural sciences. Many scientists have thoroughly studied both its properties and its numerous applications. Yet, the explicit computation of its spectrum is known to be a notoriously arduous task, and relatively few examples are known to the present day. Furthermore, calculations by direct assault hardly ever yielded results, and the pursuit of alternative methods became necessary.

For the standard Laplacian acting on smooth functions on a manifold M or, more precisely, the Laplace - Beltrami operator, the spectrum is known explicitly in the cases when M is a flat torus, a Hopf manifold, a symmetric space, a Klein bottle, $\mathbb{S}^n$, $\mathbb{CP}^n$, and $\mathbb{RP}^n$. The spectrum of the Laplace-Beltrami operator is also known for covering spaces, products, and submersions with totally geodesic fibres of the latter manifolds [2, 3]. In the case when the Laplacian acts on forms, which we shall call the Hodge-Laplacian herein, the progress regarding the explicit calculation of the spectrum is significantly humbler. At the time of writing, we are only aware of five papers that yielded results in this direction. In 1977, exploiting the full power of the representation theory of Lie groups and Lie algebras, Ikeda and Taniguchi [16] computed the spectra of the Hodge-Laplacian acting on the exterior p-form bundles over $\mathbb{S}^n$ and $\mathbb{CP}^n$. A year later, Iwasake and Katase [15] also studied the Hodge-Laplacian on the exterior p-form bundle over $\mathbb{S}^n$ by computing its spectrum by direct computations relying on homogeneous harmonic forms. They virtually extended the ideas and corrected the errors of a previous work by Gallot and Meyer [8]. The spectrum of the Hodge-Laplacian in the case of compact semi-simple Lie groups was first worked out by Beers and Millman [1] and subsequently by Fegan [10].

Notwithstanding the advances in Spectral Geometry, the study of the spectrum of the Laplacian operator remains an active field of research[1]. We shall not dwell herewith on either of the intriguing questions and methods of Spectral Geometry. We shall keep both the focus and the narrative of this paper exclusively in the business of the explicit calculation of the spectrum of the Hodge - Laplacian operator via representation theory. Nonetheless, we should only like to recommend instead the excellent concise survey by Martin Vito Cruz[2] in which the reader may learn more about this area. Also, we shall neither touch upon any considerations of the Laplace - Beltrami operator nor delve into some of the few alternatives to representation theory techniques. Our ultimate goal is to humbly revive the beautiful work of Ikeda and Taniguchi by attempting to present their ideas in as accessible a form as possible, and with the hope that more people would get interested in this matter and possibly get themselves involved in further studies. For these reasons, we shall aim at painting the broader picture, and except for the complete proof that the Hodge-Laplacian equals minus the Casimir operator, we shall avoid tedious calculations.

In 2018, the author ventured twice to lecture about the matters to be addressed herein by reading two mini-courses. The present text is motivated by and is essentially an improvement of the lectures taught at the XX School of Differential Geometry held at the Federal University of Paraíba in Brazil. Thus, this expository paper resembles the style of lecture notes more than any other style of writing mathematics. Taking this latter into account, we wish to believe that the reader might like to

[1] Neither necessarily nor exclusively in the direction of explicit computations.

[2] Alas, the only source we know this preprint is available is Professor Alan Weinstein's home page.

consider improving further the contents of these notes.

We should like to attempt to write a text that suits the palates and the varying needs of a rather diverse readership. For this reason, we ought to deviate from the immediate traditional discussion on preliminaries. Instead, we shall begin directly by posing the problem and by unravelling the general philosophy of its solution. In this way, at the end of Section 2, the reader should have the freedom of deciding to read in the subsequent background section what is only necessary to follow the general discussion on the method of Ikeda and Taniguchi. In Section 3, we elaborate in as much detail as possible on the interdisciplinary background necessary to set the context and to properly understand the solution. In particular, we define and explain concepts such as *Lie group actions*, *homogeneous manifolds*, *representation theory of Lie groups and Lie algebras*, *fibre bundles*, and develop some basics of *Hodge theory*. In Section 4, we attempt to elucidate the Ikeda and Taniguchi computation as clearly as possible, but without emphasising detailed and tedious calculations. We culminate our humble excursion in the realm of the Hodge-Laplacian by posing some questions which, we hope, might instigate possible future studies.

2 The problem and the general philosophy behind its resolution

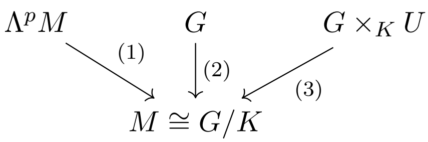
In this brief section, we discuss the general philosophy of the method of Ikeda and Taniguchi without any formal definitions and rigour. The reason for doing so is twofold. On the one hand, this will permit the reader simultaneously learn about the problem and a sketch of the path leading to its solution. On the other, it will indicate to the reader which parts of the preliminary section they need to study before delving into the method itself.

Ikeda and Taniguchi resolved the following problem in [16].

Problem 1. *Given a compact symmetric pair of Lie groups (G, K) with G semisimple, consider the homogeneous space $M = G/K$ and the Hodge-*

Laplacian Δ acting on the sections of the exterior bundle $\Lambda^p M$. Compute the spectrum of Δ.

The first thing to perceive when reading the statement of Problem 1 is that three vector bundles naturally arise in this context. These are the *exterior vector bundle* $\Lambda^p M \longrightarrow M$, the *principal K-bundle* $G \longrightarrow G/K$ and the *homogeneous bundle* $G \times_K U \longrightarrow G/K$. It is worth visualising this latter statement in terms of the following diagram.[3]



Notice that the supposition that $M = G/K$ is a Riemannian homogeneous manifold is key to the resolution of Problem 1, for it permits the natural construction of bundles (2) and (3) over M. As the reader will soon see, we shall frequently have to "jump" from one bundle to another. At any rate, the vector bundle (1) constitutes the contextual bundle of Problem 1, and it is the bundle (3) that will bring forth the solution. Bundle (2), albeit very natural, might seem redundant or unnecessary at first glance. Indeed, having as fibres copies of the subgroup K, bundle (2) looks rather abstract and not very convenient for our computational goals. However, one must not ignore the latter as the readers will soon appreciate this fact themselves. Precisely speaking, we have the following picture. The Hodge-Laplacian Δ is defined to act on the sections of the exterior bundle (1). Alas, working directly in the latter setting is known to bring difficulties. The principal bundle (2) is also inopportune because of its abstract fibres. Luckily, however, the fibres of the homogeneous bundle (3) are no longer as abstract as the fibres of (2) simply because the former are copies of the finite-dimensional vector space U which corresponds to a representation of the subgroup K. Moreover, the latter is a standard

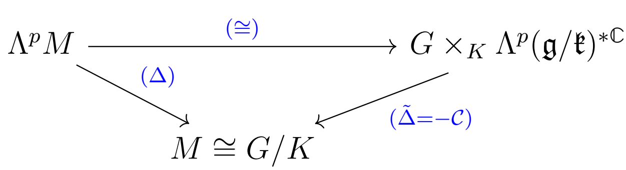
[3]For the sake of brevity, we find it convenient to keep referring to these three bundles as (1), (2), and (3) throughout the text.

tool in representation theory, for it is intimately related to the concept of *induced representations*. Thus, the homogeneous bundle (3) proves more suitable for our goals.

The solution to Problem 1 breaks down into the following four main steps. Firstly, define a suitable isomorphism between the vector bundles (1) and (3). Secondly, prove that $\Delta = -\mathcal{C}$, where $\mathcal{C}$ is the Casimir operator. The latter operator is an element of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$, and as is well-known, its spectrum is explicitly computable through the Freudenthal formula. Thirdly, decompose the space of differential forms $C^\infty(\Lambda^p M)$ into irreducible G-modules by using the Hodge decomposition theorem and the fact that the Hodge-Laplacian is a G-invariant operator. In other words, to find the eigenspaces of the Hodge-Laplacian is tantamount to computing the irreducible G-modules of $C^\infty(\Lambda^p M)$. Finally, use the Frobenius reciprocity law to reduce the computations to only irreducible K-modules. This last step is motivated by the fact that K is a compact closed subgroup of G, and the eigenspaces also have the structure of irreducible K-modules. Thus, Problem 1 amounts to the following problem in representation theory.

Problem 2. *How does an irreducible G-module decompose into irreducible K-modules?*

Luckily there is an answer to the latter problem in the literature [4, 23], and one can thus regard Problem 1 as solved. Indeed, with the solution to Problem 2 at hand, and the identification $\Delta = -\mathcal{C}$ in mind, it only remains to employ Freudenthal's formula to $\mathcal{C}$. We must notice at this juncture that the representation space U which frames the proof, is precisely the vector space $\Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively. In summary, one can visualise the solution to Problem 1 by the following simple diagram.



Notice that this diagram is not to be read as a typical commutative diagram, for it is not one! It is only of an auxiliary purpose, for it provides visual clarity of the idea of the solution. For this reason, one must interpret this diagram as follows. First, one identifies the vector bundles (1) and (3). Second, one defines the Hodge-Laplacian Δ and its "extension" $\tilde{\Delta}$ on the sections of bundles (1) and (3), respectively. Third, the identity $\tilde{\Delta} = -\mathcal{C}$ holds.

3 Some indispensable definitions and facts

The approach of Ikeda and Taniguchi is interdisciplinary and blends ideas from Differential and Riemannian geometry, Representation theory, Hodge theory, Homogeneous spaces, and Lie theory. Thus, a preliminary section is indeed necessary. Nonetheless, the impossibility of foreseeing the average mathematical background of the reader makes us a little insecure in the choice of the definitions and the propositions to be included in this preliminary section. Verily, finding such an equilibrium is but daunting task. We humbly admit the likelihood that we shall not be able to satisfy all readers' tastes for both rigour and thorough treatment, and at the same time, we might vex the highly competent readers. Making no apologies, however, we feel the obligation to take great care in elaborating the background so that even a bright undergraduate could understand in-depth the importance of the work of Ikeda and Taniguchi. Ergo, as already mentioned in the introduction, we shall aim at a preliminary section as detailed as possible, anticipating that readers from diverse mathematical backgrounds would benefit from these notes. However, despite all our efforts to write a lucid self-contained text, each of the preliminaries below will inevitably require prerequisites in its own right. The latter will be omitted for the reasons already stated. To slightly remedy this, and with the sincere hope that it would benefit the less experienced readers, we shall briefly mention in an *italic font* at the beginning of each subsection the assumed prerequisites, provide some standard references and occasionally write footnotes. Albeit the five sections below are not independent, they still might be read in any order. Thus, the less experienced readers might need to read the preliminaries entirely, whereas the more advanced readers could easily pick up only what they find necessary.

3.1 Lie Group actions and homogeneous manifolds

Our goal in this subsection is to briefly sketch some facets of the theory of *Lie groups* and *Homogeneous manifolds*. We shall presuppose that the reader is familiar with the concepts of a *differentiable manifold*, a *group*, a *subgroup*, *group quotients*, and some standard concepts from topology. The only number fields to be used in this section are $\mathbb{R}$ and $\mathbb{C}$, and we shall often denote both by $\mathbb{F}$. Two excellent references containing more details are the well-known books [9, 12].

3.1.1 Lie groups

The first indispensable concept is that of a *Lie group*.

Definition 3.1. A group G with the structure of a smooth manifold endowed with a smooth action $G \times G \longrightarrow G$ defined by $(g, h) \mapsto g^{-1}h$ is called a *Lie group*.

Notice that this definiton is tantamount to the requirement that the group multiplication and the inversion be smooth operations. It also naturally guarantees the compatibility of the group and the differentiable structures on G.

Example 3.2. Perhaps the number one pedagogical example of a Lie group is the *General Linear group*. It is defined as the group of invertible $n \times n$ matrices with entries in a field $\mathbb{F}$, with group operation being the matrix multiplication. We denote it $\mathrm{GL}(n, \mathbb{F})$.

Due to their differentiable manifold structure, Lie groups naturally enjoy topological properties such as connectedness and compactness. By borrowing the idea of *path-connected components* from the theory of manifolds we perceive that every Lie group G decomposes to the disjoint union of its path-connected components, which is $G = G_0 \cup G_1 \cup \cdots$, where $G_i \cap G_j = \emptyset$ for $i \neq j$.

Definition 3.3. The *connected component of the identity* G_0 of a Lie group G is the set of all elements $g \in G$ that can be connected with the identity element $e \in G$ by a continuous path.

It ought to be noticed that G_0 is both a Lie group itself and a normal subgroup of G. We define the notion of *compactness* as follows.

Definition 3.4. A *compact* Lie group is a Lie group whose topology is compact.

Known examples of compact Lie groups are the n-dimensional *torus* T^n, the *orthogonal group* $O(n, \mathbb{F})$, the *special orthogonal group* $SO(n, \mathbb{F})$, the *unitary group* $U(n, \mathbb{F})$, the *special unitary group* $SU(n, \mathbb{F})$ and the *symplectic group* $Sp(n, \mathbb{F})$. For our immediate objectives, it is worth recalling the following definitions.

$$O(n, \mathbb{F}) = \{A \in GL(n, \mathbb{F}) \mid AA^T = \mathbb{1}\},$$

$$SO(n, \mathbb{F}) = \{A \in O(n, \mathbb{F}) \mid \det A = 1\},$$

$$U(n, \mathbb{C}) = \{A \in O(n, \mathbb{C}) \mid AA^* = \mathbb{1}\},$$

$$SU(n, \mathbb{C}) = \{A \in U(n, \mathbb{C}) \mid \det A = 1\}.$$

The following remark is due at this juncture. All these latter examples are also known as *matrix Lie groups*. Indeed, a *closed subgroup*[4] of the general linear group $\mathrm{GL}(n, \mathbb{F})$ is, by definition, a matrix Lie group. We ought to emphasise, however, that while every matrix Lie group is a Lie group, not every Lie group is a matrix Lie group. Now, a technically important for our purposes concept is to be defined.

[4] Closed in the topological sense.

Definition 3.5. Let G be a connected Lie group with a closed subgroup $K \leq G$. The pair (G, K) is called a *symmetric pair* if there exists an involutive analytic automorphism σ[5] of G such that $(K_\sigma)_o \subset K \subset K_\sigma$, where K_σ is the set of fixed points of σ and $(K_\sigma)_o$ is the identity component of K_σ.

Recall that we required in the statement of Problem 1 that G was semisimple. Thus, we owe the reader the following definition, whose meaning would probably bear more clarity only after the discussion in the subsequent section.

Definition 3.6. A Lie group is called *semisimple (simple)* if its Lie algebra is semisimple (simple).

3.1.2 Homogeneous manifolds

The next indispensable concept is that of a *homogeneous manifold.* To define the latter, we need to understand the notion of a *group action* on a manifold.

Definition 3.7. Let M be a manifold, and let G be a Lie group. A *left action* of G on M is a smooth map $\mu : G \times M \longrightarrow M$ such that

$$\mu(gh, x) = \mu(g, \mu(h, x)), \qquad \mu(e, x) = x,$$

for all $g, h \in G$, $x \in M$ and e the identity element of G.

It is customary, and indeed more practical, to write for a left action just $\mu(g, x) = g \cdot x$ for all $g \in G$, $x \in M$, tacitly meaning that each point on the manifold M will change its coordinates upon the action of the group G. In the case of a left group action of a matrix Lie group on $\mathbb{R}^n$ the notation $g \cdot x$ will mean matrix multiplication. However, Definition 3.7 tells us something more. For a fixed g, the map $x \mapsto \mu(g, x)$ is a diffeomorphism of M. In other words, a Lie group action of G on M can be thought of as a Lie group homomorphism $G \longrightarrow \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M. The *right action* is defined analogously.

[5] Meaning that σ^2 is the identity automorphism.

Definition 3.8. The action of a Lie group G on a manifold M is called *transitive* if for each pair $x, y \in M$ there exists an element $g \in G$ such that $gx = y$.

Before we give the general definition of *homogeneous manifold*, it is perhaps worth motivating the latter by a simple and concrete example. Let us consider the standard sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ in an n-dimensional space and the rotations in $\mathbb{R}^n$. More precisely, we are interested in the implications of the action of the special orthogonal group $SO(n, \mathbb{R})$ on the $(n-1)$-sphere. By employing some basic linear algebra, one easily perceives the truth of the following fact.

Proposition 3.9. *The action of $SO(n, \mathbb{R})$ on $\mathbb{S}^{n-1}$ is transitive for $n \geq 2$.*

In order to delve a little more into the implications of the action of the special orthogonal group on the $(n-1)$ - sphere, we need the following two definitions.

Definition 3.10. Given a G-action on a manifold M we define the *stabiliser* of $x \in M$ by $G_x = \{g \in G \mid g \cdot x = x\}$.

The stabiliser is certainly a subgroup of G, often called the *isotropy group*. The importance of the latter concept manifests in the following definition and proposition.

Definition 3.11. A Lie group G acts *freely* on M if and only if $G_x = \mathbb{1}$ for all $x \in M$.

Proposition 3.12. *Let G be a Lie group acting transitively on a manifold M and H be the isotropy group at the point $m \in M$. Then H is closed and the mapping*

$$gH \to g \cdot m$$

is diffeomorphism of G/H onto M.

Now, equipped with these latter definitions and again employing some standard linear algebra, one can readily prove the following proposition.

Proposition 3.13. *The isotropy group $SO(n,\mathbb{R})_{e_n}$ of the last standard basis vector $e_n = (0, ..., 0, 1)$ consists of elements of the form*

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

where $A \in SO(n-1,\mathbb{R})$.

This proposition tells us that $SO(n,\mathbb{R})_{e_n} \cong SO(n-1,\mathbb{R})$. More importantly, by Propositions 3.9, 3.12 and 3.13 we have the diffeomorphism

$$\mathbb{S}^{n-1} \cong SO(n,\mathbb{R})/SO(n-1,\mathbb{R}).$$

In much the same spirit, one can also prove that the following diffeomorphism holds in the complex case

$$\mathbb{S}^{2n-1} \cong SU(n)/SU(n-1).$$

These latter quotient spaces are the model examples of a homogeneous manifold. In general, one adopts the following definition.

Definition 3.14. A *homogeneous manifold* for a Lie group G is a manifold M on which G acts transitively.

The following discussion convinces us that a homogeneous manifold can always be viewed as a coset space. Let G be a group acting on a manifold M. If H_o is the stabiliser at the "origin" $o \in H$, then we can conclude that $M \cong G/H_o$. Conversely, given a coset space G/H it is a homogeneous manifold for G with a distinguished point, namely the coset of the identity. The choice of origin o is unimportant due to the inner automorphisms (conjugation) of G. This means that for $g \cdot o = o'$ we have $H'_o = gH_og^{-1}$.

3.2 Lie algebras and their structure - semisimplicity and roots

In this section, we should like to take a panoramic view of the structure of Lie algebras. The latter is not only important in its own right but is also a natural prerequisite to the subsequent section in which we shall discuss the essentials of representation theory. Well-known references for what is to follow are [9, 13, 17].

3.2.1 Semisimple Lie algebras

Geometrically, a Lie algebra $\mathfrak{g}$ of a Lie group G is identified with the tangent space T_eG at the identity element of G. Algebraically, one has the following general definition.

Definition 3.15. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Then, a *Lie algebra structure* is given on a vector space $\mathfrak{g}$ over the field $\mathbb{F}$ by the operation $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies the following properties

1. The operation $[\cdot,\cdot]$ is bilinear;
2. $[X, X] = 0$;
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for any $X, Y, Z \in \mathfrak{g}$ (the Jacobi identity).

The operation $[\,,\,]$ is usually called the *Lie bracket*. It is worth noting that, albeit indifferent to our purposes, the latter definition remains valid for any field of characteristic different than 2. In fact, for the following two reasons, we shall henceforth only work over the field $\mathbb{C}$. Firstly, the latter is an algebraically closed field, whence it is naturally more amiable to work. Secondly, every Lie algebra $\mathfrak{g}$ over the real numbers $\mathbb{R}$ can be naturally complexified by taking the tensor product over the field of real numbers $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Notice that the bracket of $\mathfrak{g}_{\mathbb{C}}$ is naturally inherited by that of $\mathfrak{g}$ via $[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes (\alpha\beta)$ for any $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$. Thus, we shall henceforth only work with complex Lie algebras.

Example 3.16. The set $M_n(\mathbb{C})$ of all $n \times n$ matrices is a Lie algebra with bracket operation $[A, B] = AB - BA$ for $A, B \in M_n(\mathbb{C})$. This Lie algebra of paramount importance is denoted $\mathfrak{gl}(n, \mathbb{C})$ and is called the *general linear Lie algebra*. Naturally, this latter is the Lie algebra of the general linear group $GL(n,\mathbb{C})$ (see Example 3.2). More generally, let V be a finite-dimensional vector space over $\mathbb{C}$ and let $\mathfrak{gl}(V)$ denote the vector space of linear maps of V into itself. Then, $\mathfrak{gl}(V)$ becomes a Lie algebra over $\mathbb{C}$ with the bracket operation $[A,B]=AB-BA$ for $A,B\in\mathfrak{gl}(V)$.

The next concept is very natural and will be of importance henceforth.

Definition 3.17. Let $\mathfrak{g}$ be a Lie algebra over the filed $\mathbb{C}$. A *subalgebra* of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ (over $\mathbb{C}$) of $\mathfrak{g}$ such that $[H_1,H_2]\in\mathfrak{h}$ for all $H_1,H_2\in\mathfrak{h}$.

By Definitions 3.15 and 3.17, one easily perceives that a subalgebra of a Lie algebra is itself a Lie algebra. The notion of linear maps between Lie algebras is also very natural and probably well-known to the general reader. We have the following definition.

Definition 3.18. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\varphi:\mathfrak{g}\longrightarrow\mathfrak{h}$ is called a *Lie algebra homomorphism* if $\varphi([X,Y])=[\varphi(X),\varphi(Y)]$ for all $X,Y\in\mathfrak{g}$. If, in addition, φ is one-to-one and onto, then φ is called a *Lie algebra isomorphism.* A Lie algebra isomorphism of a Lie algebra with itself is called a *Lie algebra automorphism.*

At this juncture, the following remark is due. The importance of the general linear algebra $\mathfrak{gl}(n,\mathbb{C})$ is at least twofold. On the one hand, by dint of Ado's theorem it is guaranteed that every complex finite-dimensional Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. On the other hand, in the next section, we shall see that representations of Lie algebras are maps with images into $\mathfrak{gl}(n,\mathbb{C})$.

Before we continue with some more definitions and begin delving deeper into the structure of Lie algebras, let us consider some examples of Lie subalgebras.

Example 3.19.

- The Lie algebra $\mathfrak{sl}(n;\mathbb{C})$ of the Lie group $SL(n;\mathbb{C})$ is the space of all traceless $n\times n$ complex matrices;

- As the connected component of the orthogonal group $O(n;\mathbb{C})$ is the Lie group $SO(n;\mathbb{C})$, the Lie algebra of both $O(n;\mathbb{C})$ and $SO(n;\mathbb{C})$ is the space $\mathfrak{so}(n;\mathbb{C})$ of all $n \times n$ complex matrices, which is, matrices satisfying $X^T = -X$.

- As the connected component of the unitary group $U(n;\mathbb{C})$ is the Lie group $SU(n;\mathbb{C})$, the Lie algebra of both $U(n;\mathbb{C})$ and $SU(n;\mathbb{C})$ is the space $\mathfrak{su}(n;\mathbb{C})$ of all complex $n \times n$ matrices satisfying $X^* = -X$.

There are two remarks to be made at this juncture. First, the symbol "T" means matrix transposition, while "$*$" means transposition together with complex conjugation. Thus, the Lie algebras $\mathfrak{so}(n;\mathbb{C})$ and $\mathfrak{su}(n;\mathbb{C})$ are not the same. Second, the identities $X^T = -X$ and $X^* = -X$ force that the elements of both $\mathfrak{so}(n;\mathbb{C})$ and $\mathfrak{su}(n;\mathbb{C})$ be traceless matrices.

The following definitions are vitally important for our purposes.

Definition 3.20. Every element X of a complex Lie algebra $\mathfrak{g}$ defines the *adjoint endomorphism* (*representation*) $ad_X : \mathfrak{g} \longrightarrow \mathfrak{g}$ by $Y \mapsto [X, Y]$ for all $Y \in \mathfrak{g}$.

Definition 3.21. The *Killing form* B on a Lie algebra $\mathfrak{g}$ is a bilinear and symmetric form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ defined by $B(X, Y) = tr(ad_X \circ ad_Y)$ with the property $B(s(X), s(Y)) = B(X, Y)$ for $X, Y \in \mathfrak{g}$ and $s \in Aut(\mathfrak{g})$.

Definition 3.22. An *ideal* of a complex Lie algebra $\mathfrak{g}$ is a complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $[X, H] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and all $H \in \mathfrak{h}$.

A very important example of an ideal of a complex Lie algebra $\mathfrak{g}$ is its *centre* $Z(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [x, z] = 0 \text{ for all } x \in \mathfrak{g}\}$. The latter allows us to define the concept of an *abelian* Lie algebra which will shortly play a very important role.

Definition 3.23. A Lie algebra $\mathfrak{g}$ is abelian if and only if $Z(\mathfrak{g}) = \mathfrak{g}$.

Definition 3.24. A *simple* Lie algebra is a non-abelian Lie algebra whose only ideals are $\{0\}$ and itself.

Definition 3.25. A complex Lie algebra $\mathfrak{g}$ is called *semisimple* if its Killing form B is non-degenerate.

It is worth mentioning at this point that the notion of semisimplicity can be defined in equivalent ways, one of which is the following.

Definition 3.26. A complex Lie algebra is called *semisimple* if it is isomorphic to a direct sum of simple Lie algebras.

Perhaps some readers might be baffled at this juncture and even think that we have only mumbled a bunch of random unintelligible and unmotivated definitions. We justly owe them the following swift comments. Definition 3.20 brings forth a representation of Lie algebras which is of paramount importance in Lie theory. Definition 3.21 constitutes an indispensable tool in the classification of semisimple Lie algebras. More importantly, the concept of Killing form does play a fundamental role in the computation of the spectrum of the Hodge-Laplacian, as we shall see in Chapter 4. The latter five definitions already imply the structure of Lie algebras.

3.2.2 Cartan subalgebras

Having mentioned the structure of Lie algebras and bearing in mind the technical aspect of the method of Ikeda and Taniguchi, we feel we must offer a brief discussion on Cartan subalgebras and the root spaces. These two concepts reveal the structure of semisimple Lie algebras and are also intimately related to the representation theory of Lie algebras.

Definition 3.27. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Cartan subalgebra* of $\mathfrak{g}$ is a complex subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties:

1. $[H_1, H_2] = 0$ for all $H_1, H_2 \in \mathfrak{h}$.

2. For all $X \in \mathfrak{g}$, if $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$.

3. The adjoint endomorphism ad_H is diagonalisable for all $H \in \mathfrak{h}$.

The first two conditions mean that the Cartan subalgebra is the maximal abelian subalgebra of $\mathfrak{g}$. Notice that a Cartan subalgebra may or may not exist for an arbitrary Lie algebra. For any semisimple Lie algebra $\mathfrak{g}$, however, a Cartan subalgebra always exists and is its fundamental structural characteristic. To see this, we shall need to define the concept of a *compact real form* of a Lie algebra.

Definition 3.28. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{k}$ be a real subalgebra of $\mathfrak{g}$. The Lie subalgebra $\mathfrak{k}$ is called *compact real form* of the algebra $\mathfrak{g}$ if every $X \in \mathfrak{g}$ can be uniquely written as $X = X_1 + iX_2$ with $X_1, X_2 \in \mathfrak{k}$ and such that there is compact simply-connected matrix Lie group K_1 such that the Lie algebra $\mathfrak{k}_1$ of K_1 is isomorphic to $\mathfrak{k}$.

Now, Definition 3.28 is crucial in the proof of the following proposition.

Proposition 3.29. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{k}$ be a compact real form on $\mathfrak{g}$, and let $\mathfrak{t}$ be any maximal abelian subalgebra of $\mathfrak{k}$. Define $\mathfrak{h} \subset \mathfrak{g}$ to be $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$. Then, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.*

It is worth noting that Definition 3.28 was not merely artificially invented to yield the proof of Proposition 3.29. Indeed, the concept of a compact real form naturally manifests itself in semisimplicity. Most importantly, every complex semisimple Lie algebra has a unique compact real form up to conjugation. This latter fact is a crucial ingredient not only in the proof of Proposition 3.29 but also in the proofs of various propositions in the structure theory of Lie algebras. A primary example of this is the following theorem.

Theorem 3.30. *A complex Lie algebra is semisimple if and only if it is isomorphic to the complexification of the Lie algebra of a simply-connected compact matrix Lie group.*

The reader might like to compare this latter theorem with Definition 3.6.

3.2.3 Root spaces

We are about to conclude this section with a brief discussion on the *root spaces*. These latter fully reveal the structure of complex semisimple Lie algebras and constitute the stepping stone for developing their representation theory. In what follows, we shall always assume that our Lie algebras are semisimple. In other words, there will always be a Cartan subalgebra constructed utilising Proposition 3.29.

Definition 3.31. A *root* of $\mathfrak{g}$ (relative to the Cartan subalgebra $\mathfrak{h}$) is a nonzero linear functional α on $\mathfrak{h}$ such that there exists a nonzero element $X \in \mathfrak{g}$ with

$$[H, X] = \alpha(H)X$$

for all $H \in \mathfrak{h}$. The set of all roots is denoted R.

Definition 3.32. If α is a root, then the *root space* $\mathfrak{g}_\alpha$ is the space of all $X \in \mathfrak{g}$ for which $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{h}$. An element of $\mathfrak{g}_\alpha$ is called a *root vector* (for the root α).

As the roots are essentially the eigenvalues of the adjoint representation ad_H, the following proposition is indeed natural to expect. It is often referred to as the *Cartan decomposition* and will play a vital role in Section 4.

Proposition 3.33. *The Lie algebra $\mathfrak{g}$ can be decomposed as a direct sum as follows:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

It is noteworthy that the following theorem, together with this latter proposition, indeed reveals the structure of semisimple Lie algebras.

Theorem 3.34. *Let $\mathfrak{g}$ be a simisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. Then the following hold true:*
(i) If α is a root, then the only multiples of α that are roots are α and $-\alpha$.

(ii) If α is a root, then the root space $\mathfrak{g}_\alpha$ is one dimensional.

(iii) For each root α, there exist nonzero elements $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in \mathfrak{h}$ such that

$$[H_\alpha, X_\alpha] = 2X_\alpha$$
$$[H_\alpha, Y_\alpha] = -2Y_\alpha$$
$$[X_\alpha, Y_\alpha] = H_\alpha$$

The elements H_α of $\mathfrak{h}$ are called *co-roots*. The reader might like to try to prove that, for a root α, the element H_α is unique in the sense that it is independent of the choice of X_α and Y_α. It is worth noticing that the latter theorem is a precursor to the general theory of *root systems*. For completeness, we shall give below some further definitions.

Definition 3.35. A subset Φ of an euclidean space E with inner product $\langle \cdot, \cdot \rangle$ is called a *root system* in E if the following axioms are satisfied:

(R1) Φ is finite, spans E, and does not contain 0.

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

The reflection is naturally defined as the map $\sigma_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$, for all $\beta \in \Phi$. The following two implications of the latter definition are ensuing. Firstly, axioms $(R1)$ and $(R2)$ imply that $\Phi = -\Phi$. Secondly, given a basis $\mathcal{B} = \{\alpha_1, ..., \alpha_r\}$ of Φ we have for any root $\alpha \in \Phi$ that $\alpha = \sum_{i=1}^{r} n_j \alpha_j$, where the n_j's are integers and either greater than or equal to zero or all less than or equal to zero. Whence we have the following definition.

Definition 3.36. For a base $\mathcal{B}$ of a root system Φ, the α's for which $n_j \geq 0$ are called the *positive roots* with respect to the given choice of $\mathcal{B}$

and the α's with $n_j \leq 0$ are called *negative roots*. The elements of $\mathcal{B}$ itself are called *positive simple roots*.

To close, we must mention that one does not need to know all the roots to understand the structure of a semisimple Lie algebra, as the following theorem ascertains.

Theorem 3.37. *For any root system, a base exists.*

3.3 Some basic Representation Theory of Lie Groups and Lie algebras

The main computational tools employed in [16] come from the Representation theory of Lie groups and Lie algebras. Ergo, a few introductory words on this matter are indispensable. Whence, this section discusses the basics of the representation theory of Lie groups and Lie algebras and elucidates the intimate relationship between the two. We shall deliberate in this section on the concepts of *representation*, *module*, *fundamental* and *highest weight*, *dominant integral elements*, and the *universal enveloping algebra*, for all they are essential technical prerequisites for Chapter 4. It ought to be stressed at the very beginning that, as it will be clear shortly, the concepts of a G-*representation* and a G-*module* are equivalent. Thus, the latter two are to be used interchangeably throughout the text. Since the definitions below are valid for real and complex vector spaces, we shall only work with the latter case. Standard references on representation theory are the books [9, 11, 13].

3.3.1 On representations of Lie groups

Definition 3.38. A finite-dimensional *complex representation* of a Lie group G is a Lie group homomorphism

$$P : G \longrightarrow GL(n; \mathbb{C})$$

or, more generally, a Lie group homomorphism

$$P : G \longrightarrow GL(V)$$

where V is a finite-dimensinonal complex vector space with complex dimension.

For the sake of brevity, P will sometimes be called a *G-representation*. The *degree* of the representation P is the dimension of the vector space V and the G-action on V is given by $g \cdot v = P(g)v$. Notice that the trivial representation with $V = \{0\}$ is naturally a complex representation. Furthermore, the representation P naturally gives V the structure of a G-module. Ipso facto, one must always think of representations as linear actions of a Lie group (or a Lie algebra) on a vector space. Indeed, the truthfulness of this latter observation stems from the following definition.

Definition 3.39. Let G be a Lie group. A *left G-module* consists of a vector space V together with a $\mathbb{C}$-linear left group action $\varphi : G \times V \longrightarrow V$ given by $\varphi(g \cdot v) = g \cdot v$.

Equivalently, one may just write $g \cdot (\alpha v + w) = \alpha g \cdot v + g \cdot w$ for any $v, w \in V$, $g \in G$ and $\alpha \in \mathbb{C}$. Similarly, if the group action was on the right, we then speak of a *right G-module.* In what follows, we shall sometimes use the term G-module whenever the group action is clearly understood to be left or right.

We shall need to know for our purposes more about the structure of the representation spaces as well as whether or not two representations are equivalent. Thus, the following two definitions are of paramount importance.

Definition 3.40. Let P be a finite-dimensional representation of a Lie group G, acting on a vector space V. A subspace $W \subset V$ is called *G-invariant* if $P(g)w \in W$ for all $w \in W$ and all $g \in G$. A representation P is called *irreducible* if the only G-invariant subspaces of V are $\{0\}$ and V itself.

Definition 3.41. Let G be a matrix Lie group. Let P_1 be a representation of G acting on the space V, and let P_2 be a reresentation of G acting on the space W. A linear map $\varphi : V \longrightarrow W$ is called a *homomorphism* of representations if

$$\varphi(P_1(g)v) = P_2(g)\varphi(v)$$

for all $g \in G$ and all $v \in V$. If φ is in addition invertible, then it is said to be an *isomorphism* of representantions. If there exists an isomorphism between V and W, then the representations are said to be *isomorphic*.

We shall also necessarily need to be able to distinguish and identify G-modules. For this purpose, we shall utilise the following definition.

Definition 3.42. Let A and B be two G-modules. A function $f : A \longrightarrow B$ is called a *morphism* of G-modules, or *G-homomorphism*, if it is both homomorphism and $fG = Gf$.

By the latter condition we mean that $f(g \cdot x) = gf(x)$. A map f satisfying such a property is called *G-equivariant*. If a G-homomorphism happens to be an isomorphism it will be denoted by the following symbol $\widetilde{\rightarrow}$.

Since we shall be naturally interested in the structure of G-modules, we also need to define the concepts of a *submodule* and a *quotient module*.

Definition 3.43. Let V be a G-module. A *submodule* of V is a subspace $U \subset V$ such that $g \cdot u \in U$ for all $g \in G$ and all $u \in U$. Furthermore, given a submodule U of V, the *quotient module* V/U is just the quotient space V/U with the action $g \cdot (v + U) = g \cdot v + U$, $v \in V$.

At this juncture, the time is ripe to comment on the Frobenius reciprocity law, for it will manifest itself in the sequel. Albeit the latter was originally formulated in the context of finite groups, we shall hereby only resort to Raoul Bott's version in the context of Lie groups [6]. Let K be a closed subgroup of a Lie group G. Clearly, G-modules are naturally K-modules. Conversely, starting from a K-module we can construct a G-module as follows. Let E be a K-module. There exists a natural transformation I that assigns to the K-module E the G-module $I \cdot E$, which consists of all the functions $f : G \longrightarrow E$ satisfying the identity

$$f(gk) = k^{-1} \cdot f(g), \quad g \in G,\, k \in K.$$

A G-module structure is given on $I \cdot E$ by the action

$$(gf)(x) = f(g^{-1}h) \quad g, h \in G,\, f \in I \cdot E.$$

Write $Hom_G(\cdot, \cdot)$ and $Hom_K(\cdot, \cdot)$ for the G- and K-equivariant maps, respectivelly. Then, the Frobenius reciprocity states.

Proposition 3.44. *Let K be a closed subgroup of a Lie group H. Let E be a K-module, and W be a G-module. Then*

$$Hom_G(W, I \cdot E) = Hom_K(W, E).$$

The more advanced readers must have noticed that this construction is categorical and that I is a functor. We shall not dwell on any more details on this matter but refer the reader to the papers [5, 6].

3.3.2 On representations of Lie algebras

Let us now turn our attention to representations of Lie algebras. As all the definitions above have their immediate analogues for Lie algebras, we shall only explicitly discuss some of them. Given a complex vector space V, one can naturally define the general linear Lie Algebra $\mathfrak{gl}(V)$ of V as the Lie algebra of the endomorphisms of V with Lie bracket given by the commutator of endomorphisms. In the same vein as Definition 3.38 one has the following.

Definition 3.45. A complex finite-dimensional *representation* of a finite dimensional Lie algebra $\mathfrak{g}$ in the finite-dimensional complex vector space V is a linear map

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathfrak{gl}(V) \\ X &\mapsto \rho(X), \end{aligned}$$

such that for any pair $X, Y \in \mathfrak{g}$ the following homomorphism of Lie algebras is satisfied

$$\rho\big([X,Y]\big) = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

Similarly to the foregoing subsection, we shall sometimes call ρ a $\mathfrak{g}$-*representation*. The concept of a $\mathfrak{g}$-module for the Lie algebra $\mathfrak{g}$ is defined as follows.

Definition 3.46. Let $\mathfrak{g}$ be a complex Lie algebra acting on a vector space V through the action $\mathfrak{g} \times V \longrightarrow V$ given by $(X, v) \mapsto Xv$. The vector space V is called a $\mathfrak{g}$-*module* if, for any $a, b \in \mathbb{C}$, $X, Y \in \mathfrak{g}$, $v, w \in V$, the following conditions are satisfied:

(M1) $(aX + bY)v = a(Xv) + b(Yv)$;

(M2) $X(av + bw) = a(Xv) + b(Xw)$;

(M3) $[X,Y]v = XYv - YXv$.

Just as in the case of Lie groups, a representation $\rho : \mathfrak{g} \longrightarrow V$ defines the $\mathfrak{g}$-module structure on the space V via the action $X \cdot v = \rho(X)v$ and vice-versa. A linear map $\phi : V \longrightarrow W$ such that $\phi(X \cdot v) = X \cdot \phi(v)$ for $X \in \mathfrak{g}$ and $v \in V$ is called a homomorphism of $\mathfrak{g}$-modules. When ϕ is an isomorphism of vector spaces it is also called a $\mathfrak{g}$-module isomorphism. If two $\mathfrak{g}$-modules are isomorphic, we say they yield equivalent representations of $\mathfrak{g}$.

Let us stop for a moment to reflect on what has been done so far in this section. We have defined the representations of Lie groups and Lie algebras as the homomorphisms $P : G \longrightarrow GL(V)$ and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, respectively. Due to the geometric relationship of the Lie group G and its Lie algebra $\mathfrak{g}$, the homomorphisms P and ρ are related as follows.

Proposition 3.47. *Let G be a Lie group with a Lie algebra $\mathfrak{g}$. If P is a G-representation, than its differential $d\,P$ is a $\mathfrak{g}$-representation.*

This relationship will be essential to Freudenthal's formula discussed in Chapter 4. It also somewhat justifies the slight abuse of notation we shall adopt henceforth - ρ will simultaneously denote the representations of a Lie group and a Lie algebra.

In the vein of Definition 3.40, we define the irreducibility of $\mathfrak{g}$-modules as follows.

Definition 3.48. We say that a $\mathfrak{g}$-module V is *irreducible* if its only $\mathfrak{g}$-submodules are itself and $\{0\}$.

The concept of *complete reducibility* is indeed very natural and is defined as follows.

Definition 3.49. A $\mathfrak{g}$-module V is called *completely reducible*, or *fully reducible*, if it is a direct sum of irreducible $\mathfrak{g}$-modules.

Now, with this latter definition and the discussion on the semisimple Lie algebras in the previous section, we can state the following natural theorem that indeed plays its role in what follows.

Theorem 3.50. *Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.*

The study of irreducible representations is one of the principal goals of representation theory. A fundamental result in this context is the celebrated Schur's Lemma. The reader may wish to consult [9] for proof.

Theorem 3.51. *(Schur's Lemma)*

1. *Let V and W be irreducible real or complex representations of a Lie group or a Lie algebra and let $\varphi : V \longrightarrow W$ be a homomorphism. Then, either $\varphi = 0$ or φ is an isomorphism.*

2. *Let V be an irreducible complex representation of a Lie group or a Lie algebra and let $\varphi : V \longrightarrow V$ be a homomorphism of V with itself. Then, $\varphi = \lambda I$, for some $\lambda \in \mathbb{C}$.*

3. *Let V and W be irreducible complex representations of a Lie group or a Lie algebra and let $\varphi_1, \varphi_2 : V \longrightarrow W$ be nonzero homomorphisms. Then, $\varphi_1 = \lambda\varphi_2$, for some $\lambda \in \mathbb{C}$.*

It must be clear at this point that irreducible representations are the building blocks of representations. Indeed, to understand in-depth any given representation, it suffices to study its irreducible representations. Remarkably, the irreducible representations of a Lie algebra are in a natural correspondence with the structure theory of semisimple Lie algebras. Thus, we shall have to turn back to the concept of a root system to be able to delve into the representation theory of Lie algebras. Ergo, in what follows, $\mathfrak{h}$ is always to be considered a Cartan subalgebra of some semisimple Lie algebra $\mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ will be the inner product on $\mathfrak{g}$ that is invariant under the adjoint action of K and takes real values on $\mathfrak{k}$ (see the discussion on Cartan subalgebras). We shall define particular concepts as the *dominant integral element, highest weight,* and the *fundamental weight.*

Definition 3.52. An element μ of $\mathfrak{h}$ is called an *integral element* if $\langle \mu, H_\alpha \rangle$ is an integer for each root α.

Definition 3.53. An element μ of $\mathfrak{h}$ is called a *dominant integral element* if $\langle \mu, H_\alpha \rangle$ is a non-negative integer for each positive simple root α. Equivalently, μ is a dominant integral element if

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is a non-negative integer for each positive simple root α.

Definition 3.54. Let ρ be a finite-dimensional representation of $\mathfrak{g}$ on a vector space V. Then, $\mu \in \mathfrak{h}$ is called a *weight* for ρ if there exists a non-zero vector $v \in V$ such that

$$\rho(H)v = \langle \mu, H \rangle v \tag{3.1}$$

for all $H \in \mathfrak{h}$. A nonzero vector satisfying (3.1) is called a *weight vector* for the weight μ, and the set of all weight vectors satisfying (3.1), both zero and nonzero, is called the *weight space* with weight μ. The dimension of the weight space is called the *multiplicity* of the weight.

This latter definition necessitates the following comments. A weight vector is a simultaneous eigenvector for all the $\rho(H)$'s. To see this, assume that v is a simultaneous eigenvector for each $\rho(H)$, which is, there is a number λ_H such that $\rho(H) = \lambda_H v$ for $v \in V$, $H \in \mathfrak{h}$. As the representation ρ is linear, the number λ_H also depends linearly on H. In other words, the map $H \longmapsto \lambda_H$ is a linear functional on $\mathfrak{h}$ such that $\lambda_H = \langle \mu, H \rangle$. Observe that this latter argument is yet another manifestation of the Riesz representation theorem, see Theorem 3.77. Furthermore, Definition 3.54 is the necessary ingredient which naturally relates to the structure theory of semisimple Lie algebras and their representations. Indeed, as an immediate consequence, one has that the roots of the Lie algebra $\mathfrak{g}$ are exactly the nonzero weights of the adjoint representation of $\mathfrak{g}$, and that two equivalent representations have the same weights and multiplicities.Two important properties of weights and weight spaces are the following.

Proposition 3.55. *If $\mu \in \mathfrak{h}$ is a weight of some finite-dimensional representation (ρ, V) of $\mathfrak{g}$, then μ is an integral element.*

Proposition 3.56. *Every finite-dimensional representation (ρ, V) is the direct sum of its weight spaces.*

Notice that Proposition 3.56 tells us that the set of operators of the form $\rho(H)$, $H \in \mathfrak{h}$, are simultaneously diagonalisable in every finite-dimensional representation.

In the spirit of the present discussion, it is now evident that the weights and weight spaces reveal the structure of a representation space of a Lie algebra. Before we state this latter as a theorem, however, we need to define the concept of the *highest weight.*

Definition 3.57. Let μ_1 and μ_2 be two elements of $\mathfrak{h}$. Then μ_1 is *higher* than μ_2 (or, equivalently, μ_2 is *lower* than μ_1) if there exist non-negative real numbers $a_1, ..., a_r$ such that

$$\mu_1 - \mu_2 = a_1\alpha_1 + \cdots + a_r\alpha_r,$$

where $\alpha_1, ..., \alpha_r$ is the set of positive simple roots. One writes $\mu_2 \preceq \mu_1$ or $\mu_1 \succeq \mu_2$. Given a representation ρ of $\mathfrak{g}$ the weight μ_0 for ρ is said to be a *highest weight* if for all weights μ of ρ, $\mu \preceq \mu_0$.

Thus, we can now see the full power of irreducible representations revealed in terms of the highest weights in the following theorem.

Theorem 3.58.
1. Every irreducible representation has a highest weight.
2. Two irreducible representations with the same highest weight are equivalent.
3. The highest weight of every irreducible representation is a dominant integral element.
4. Every dominant integral element occurs as the highest weight of an irreducible representation.

Recall that we already know that the roots describe the structure of semisimple Lie algebras and that it suffices to work with a base of a given root system. In the same spirit, and for later purposes, we shall define the notion of the *fundamental weight.*

Definition 3.59. Let $\mathcal{B} = \alpha_1, ..., \alpha_r$ be a base. Then, the *fundamental weights* relative to $\mathcal{B}$ are the elements $\mu_1, ..., \mu_r$ with the property

$$2\frac{\langle \mu_k, \alpha_l \rangle}{\langle \alpha_l, \alpha_l \rangle} = \delta_{kl},$$

for $k, l = 1, ..., r$.

3.3.3 The universal enveloping algebra

Another notion of paramount importance for our purposes is the concept of *universal enveloping algebra.* In order to define it, one first needs to delve into tensor algebras.

Definition 3.60. Given a vector space V over a field $\mathbb{F}$ we define the tensor algebra

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

with multiplication

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \cdot (w_1 \otimes w_2 \otimes \cdots \otimes w_l) = v_1 \otimes v_2 \otimes \cdots \otimes v_k \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_l.$$

Let us now consider the particular case $V = \mathfrak{g}$. We can take the two-sided ideal J of $T(\mathfrak{g})$ generated by the elements of the form

$$X \otimes Y - Y \otimes X - [X, Y],$$

or more precisely,

$$J = \operatorname{span}\big\{A \otimes \big(X \otimes Y - Y \otimes X - [X, Y]\big) \otimes B \mid X, Y \in \mathfrak{g}, A, B \in T(\mathfrak{g})\big\}.$$

Definition 3.61. The factor algebra $U_{\mathfrak{g}} = T(\mathfrak{g})/J$ is called the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

In effect, we have constructed an associative algebra starting from a Lie algebra, which itself is not. One reason the universal enveloping algebra is so important is that there is a one-to-one correspondence between its representations and those of the Lie algebra $\mathfrak{g}$. This latter statement is a consequence of the following observation. Starting from a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, we can naturally define a representation of the tensor algebra

$$\tilde{\rho} : T(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$$

by

$$\tilde{\rho}(X_1 \otimes X_2 \otimes \cdots \otimes X_k) = \rho(X_1) \cdot \rho(X_2) \cdots \rho(X_k).$$

Now, by the definition of a representation of a Lie algebra, it is evident that $\tilde{\rho}(J) = 0$. Then, the representation $\tilde{\rho}$ defines the representation

$$\hat{\rho} : U_{\mathfrak{g}} \longrightarrow \mathfrak{gl}(V)$$

by

$$\hat{\rho}(a) = \tilde{\rho}(A) \quad a \equiv A \mod J,$$

for $a \in U_{\mathfrak{g}}$ and $A \in T(\mathfrak{g})$. Another important observation of the significance of the universal enveloping algebra is the following. Let $D(g)$ be the algebra of operators on $C^\infty(G)$[6] generated by all the left-invariant vector fields on G and the identity I on $C^\infty(G)$. Then, the following well-known isomorphism holds.

Proposition 3.62. $D(\mathfrak{g}) \cong U_{\mathfrak{g}}$.

The truthfulness of this latter proposition is an immediate consequence of the construction of the universal enveloping algebra. Indeed, we may naturally think of the elements of the Lie algebra $\mathfrak{g}$ as the left-invariant first order homogeneous differential operators on the Lie group G. Thus, $U_{\mathfrak{g}}$ consists of all left-invariant differential operators (of any order) of G. In particular, if $D^n(\mathfrak{g})$ denotes the algebra of differential operators with a total degree at most n, then the quotient $D^n(\mathfrak{g})/D^{n-1}(\mathfrak{g})$ consists only of homogeneous differential operators of degree n. These latter are precisely the left-invariant homogeneous polynomials with constant coefficients.

Albeit easy to understand, this latter proposition is a profound result. It enables us to study the operators acting on $C^\infty(G)$ from a very convenient perspective, namely, as elements of the universal enveloping algebra. Indeed, we shall prove in Section 4.2 the remarkable fact that the Hodge-Laplacian, clearly a differential operator, is equal to minus the Casimir operator, which is an element of $U_{\mathfrak{g}}$ (see Proposition 4.6). And this is precisely the key to the method of Ikeda and Taniguchi, for the spectrum of the Casimir operator is known and fully computable by the Freudenthal's formula (Proposition 4.12).

[6]By $C^\infty(G)$ we mean the smooth functions on G. In the next section, however, the symbol $C^\infty(\cdot)$ shall denote the sheaf of sections of a bundle.

3.4 Fibre Bundles

We make a brief review of fibre bundles in this section. We begin with some basic definitions of *Vector* and *Principal bundles*, followed by detailed definitions of the bundles (1), (2), and (3). We shall assume that the reader is familiar with the standard definition of the *tangent bundle*, which appears in any textbook treating differentiable manifolds. Should the reader need to learn more about these matters, we should like to recommend the books [14, 19].

3.4.1 Essentials of Fibre and Vector bundles

Considering the disjoint union of all tangent spaces of a differentiable manifold as a whole, one naturally arrives at the following general picture drawn by the following definitions.

Definition 3.63. Let E and B be differentiable manifolds. A *bundle* is a triple $\xi = (E, \pi, B)$ where $\pi : E \longrightarrow B$ is a surjective map. For each $b \in B$ the space $\pi^{-1}(b)$ is called *fibre* of the bundle over the point $b \in B$. E is called the *total space*, B the *base space* and the map π *projection.*

The fibre of the bundle ξ over a point b is often denoted $F_b(\xi)$.

Definition 3.64. Given a bundle (E, π, B) we define its *sections* as the maps $s : B \longrightarrow E$ satisfying $\pi \circ s = \mathbb{1}_B$.

This latter definition simply tells us that $s(b) \in \pi^{-1}(b)$.

Definition 3.65. A *rank k vector bundle* ξ over a field $\mathbb{F}$ is a bundle (E, π, B) whose fibres $\pi^{-1}(b)$ have the structure of a vector space over the field $\mathbb{F}$ and the following *local triviality condition* is satisfied. For each $b \in B$ there is an open neighbourhood U and a diffeomorphism

$$h : U \times \mathbb{F}^k \longrightarrow \pi^{-1}(U)$$

such that the restriction

$$b \times \mathbb{F}^k \longrightarrow \pi^{-1}(b)$$

is a vector space isomorphism, and $\pi \circ h = p_1$ where $p_1 : U \times \mathbb{F}^k \longrightarrow U$ is the canonical projection on the first factor. The isomorphism h is called a *local trivialisation* of the bundle ξ.

The natural number k is the dimension of the fibres and is called the *rank* of the vector bundle ξ. The following are well-known examples of vector bundles. The first example is most natural and justifies the necessity of the local triviality condition just mentioned in Definition 3.65.

Example 3.66. The *trivial vector bundle* is $B \times \mathbb{F}^n$. It is evident that in this case the bundle trivialisation is globally defined.

The second example, as mentioned earlier, may be thought of as the primordial prototype of vector bundles. We readily omit the details as they are found virtually in every book on differentiable manifolds.

Example 3.67. Every differentiable manifold M has a *tangent bundle* TM. The fibres are the tangent spaces T_pM and sections are the vector fields $X : M \longrightarrow TM$.

Definition 3.68. Let ξ and η be two vector bundles over the same base space. We say that ξ is isomorphic to η, written $\xi \cong \eta$, if there exists a homeomorphism of the total spaces $f : E(\xi) \longrightarrow F(\eta)$ which maps each fibre $F_b(\xi)$ isomorphically onto the corresponding fibre $F_b(\eta)$.

Having defined the very basics of fibre and vector bundles, we are now ready to discuss in detail bundles (1), (2) and (3).

3.4.2 The exterior power bundle

In this subsection, we shall construct bundle (1). It constitutes the context of Problem 1, is called the *exterior power bundle*, and is oft denoted $\Lambda^p M$. It is naturally defined over any differentiable manifold M and has sections the differential p-forms on M denoted $C^\infty(\Lambda^p M)$.

It is worth beginning with the following general construction. Let V be a vector space over a field $\mathbb{F}$ and let $T(V)$ be the tensor algebra as defined in subsection 3.3.3. Take the two-sided ideal $\mathfrak{J}$ generated by all elements of the form $v \otimes v$ for all $v \in V$. Then, the *exterior algebra* of the vector space V is defined to be the quotient algebra

$$\Lambda(V) = T(V)/\mathfrak{J}.$$

This is an associative algebra with a multiplication called the *exterior product*. Evidently, the latter is induced by the tensor product $\otimes$ on $T(V)$ and is denoted $\wedge$. Observe that the exterior product is by construction *alternating* on the elements on V, which is, $v \wedge v = 0$ for all $v \in V$. Furthermore, consider the ideal $\mathfrak{K}$ generated by the elements of the form $v \otimes w + w \otimes v = (v+w) \otimes (v+w) - v \otimes v - w \otimes w$ for $v, w \in V$. Since we only work with fields of characteristic zero, the ideals $\mathfrak{J}$ and $\mathfrak{K}$ coincide. Whence we deduce the crucial property of the exterior product $v \wedge w = -w \wedge v$, for any $v, w \in V$. Now, with these properties of the exterior product at hand we are ready to proceed with our construction.

Consider the standard coordinates $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. It is immediately clear that the differentials $\mathrm{d}x_1$,...,$\mathrm{d}x_n$ with the relations

$$\begin{cases} \mathrm{d}x_i \wedge \mathrm{d}x_i = 0 \\ \mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i \quad (i \neq j), \end{cases} \tag{3.2}$$

define the exterior algebra algebra $\Lambda(\mathbb{R}^n)$ over $\mathbb{R}$. As a vector space, the latter has the following basis

$$1, \mathrm{d}x_{i_1}, \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2}, \cdots, \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_n},$$

with the index ordering $i_1 < i_2 < \cdots < i_n$. The *differential forms* on $\mathbb{R}^n$ are then defined as the elements of $\Lambda^*(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \Lambda(\mathbb{R}^n)$, where $C^\infty(\mathbb{R}^n)$ are the smooth functions on $\mathbb{R}^n$. More explicitly, every differential p-form can be uniquely written as $\omega = \sum f_{i_1,\dots,i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$. The latter is often shorthanded as $\omega = \sum f_I \mathrm{d}x_I$, for some index set $I = \{i_1, ..., i_p\}$. It is also worth mentioning at this juncture that the algebra $\Lambda^*(\mathbb{R}^n)$ is naturally *graded*, which means that $\Lambda^*(\mathbb{R}^n) = \bigoplus_{p=0}^{n} \mathcal{C}^\infty(\Lambda^p \mathbb{R}^n)$

and $C^\infty(\Lambda^p\mathbb{R}^n)\wedge C^\infty(\Lambda^q\mathbb{R}^n) \subseteq C^\infty(\Lambda^{p+q}\mathbb{R}^n)$, where $C^\infty(\Lambda^p\mathbb{R}^n)$ denotes the differential p-forms on $\mathbb{R}^n$. Now, to define differential forms on an arbitrary differentiable manifold, one only must substitute the vector space $\mathbb{R}^n$ with the cotangent space T_pM^*, and the latter will yield exactly a typical fibre of the exterior power bundle Λ^pM. The reader might now perceive the little notational sin we have committed - the adequate notation for the exterior p-bundle is verily $\Lambda^p(T_pM)^*$.

3.4.3 The principal bundle

The goal of this subsection is to define bundle (2). Since the latter is a special kind of a principal bundle, we ought to prepare the ground with some general definitions. It is worth remarking that there is a noticeable difference between a vector bundle and a principal bundle. Indeed, not only the fibres of the latter are copies of the group G, but both the total space and the base space of the principal bundle are subject to a very specific G-action. This latter fact will allow us to construct the *homogeneous bundle* in the next section and invoke representation theory sometime later.

Definition 3.69. Let G be a Lie group. A *left G-manifold M* is a smooth manifold endowed with a smooth left action $G \times M \longrightarrow M$. Similarly, a *right G-manifold* is a smooth manifold M with a smooth right action $M \times G \longrightarrow M$.

Definition 3.70. Consider a map $\phi : M \longrightarrow N$ between two manifolds. If M and N are left G-manifolds and $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in M$, then the map ϕ is called *left G-equivariant.* If M and N are right G-manifolds and $\phi(xg) = \phi(x)g$ for all $g \in G$ and $x \in M$, then the map ϕ is called *right G-equivariant.*

Definition 3.71. Let B be a smooth manifold. Consider a right G-manifold P equipped with a right G-equivariant map $\pi : P \longrightarrow B$ such that G acts trivially on B, which is, $bg = b$ for all $g \in G$ and $b \in B$. We say that (P, π) is a *principal G-bundle* over B if π satisfies the following *local triviality condition*: B has a covering of open sets U such that there exists a G-equivariant diffeomorphism $\phi_U : \pi^{-1}(U) \longrightarrow U \times G$ so that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ & \searrow^{\pi} & \downarrow p \\ & & U \end{array}$$

Observe that (P, π) is a local product over B with fibre G. Furthermore, as G acts trivially on B, we perceive that the right G-action on $U \times G$ is given by $(u, g)h = (u, gh)$. This latter condition means two things. First, G acts freely on P. Second, π factors through a diffeomorphism $\bar{\pi} : P/G \longrightarrow B$ and therefore B may be thought of as the orbit space of P. Note that G acts freely and transitively on the fibres.

We define the notion of a section of a principal bundle just as in the case of vector bundles (see Definition 3.64). The trivial principal G-bundle is also defined analogously, namely.

Definition 3.72. A principal G-bundle is *trivial* if it is isomorphic to the product principal bundle $B \times G \longrightarrow B$.

We ought to remark that trivial principal G-bundles and the existence of sections are intimately related. Unlike vector bundles, principal bundles do not always admit sections. The following definition and propositions establish the veracity of this latter fact.

Definition 3.73. A *morphism* of two G-principal bundles P and Q over the same base B is a G-equivariant map $\sigma : P \longrightarrow Q$.

Proposition 3.74. *Any morphism of principal G-bundles is an isomorphism.*

Proposition 3.75. *A principal G-bundle $\pi : P \longrightarrow B$ is trivial if and only if it admits a section.*

Proof. If the bundle is trivial there is a section. Conversely, let there be a section $s : B \longrightarrow P$. Consider the map $\phi : B \times G \longrightarrow P$ defined by $\phi(b, g) = s(b)g$. One can effortlessly check that ϕ is a G-equivariant map of bundles over B, which is, a morphism of principal bundles. Thus, by Proposition 3.74, it is an isomorphism and therefore P is trivial. □

Once we have set the context, we are in the position to describe in detail the principal bundle $G \longrightarrow G/K$. We know that, for a Lie group G and a closed subgroup K, we always can consider the homogeneous space G/K. However, the latter is also a principal K-bundle. More precisely, we certainly have $\pi : G \longrightarrow G/K$, where π is the natural projection given by $\pi(g) = gK$ for all $g \in G$. We observe that the fibre over a point of G/K is a copy of the subgroup K. This latter observation allows us to think of the group G as the family of copies of the subgroup K, parametrised by point in the base space G/K. Let us now explain why this bundle is rather special. There are two commuting group actions on the total space G. The first one is the right free action of K, which defines the principal bundle. The second is the left action of G itself. We remark that this latter G-action is transitive on both the base and the total space of the principal K-bundle under consideration. Indeed, the base G/K is a homogeneous manifold, and the left G-action on itself is verily transitive. In general, however, a principal K-bundle does not admit a transitive action. Furthermore, our principal K-bundle does not even admit sections. Indeed, if it did, then by Proposition 3.75, we must have had the veracity of the following product $G = G/K \times K$. But for topological obstructions, the latter identity is generally false. This falsity can be seen easily in the following important example. Consider the *Hopf fibration*, which is a particular example of our bundle (2) for $G = SU(2) \cong \mathbb{S}^3$, $K = U(1) \cong \mathbb{S}^1$ and $G/K \cong \mathbb{S}^2$. Now, the identity $\mathbb{S}^3 = \mathbb{S}^2 \times \mathbb{S}^1$ is a topological absurd. The moral of the story is this. Bundle (2) is perhaps the most natural to think of when considering a homogeneous space G/K, and it constitutes a fundamental part of the general picture of Problem 1. Alas, for its abstract fibres and the lack of sections, it is certainly of little use for concrete computations. For this reason, we shall endeavour to construct bundle (3).

3.4.4 The homogeneous bundle

Let ρ be a representation of K on a vector space U. Consider the space $G \times U$ and the right K-action on it defined by

$$(g, v) \mapsto (gk, \rho(k^{-1})v), \tag{3.3}$$

for $(g, v) \in G \times U$ and $k \in K$. We wish to construct a new vector bundle over G/K. To do this, we only have to quotient the space $G \times U$ by the K-action (3.3). In other words, we identify all the points in $G \times U$ related by an element $k \in K$ through the action (3.3). We denote thus obtained quotient space by $G \times_K U$, whence we have just constructed the vector bundle $\pi : G \times_K U \longrightarrow G/K$ with projection map $\pi(g, v) = gK$. Notice that the fibres are by construction copies of the representation space U of the subgroup K, and the space of sections is defined as

$$C^\infty(G \times_K U) = \{s : G/K \longrightarrow G \times_K U \mid s(gK) = (g, v),\ v \in U\}.$$

Thus, we have just constructed a vector bundle which is more suitable for our computational purposes. It is called the *homogeneous bundle*, or sometimes the *associated bundle* defined by the representation ρ. Albeit it must be evident by its construction, we are in a hurry to emphasise the dependence of this bundle of the representation ρ. It is called homogeneous, for the transitive left action of G induces the group action $(g_0, v) \mapsto (gg_0, v)$ on $G \times_K U$, which identifies the fibres over any two points on the base space G/K. Moreover, as we shall see in a moment, this latter action will induce a representation of G on the space of sections of the homogeneous bundle.

Remarkably, the homogeneous bundle is intimately related to the concept of *induced representation*, which makes it the bridge towards representation theory. Indeed, we shall now demonstrate how the concept of induced representation naturally pertains to homogeneous bundles. The idea is simple - construct a representation of the group G by starting from a representation ρ of the subgroup K. To achieve this latter, we shall need another simple idea, which is to generalise the space of complex functions on the base space G/K. This is done by defining the space of *twisted functions* by

$$\{f : G \longrightarrow U \mid f(gk) = \rho(k^{-1})f(g)\ \forall\, g \in G,\ k \in K\}. \tag{3.4}$$

Notice that the latter space, which we shall denote $C^\infty(G,K;U)$, has a natural structure of a G-submodule. It should be easy to perceive that, in the special case of $U = \mathbb{C}$ and the trivial representation of K, the set of twisted functions coincides with the complex functions on G/K. Now, it is the K-action (3.3) that allows us to identify the space of sections $C^\infty(G \times_K U)$ with the set of twisted functions $C^\infty(G,K;U)$. Indeed, one attains this identification by associating to each twisted function f the map $s : G/K \longrightarrow G \times_K U$ defined by $s(gK) = (g, f(g))$. Now, the induced left G-action on $G \times_K U$ is given by $(g_0, v) \mapsto (gg_0, v)$. It induces a representation of G on $C^\infty(G \times_K U)$ in the following manner. For each $g \in G$ there is a linear map $A(g) : C^\infty(G \times_K U) \longrightarrow C^\infty(G \times_K U)$ given by $\big(A(g)f\big)(x) = g \cdot f(g^{-1}x)$. Evidently, for a given U the action $A(g)$ is uniquely defined. We call the latter the *induced representation*.

3.5 The Hodge-Laplacian operator

In this last preliminary subsection, we define and discuss some of the basic properties of both the Hodge $*$ - operator and the Hodge-Laplacian operator[7]. Familiarity with some basic concepts from *Riemannian geometry*, such as *orientability and integration on manifolds*, is to be assumed. We mentioned in the introduction that the Laplace - Beltrami operator does not constitute any particular interest for our purposes. However, for the sake of completeness, we recall that it is a second-order, elliptic, self-adjoint, partial differential operator which acts on the space of smooth functions of a given manifold and is defined by $\Delta f = -\operatorname{div}\operatorname{grad} f$. The

[7]It is necessary to be stressed here that some authors still call this operator the Laplace - Beltrami operator.

minus sign in the latter definition is a crucial requirement so that the eigenvalues are all non-negative and form an increasing infinite sequence. It also ought to be noticed herewith that the Laplace - Beltrami operator is the prototype of the Hodge - Laplacian, for the two indeed coincide in the space of smooth functions on the manifold M. The spectral geometry of the Laplace - Beltrami operator is studied in great depth and detail in the books [7, 21], to which we are not at all hesitant to refer the reader.

We shall consider an oriented manifold M and its bundle of exterior forms $\Lambda^p M$, and write $C^\infty(\Lambda^p M)$ for the sheaf of smooth sections of $\Lambda^p M$. Two concepts ought to be recalled before giving our first definition. Firstly, given a Riemannian manifold (M, g), for any local coordinate system $(x_1, \cdots x_n)$ one defines an n-form ω, called the *volume form*, by $\omega = \sqrt{det\,(g_{ij})}dx_1 \wedge \cdots \wedge dx_n$. Secondly, the inner product of two p-forms $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_p$ is defined as $\langle \alpha, \beta \rangle = det(\langle \alpha_i, \beta_j \rangle)$. With these latter in mind, we should also like to recall that $|\omega| = \sqrt{\langle \omega, \omega \rangle} = 1$.

Definition 3.76. Let M be an oriented manifold of dimension n and $\alpha, \beta \in C^\infty(\Lambda^p M)$. The map $* : C^\infty(\Lambda^p M) \longrightarrow C^\infty(\Lambda^{n-p} M)$ defined by $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$ is called the *Hodge $*$-operator*.

It is of utmost importance to remark that the Hodge $*$-operator manifests itself in duality. Notwithstanding few such dualities will naturally arise in Section 4.1, it is worth mentioning that this manifestation in duality is a mere consequence of the Riesz representation theorem.

Theorem 3.77. *(Riesz) Let W be a finite-dimensional vector space with inner product $\langle\, , \rangle_W$. Then, for each functional $f \in W^*$ exists unique $v \in W$ such that $f(w) = \langle w, v \rangle_W$ for all $w \in W$.*

Let us see how this latter theorem implies the duality of the Hodge $*$-operator. It tells us that there is the vector space isomorphism $W^* \cong W$ defined by $f \mapsto v$. Consider a vector space V with basis $(e_1, ..., e_n)$ and construct the exterior power spaces $\Lambda^p V$ and $\Lambda^{n-p} V$ for $0 \leq p \leq n$. Let

$\lambda \in \Lambda^p V$ and $\theta \in \Lambda^{n-p} V$. The only n-form, up to scalar multiplication, is $\lambda \wedge \theta$. Thus, for a fixed p-form λ, the existence of the unique linear functional $f_\lambda \in \Lambda^{n-p} V^*$ will imply the uniqueness of the representation $\lambda \wedge \theta = f_\lambda(\theta) e_1 \wedge \cdots \wedge e_n$. Denote the inner product on $\Lambda^{n-p} V$ by $\langle \, , \, \rangle$. Then, by the Riesz representation theorem there exists unique $\xi \in \Lambda^{n-p} V$ such that $f_\lambda = \langle \theta, \xi \rangle$. Write $\xi = *\lambda$ and call it the *Hodge dual* of λ. Observe that $*\lambda$ is the image of f_λ under the isomorphism induced by the inner product, namely, by the following isomorphism

$$\left(\Lambda^{n-p} V \right)^* \cong \Lambda^{n-p} V.$$

The intimate relationship between the metric g and the Hodge $*$-operator must now be evident. In the tensorial context, the Hodge $*$-operator is also related to the Levi-Civita symbol $\mathcal{E}_{a_1 \ldots\ldots a_n}$ - a fact which we shall not explain here. Given the orthonormal frame $\left\{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\}$, a more practical definition of the Hodge $*$-operator is the following:

$$*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

where $i_1 < \cdots < i_k < j_1 < \cdots < j_{n-k}$, we identify $(i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$ with $(1, 2, 3, \cdots, n)$ and σ gives the parity of the permutations of the latter. One can indeed compute various concrete examples using this definition. The reader may wish to do an easy exercise and check that the volume form $\omega \in \Lambda^n V$ and $1 \in \Lambda^0 V$ are Hodge duals to one another, which is, $*1 = \omega$ and $*\,\omega = 1$.

The following property of the Hodge $*$ - operator, albeit not explicitly used herein, is too important not to be at least mentioned. Indeed, one of its immediate implications would be the definition of $*^{-1}$.

Proposition 3.78. $*^2 = (-1)^{p(n-p)} \mathbb{1}$.

We can now proceed further with the definition of the Hodge-Laplacian operator. As is well-known, one of the most important operators acting on p-forms is the *exterior derivative*, or the *differential*. More precisely, the differential operator is the *anti-derivation* $d : C^\infty(\Lambda^p M) \longrightarrow C^\infty(\Lambda^{p+1} M)$

with the property $d^2 = 0$. By anti-derivation one understands the following:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta,$$

for $\alpha \in C^\infty(\Lambda^p M)$, and β a form of any degree. For smooth functions $f \in C^\infty(M)$ and vector fields $X \in \mathfrak{X}(M)$ the exterior derivative d is defined as $dX(f) = X(f)$. The dual of the differential is defined by means of the Hodge $*$-operator.

Definition 3.79. The map $\delta : C^\infty(\Lambda^p M) \longrightarrow C^\infty(\Lambda^{p-1} M)$ defined by the identity $\delta = (-1)^{np+p+1} * d*$ is called the *codifferential.*

A direct computation immediately yields $\delta^2 = 0$. We can finally introduce the protagonist of this story.

Definition 3.80. The *Hodge Laplacian* $\Delta : C^\infty(\Lambda^p M) \longrightarrow C^\infty(\Lambda^p M)$ is defined by $\Delta = d\delta + \delta d$.

Perhaps one observation worth making after the latter three definitions is the following. Both the differential and the codifferential commute with the Hodge - Laplacian. Indeed, due to $d^2 = 0$ and $\delta^2 = 0$, the following are decidedly true:

$$d\,\Delta = d(d\delta + \delta d) = d\,\delta\,d = (d\delta + \delta d)d = \Delta\,d,$$

$$\delta\,\Delta = \delta(d\delta + \delta d) = \delta\,d\,\delta = (d\delta + \delta d)\delta = \Delta\,\delta.$$

This latter observation will manifest itself in the fact that the differential and the codifferential preserve the eigenspaces of Δ, see equations (4.1), (4.2), and (4.3).

Let us now peer at some of the properties of Δ. It turns out that one can naturally define an inner product of forms. This is indeed imperative, as it will be evident in the immediate sequel.

Definition 3.81. Given an oriented manifold M the *inner product* on the sections of the exterior bundle $\Lambda^p M$ is defined by

$$(\eta, \xi)_M = \int_M \eta \wedge *\xi.$$

Before we continue, we make the following remark. Definition 3.76 suggests that we can equivalently write $(\eta, \xi)_M = \int_M \langle \eta, \xi \rangle \omega$ for the inner product we have just defined. Thus, we shall henceforth use both versions of this latter definition. For instance, in the proof of the duality of δ below, we shall prefer its first version, whereas later in Chapter 4 - its latter version.

Lemma 3.82. $(\eta, \delta\xi)_M = (d\eta, \xi)_M$.

Proof. Let $\eta \in C^\infty(\Lambda^p M)$ and $\xi \in C^\infty(\Lambda^{p+1} M)$. By dint of Stoke's theorem we have

$$0 = \int_M d(\eta \wedge *\xi) = \int_M d\eta \wedge *\xi + (-1)^p \int_M \eta \wedge d(*\xi).$$

The rest of the proof is a straightforward calculation. □

The following proposition represents two important properties of Δ.

Proposition 3.83. *The Hodge Laplacian is a self-adjoint operator that commutes with the Hodge $*$-operator.*

In other words, we have that $(\Delta\eta, \xi)_M = (\eta, \Delta\xi)_M$ and $*\Delta = \Delta*$. It ought to be clear that the proofs of the latter readily follow from definitions and Lemma 3.82.

4 The method of Ikeda and Taniguchi

In as much detail as the volume of this text permits, we explain below the solution to Problem 1. More precisely, we develop the method for the case when M is an n-sphere. The present section is indeed a natural extension of Section 2, which also presupposes the preliminaries just deliberated in Section 3. For convenience, we shall refer to the method explained herein as the method of Ikeda and Taniguchi. Notice, however, that we do not know whether or not the latter authors were the true discoverers of this method, for various others researchers at the time have been already aware of it. We shall begin by studying the structure of the spectrum of the Hodge-Laplacian. Firstly, we shall define the latter and discuss how it decomposes into G-modules via the Hodge decomposition theorem. Secondly, we shall see how representation theory naturally constitutes an integral part of our context. Thirdly, we shall prove that the Casimir operator differs from the Hodge-Laplacian only by a negative sign. Notice that this latter result is of fundamental importance for the method under consideration as we know that the eigenvalues of the Casimir operator are computable in terms of Freudhental's formula. Lastly, by invoking yet another result from representation theory, we elaborate on the spectrum of the Hodge-Laplacian on spheres.

4.1 The spectrum of the Hodge-Laplacian and its structure

The ultimate objective of this section is to understand the structure of the spectrum of the Hodge-Laplacian. Firstly, we shall define the latter and study how it decomposes into its eigenspaces. Secondly, we shall invoke representation theory and show how the eigenspaces of the operator Δ are appropriately related to G-modules. Thirdly, we shall introduce the G- and K - modules to be used shortly. Finally, we shall briefly comment on the multiplicities of the eigenvalues.

Definition 4.1. Let Δ be the Hodge-Laplacian acting on $C^\infty(\Lambda^p M)$. For each p the *spectrum* of Δ is the set

$$\mathrm{spec}^p(\Delta) = \{\lambda^p \mid \Delta\varphi = \lambda^p\varphi,\, \varphi \in \Omega^p(M)\}.$$

Notice that here, p is the degree of the differential forms and must be confused neither with multiplicities nor powers of the eigenvalues. Just as the usual Laplacian, the Hodge-Laplacian has a discrete spectrum, that is, $\mathrm{spec}^p(\Delta) = \{0 \leq \lambda_1^p < \lambda_2^p < \cdots \to \infty\}$ for each p. The eigenspaces are denoted $E^p_{\lambda_i^p}$. Now, the structure of the spectrum, or in other words, the configuration of the eigenspaces of Δ within $C^\infty(\Lambda^p M)$ is not difficult to be perceived as the following discussion unveils. On the one hand, we have the famous Hodge decomposition theorem[8] which asserts that

$$C^\infty(\Lambda^p M) = E_0^p \oplus dC^\infty(\Lambda^{p-1}M) \oplus \delta C^\infty(\Lambda^{p+1}M).$$

Here $E_0^p = \{\phi \in \Lambda^p M \mid \Delta\phi = 0\}$ is the space of *harmonic* p-forms. On the other hand, the operator Δ is a G-invariant differential operator, which means that its eigenspaces are G-modules. Ergo, the Hodge decomposition above is a direct sum of G-modules. More precisely, the eigenspaces $E^p_{\lambda_i^p}$ of the Hodge-Laplacian are finite-dimensional G-submodules of the G-modules in the Hodge decomposition, and more importantly, the algebraic sum $\sum_{i=0}^{\infty} E^p_{\lambda_i^p}$ is a dense subspace of $C^\infty(\Lambda^p M)$ concerning the topology defined by the inner product on $C^\infty(\Lambda^p M)$.

To perceive the importance of the present discussion, let us look a little bit deeper into the structure of $\mathrm{spec}^p(\Delta)$. For each eigenvalue λ_i^p we set the following G-submodules of $E^p_{\lambda_i^p}$

$${}'E^p_{\lambda_i^p} = \{\phi \in E^p_{\lambda_i^p} \mid d\phi = 0\} \quad \text{and} \quad {}''E^p_{\lambda_i^p} = \{\phi \in E^p_{\lambda_i^p} \mid \delta\phi = 0\},$$

and the following decompositions become evident

$$\begin{cases} E^p_{\lambda_i^p} = {}'E^p_{\lambda_i^p} \oplus {}''E^p_{\lambda_i^p} & \text{for } \lambda_i^p \neq 0, \\ E^p_{\lambda_i^p} = {}'E^p_{\lambda_i^p} = {}''E^p_{\lambda_i^p} & \text{for } \lambda_i^p = 0. \end{cases} \tag{4.1}$$

It is important to observe that the differential and the codifferential define the following G-isomorphisms

$$d : {}''E^p_{\lambda_i^p} \xrightarrow{\sim} {}'E^{p+1}_{\lambda_i^p} \quad \text{and} \quad \delta : {}'E^p_{\lambda_i^p} \xrightarrow{\sim} {}''E^{p-1}_{\lambda_i^p} \quad \text{for } \lambda_i^p \neq 0. \tag{4.2}$$

By $*\Delta = \Delta *$ and the duality imposed by the Hodge $*$-operator, it is readily seen that

$$\begin{cases} \lambda_i^p = \lambda_i^{n-p} & (i = 1, 2, \cdots) \text{ and} \\ * : {}'E^p_\lambda \xrightarrow{\sim} {}''E^{n-p}_\lambda & (p = 0, 1, \cdots, n). \end{cases} \tag{4.3}$$

[8]For the proof of this remarkable theorem, the reader may care to consult the book of Frank Warner [22].

We, therefore, perceive that the Hodge decomposition theorem naturally sheds some light on the structure of $\mathrm{spec}^p(\Delta)$. To understand the true nature of the just defined G-modules, however, we ought to swiftly bring forward some more facts and ideas from representation theory. More concretely, we shall have to invoke the concept of *complete reducibility* (compare with Definition 3.49).

Definition 4.2. A Lie group or a Lie algebra is said to have the *complete reducibility property* if every finite-dimensional representation of it is completely reducible.

Furthermore, as we only consider compact symmetric pairs (G, K), we can safely resort to the following well-known proposition.

Proposition 4.3. *Any compact Lie group G has the complete reducibility property.*

We shall also need the notion of a *complete set of irreducible representations.* We opt to define the latter in the more convenient terms of G-modules.

Definition 4.4. We say that the irreducible G-modules $V_1, ..., V_k$ form a *complete set of irreducible* G-modules if every irreducible G-module is isomorphic to some V_i, and no two of $V_1, ..., V_k$ are isomorphic.

Now, let $\mathcal{J}_G$ be the complete set of inequivalent irreducible representations of G over $\mathbb{C}$. Then, given an irreducible representation $\rho \in \mathcal{J}_G$ with representation space V_ρ, we define an injective G-homomorphism

$$\imath_\rho : \mathrm{Hom}_G(V_\rho, C^\infty(\Lambda^p M)) \otimes_{\mathbb{C}} V_\rho \longrightarrow C^\infty(\Lambda^p M) \tag{4.4}$$

by $\phi \otimes v \mapsto \phi(v)$. Set $\mathrm{Im}(\imath_\rho) = \Gamma^p_\rho$ and $\mu_\rho = \dim_{\mathbb{C}} \mathrm{Hom}_G(V_\rho, C^\infty(\Lambda^p M))$. Observe that the latter quantities only depend on the equivalence class of ρ and that Γ^p_ρ is isomorphic to the direct sum of μ_ρ-copies of V_ρ. This injective G-homomorphism allows us to conclude two things. Since each $E^p_{\lambda_{p_i}}$ is a finite dimensional G-module, it is necessarily completely reducible and therefore by Proposition 4.3 it is contained in $\sum_{\rho\in\mathcal{J}_G}\Gamma^p_\rho$. In other words, the eigenspaces of Δ and Γ^p_ρ are in the following natural relationship

$$\sum_i^{\infty} E^p_{\lambda^p_i} = \sum_{\rho\in\mathcal{J}_G}\Gamma^p_\rho. \tag{4.5}$$

Moreover, as the inner product $(\cdot,\cdot)_M$ on $C^\infty(\Lambda^p M)$ is G-invariant, one can conclude that the images Γ^p_ρ of the G-homomorphism $\imath_\rho$ are orthogonal to each other. Indeed, one can safely say at this juncture that the identity (4.5) bridges between Spectral Geometry and the Representation theory of Lie groups and algebras. Remarkably, as will be perceived shortly, when (G,K) is a compact symmetric pair with a semisimple Lie group G, we have each Γ^p_ρ contained in a certain eigenspace $E^p_{\lambda_i}$.

Let us briefly introduce some of the actual G-module and K- module structures we shall use in our context. The space of complex p-forms $C^\infty(\Lambda^p M)$ has a natural G-module structure over $\mathbb{C}$. This latter is defined by the action $(g\cdot s)(x) = g\cdot s(g^{-1}x)$ for $g\in G$, $s\in C^\infty(\Lambda^p M)$ and $x\in M$. It is worth mentioning that the latter G-action is naturally given by the pullbacks of the left G-action on $M = G/K$. Given a finite-dimensional vector space U we shall write $C^\infty(G;U)$ for the vector space of all smooth functions on G with values in U. The space $C^\infty(G;U)$ also has a natural G-module structure which is given by the action $(g\cdot f)(x) = f(g^{-1}h)$ for $g,h\in G$ and $f\in C^\infty(G;U)$. Furthermore, looking at U as a K-module, one readily perceives the following implications. On the one hand, we have already defined the space of twisted functions $C^\infty(G,K;U)$ in (3.4), and we know it is a G-submodule. On the other hand, the K-module U determines a homogeneous (associate) vector bundle $G\times_K U$ over G/K. Then, we can identify the G-module $C^\infty(G,K;U)$ with the space of sections of the homogeneous bundle $C^\infty(G\times_K U)$. The reader may wish to recall the discussions on the Frobenius reciprocity law (Proposition 3.44) and the space of twisted functions (3.4). Vitally important in the sequel will be the following particular case. Recall that $M = G/K$ and set

$$(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}} = \{\varphi : \mathfrak{g}\longrightarrow\mathbb{C} \mid \varphi(X) = 0\,\text{for all}\, X\in\mathfrak{k}\},$$

where φ is $\mathbb{C}$-linear. Observe that $\mathfrak{g}/\mathfrak{k}$ is naturally a K-module by the adjoint action of K in $\mathfrak{g}$. Furthermore, by choosing a base point, one may construct the isomorphism $\Lambda^p M \cong G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$, which would imply the identidification of the G-modules $C^\infty(\Lambda^p M)$ and $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$. Since this latter identification of G-modules will constitute the core hypothesis of Theorem 4.6, we shall explicitly define it at the beginning of the proof of Lemma 4.7. Precisely speaking, in the subsequent section we shall perceive the following identifications of G-modules

$$C^\infty(\Lambda^p M) \cong C^\infty(G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}) \cong C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}).$$

Recall that the Frobenius reciprocity law allows us to make the passage between G-module and K-module structures and to compute μ_ρ. Indeed, by applying Proposition 3.44 in our particular case we have the following isomorphism

$$\mathrm{Hom}_G(V_\rho, C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}))) \cong \mathrm{Hom}_K(V_\rho, \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})).$$

Furthermore, here μ_ρ is finite, it counts the multiplicity of an irreducible representation V_ρ of G in $C^\infty(\Lambda^p M)$ and is calculated by the formula

$$\mu_\rho = \dim \mathrm{Hom}_K(V_\rho, \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}) \quad \text{for any} \quad \rho \in \mathcal{J}_G.$$

4.2 The Casimir Operator vs The Hodge-Laplacian

We have just arrived at the most crucial part of the method of proof of Ikeda and Taniguchi. We begin with the following definition.

Definition 4.5. Let G be a compact semisimple Lie group and B the Killing form of its Lie algebra $\mathfrak{g}$. Given a basis $\{X_1, ..., X_N\}$ of $\mathfrak{g}$ we define the *Casimir operator* by

$$\mathcal{C} = \sum_{i,j} C^{ij} X_i \cdot X_j \qquad \text{where} \qquad C^{ij} = \Big(B(X_i, X_j)\Big)^{-1}. \tag{4.6}$$

Notice that $\mathcal{C}$ is an element in the universal enveloping algebra $U_\mathfrak{g}$ of $\mathfrak{g}$. More precisely, it lies in the centre of $U_\mathfrak{g}$, which is, $\mathcal{C} \otimes a = a \otimes \mathcal{C}$ for all $a \in U_{\mathfrak{g}}$. Moreover, it is a two-sided invariant second-order differential operator acting on G. Thus, by dint of Proposition 3.62, one perceives the Laplacian operator as the first candidate that appears, purporting the possibility of being identified with the Casimir operator just defined. In this section, we shall prove the verity of the latter identification.

Recall that (G, K) is a symmetric pair with G being a compact connected semisimple Lie group. This fact allows us to define an inner product on the sections of the exterior power bundle $\Lambda^p(G/K) \longrightarrow G/K$ in a natural manner. It follows from the general theory of vector bundles that an inner product on $C^\infty(\Lambda^p M)$ is naturally induced by a metric on the base space G/K. Since our goal is to exploit the full power of the representation theory of semisimple Lie algebras, we shall define a metric on the homogenous space G/K through the Killing form. This is done in the following natural way. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K, respectively. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form, which essentially is $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$. Then, using the Cartan decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$$

and restricting the Killing form on $\mathfrak{g}$ sign changed to $\mathfrak{m}$, one naturally defines a G-invariant Riemannian metric on G/K. Should the complex case be of interest, one must first complexify $\Lambda^p M$ and then extend the latter metric canonically to a hermitian metric on M. In both cases, the metric just defined will be denoted by $\langle \, , \, \rangle$. Recalling Definition 3.81 and our comment about it, we write for the inner product $(\, , \,)_M$ on $C^\infty(\Lambda^p M)$

$$(\varphi, \psi)_M = \int_M \langle \varphi, \psi \rangle \, dm \qquad \varphi, \psi \in C^\infty(\Lambda^p M), \tag{4.7}$$

where dm is the smooth measure on M defined by the G-invariant Riemannian metric on the base space G/K. By construction, we indeed have that the G-action preserves $(\, , \,)_M$.

With the aforementioned preparatory remarks, we are ready to "identify" the Hodge-Laplacian Δ with minus the Casimir operator $\mathcal{C}$.

Theorem 4.6. *Let (G, K) be a compact symmetric pair with G semisimple. Consider the homogeneous manifold $M = G/K$ and the Hodge-Lapalcian Δ defined by the metric (4.7). Then, under the identification $C^\infty(\Lambda^p M) \cong C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$, we have $\Delta = -\mathcal{C}$.*

We hold the opinion that this latter theorem lies at the very heart of the method of Ikeda and Taniguchi. Indeed, as elegant as it may be, it simultaneously pertains to areas such as spectral geometry, Lie theory, representation theory and Riemannian geometry in a very natural way. While the literature indicates that this result was known to other contemporaries of Ikeda and Taniguchi, to the best of our knowledge, most papers only stated it without proof. For these reasons, we shall reproduce below the proof of Theorem 4.6 in its entirety. Of course, notwithstanding some slight differences in organisation and exposition, the full credit for the latter is due to Ikeda, Taniguchi, et al.

The following notational remark is necessary before we start. The symbol $*$ must not be confused henceforth with the Hodge $*$-operator. The latter symbol will denote the dual of a vector space, the dual of an operator, or the pull-back of a map. Hopefully, this light abuse of notation will not bring forth much inconvenience.

The proof of Theorem 4.6 comprises three lemmas. The first one is conceptual, whereas the latter two are technical. Verily, the first lemma is crucial for the following reason. Bundle (3) a priori does not have sections differential forms, which implies that we do not have an intrinsic definition of the Hodge-Laplacian. However, as we have already seen, it is a bundle (3) which sets the representation theory context. Lemma 4.7 remedies this incoherence, for it provides a natural extension of the action of the Hodge-Laplacian on the sections of the bundle (3).

Lemma 4.7. *Let (G, K) be a compact symmetric pair with G semisimple, and let Δ be the Hodge-Lapalcian defined on the sections of $\Lambda^p M$. Then there is a vector bundle isomorphism $\zeta : \Lambda^p M \longrightarrow G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}$ such that the action of Δ is extended on the sections of $G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}$.*

Proof. For our immediate purposes, we shall only be concerned about the isomorphism between the sections of $\Lambda^p M$ and $G\times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}$. This latter isomorphism is naturally defined as the identification

$$C^\infty(\Lambda^p M) \cong C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}), \tag{4.8}$$

denoted $\alpha \mapsto \tilde{\alpha}$, by

$$(\tilde{\alpha}(g))(Y_1, ..., Y_p) = (\pi^*\alpha)(Y_1, ..., Y_p)g,$$

where $g \in G$, and $Y_1, ..., Y_p \in \mathfrak{g}$. The map π^* is the pull-back induced by the projection $\pi : G \longrightarrow G/K$. Notice that the map (4.8) is the restriction of ζ on the fibres. It is best to think of it as a G-module isomorphism.

We shall now show how the Hodge-Laplacian Δ, acting on the G-module $C^\infty(\Lambda^p M)$, can be naturally "lifted" to $\widetilde{\Delta}$, acting on the G-module $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$. Let $X_1, ..., X_n, X_{n+1}, ..., X_N$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form such that $X_1, ..., X_n$ forms a basis of $\mathfrak{m}$ and $X_{n+1}, ..., X_N$ a basis of $\mathfrak{k}$. For $\eta \in C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$, we define the linear map

$$D : C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}) \longrightarrow C^\infty(G, K; \Lambda^{p+1}(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$$

by

$$(D\eta)(X_{i_1}, ..., X_{i_p+1}) = \sum_{u=1}^{p+1}(-1)^{u-1} X_{i_u}\eta(X_{i_1}, ..., \hat{X}_{i_u}, ..., X_{i_p+1}), \tag{4.9}$$

with the order of indices $1 \leq i_1 < \cdots < i_{p+1} \leq n$. By dint of the identification (4.8) and the fact that $[X_{i_u}, X_{i_v}] \in \mathfrak{k}$, the following identity holds for any $X_{i_u}, X_{i_v} \in \mathfrak{m}$

$$\widetilde{d\alpha} = D\tilde{\alpha} \qquad \alpha \in C^\infty(\Lambda^p M). \tag{4.10}$$

In order to define the adjoint (the dual) of D, we first define an inner product $(\, , \,)^*$ on $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$ using the K-invariant inner product on $\Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}$ by

$$(\xi, \eta)^* = \int_G \langle \xi(g), \eta(g) \rangle dg, \tag{4.11}$$

for $\xi, \eta \in C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$. Here dg stands for a G-invariant smooth measure on G. It is now a matter of direct computation to prove that the adjoint operator D^* of D is given by

$$(D^*\xi)(X_{i_1}, ..., X_{i_{p-1}}) = -\sum_{k=1}^{n} X_k \xi(X_k, X_{i_1}, ..., X_{i_{p-1}}), \tag{4.12}$$

for $\xi \in C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$ and $1 \leq i_1 < \cdots < i_{p-1} \leq n$. Precisely speaking, we compute the adjoint operator D^* starting from (4.9) as follows. For any $\eta \in C^\infty(G, K; \Lambda^{p-1}(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$, we have

$$\begin{aligned}
&(\xi, D\eta)^* = \\
&= \int_G \frac{1}{p!} \sum_{i_1,..,i_p=1}^{n} \xi(X_{i_1}, \cdots, X_{i_p}) \cdot \sum_{u=1}^{p} (-1)^{u-1} X_{i_u} \eta(X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p}))dg \\
&= -\frac{1}{p!} \sum_{i_1,..,i_p=1}^{n} \int_G \sum_{u=1}^{p} (-1)^{u-1} X_{i_u} \cdot \xi(X_{i_1}, \cdots, X_{i_p}) \cdot \eta(X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p}))dg \\
&= -\frac{1}{(p-1)!} \sum_{i_1,\cdot\cdot,i_{p-1}=1}^{n} \int_G \sum_{k=1}^{n} X_k \xi(X_k, X_{j_1}, \cdots, X_{j_{p-1}}) \cdot \eta(X_{j_1}, \cdots, X_{j_{p-1}})dg.
\end{aligned}$$

We can now define the Hodge-Laplacian acting on $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}})$ by setting $\Delta^\circ = DD^* + D^*D$. Notice that, analogously to 4.10, the G-module isomorphism 4.8 will imply

$$\widetilde{\delta\alpha} = D^*\tilde{\alpha} \qquad \alpha \in C^\infty(\Lambda^p M).$$

Moreover, by playing a bit with the inner products (4.7) and (4.11) one can easily perceive the truth of

$$(\alpha, \beta)_M = c \cdot (\tilde{\alpha}, \tilde{\beta})^* \qquad \alpha, \beta \in C^\infty(\Lambda^p M) \tag{4.13}$$

for some constant c, which in turn implies

$$(\Delta\alpha, \beta)_M = c \cdot (\Delta^\circ\tilde{\alpha}, \tilde{\beta})^* \quad \text{for} \quad \alpha, \beta \in C^\infty(\Lambda^p M).$$

We have just proven the equality $\widetilde{\Delta\alpha} = \Delta^\circ\tilde{\alpha}$. □

Before we continue, we should like to make the following remark. The isomorphism ζ implies that $spec(\widetilde{\Delta}) = spec(\Delta)$. In other words, we may indeed identify $\widetilde{\Delta}$ with Δ, written $\widetilde{\Delta} \equiv \Delta$. What we shall prove below is, in point of fact, the identity $\widetilde{\Delta} = -\mathcal{C}$. Thus, these latter two observations will imply the desired $\Delta = -\mathcal{C}$.

The second lemma gives Δ° in terms of left-invariant first-order differential operators (vector fields).

Lemma 4.8.

$$(\Delta^\circ \tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{k=1}^{n} X_k^2 \tilde{\alpha}(X_{i_1}, \cdots, X_{i_p})$$
$$-\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^{u-1}[X_{i_u}, X_{i_k}]\tilde{\alpha}(X_k, X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p}).$$

Proof. Take the vector fields $X_{i_1}, \cdots X_{i_p}$ with $1 \leq i_1 < \cdots < i_p \leq n$. To prove the lemma is to calculate the quantities $(DD^*\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p})$ and $(D^*D\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p})$. Definitions 4.9 and 4.12 immediately imply

$$(DD^*\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} X_{i_u}X_k\tilde{\alpha}(X_k, X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p}).$$

It takes a little more effort to compute D^*D. We have the following

$$(D^*D\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{k=1}^{n} X_k(D\tilde{\alpha})(X_k, X_{i_1}, \cdots, X_{i_p}) =$$
$$-\sum_{k=1}^{n} X_k^2\tilde{\alpha}(X_{i_1}, \cdots, X_{i_p})$$
$$-\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^u X_k X_{i_u}\tilde{\alpha}(X_k, X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p})$$
$$-\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^u X_k\tilde{\alpha}([X_k, X_{i_u}], X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p})$$
$$-\sum_{k=1}^{n}\sum_{u<v}(-1)^{u+v} X_k\tilde{\alpha}([X_{i_u}, X_{i_v}], X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, \hat{X}_{i_v}, \cdots, X_{i_p}).$$

Since $[X_{i_u}, X_k] \in \mathfrak{k}$, we readilly obtain that

$$(DD^*\tilde{\alpha} + D^*D\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{k=1}^{n} X_k^2 \tilde{\alpha}(X_{i_1}, \cdots, X_{i_p}) + A,$$

where

$$A = -\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^{u-1}[X_{i_u}, X_{i_k}]\tilde{\alpha}(X_k, X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p})$$

□

This last lemma is a refinement of the former and completes the proof of Theorem 4.6.

Lemma 4.9.

$$\widetilde{\Delta\alpha} = -\sum_{k=1}^{N} X_k^2 \tilde{\alpha} \quad \text{for} \quad \alpha \in C^\infty(\Lambda^p M). \tag{4.14}$$

Proof. With Lemma 4.7 in mind, we must prove that $\Delta^\circ \tilde{\alpha} = -\sum_{k=1}^{N} X_k^2 \tilde{\alpha}$. Write $[X_i, X_k] = \sum_{j=1}^{N} b_{ik}^j X_j$, where b_{jk}^i are the structure constants of G. The compactness of G implies that the structure constants are skew-symmetric with respect to the indices i, j, k, which in the computation below will manifest in the equality $b_{ji_u}^k = b_{i_u k}^j$. On the one hand, we can rewrite the term A from Lemma 4.8 as

$$A = -\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^{u-1} \sum_{j=n+1}^{N} b_{i_u k}^j X_j \tilde{\alpha}(X_k, X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_p}).$$

On the other hand, for $j = n+1, \cdots, N$ and $g \in G$, we compute the following

$$\begin{aligned} X_j\tilde{\alpha}(X_{i_1},\cdots,X_{i_p})(g) &= \frac{d}{dt}\Big|_{t=0}\tilde{\alpha}(g\cdot\exp tX_j)(X_{i_1},\cdots,X_{i_p}) \\ &= \frac{d}{dt}\Big|_{t=0}\tilde{\alpha}(g)(\exp tX_j\cdot X_{i_1},\cdots,\exp tX_j\cdot X_{i_p}) \\ &= \sum_{u=1}^{p}(-1)^{u-1}\tilde{\alpha}(g)([X_j,X_{i_u}],X_{i_1},\cdots,\hat{X}_{i_u},\cdots,X_{i_p}) \\ &= \sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} b_{ji_u}^{k}\tilde{\alpha}(X_k,X_{i_1},\cdots,\hat{X}_{i_u},\cdots,X_{i_p}) \\ &= \sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} b_{i_u k}^{j}\tilde{\alpha}(X_k,X_{i_1},\cdots,\hat{X}_{i_u},\cdots,X_{i_p}). \end{aligned}$$

Applying again the vector field X_j to the latter expression we obtain

$$X_j^2\tilde{\alpha}(X_{i_1},\cdots,X_{i_p})(g)=\sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} b_{i_u k}^{j}X_j\tilde{\alpha}(X_k,X_{i_1},\cdots,\hat{X}_{i_u},\cdots,X_{i_p}).$$

Now, by taking sum from $j=n+1$ to N, we have

$$A=-\sum_{j=n+1}^{N}X_j^2\tilde{\alpha}(X_{i_1},\cdots,X_{i_p}),$$

and therefore (4.14) holds. □

We should like to close this subsection with the following remark. As already seen at few instances, the compactness of the group G is essential for the method presented herein. However, it deserves to be remarked here that an analogous formula for the non-compact case was given by Y. Matsushima and S. Murakami in [18]. As a matter of fact, according to Ikeda and Taniguchi, the proof we have just presented follows closely the techniques developed in the latter paper.

4.3 The spectra of the Hodge-Laplacian on the spheres

In virtue of the discussions in the preceding two sections, we are finally in the position to discuss the solution to Problem 2 and to show how it implies the spectrum of Δ on the spheres. We shall do three things in this section. First, with the celebrated Freudenthal formula and Theorem 4.6 at hand, we shall see that the computation of the eigenvalues of the Hodge-Laplacian reduces to the calculation of the highest weights of irreducible representations. Second, we shall comment on a known result from representation theory which offers the solution to Problem 2 in our particular case. Finally, we shall show how the latter is naturally applied in our context and thus perceive the truth of the theorem of Ikeda and Taniguchi, see Theorem 4.14.

4.3.1 The Freudenthal's formula

Recall that G is a compact semisimple Lie group and $\mathcal{C}$ is the Casimir operator. A very natural proposition is the following.

Proposition 4.10. *Let U be a finite-dimensional vector space over $\mathbb{C}$. Then, for any finite-dimensional G-submodule (V,ρ) of $C^\infty(G;U)$, we have*

$$\mathcal{C}f = \rho(\mathcal{C})f, \tag{4.15}$$

for all $f \in V$ and ρ being extended to the representation of the universal enveloping algebra $U_\mathfrak{g}$.

Notice that this latter proposition gives us precisely the action of the Casimir operator on the G-submodule V. To perceive its veracity, it suffices to recall our discussions in Subsections 3.3.2 and 3.3.3. Indeed, we know that any representation $\rho : G \longrightarrow GL(V)$ of a Lie group G naturally induces a representation of its Lie algebra $\mathrm{d}\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, which itself naturally extends to a representation of the universal enveloping algebra $\widehat{\mathrm{d}\rho} : U_\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

We also have the following corollary of Proposition 4.10 that neatly agrees with the discussion in Section 4.1.

Corollary 4.11. *Each finite-dimensional G-submodule of $C^\infty(G;U)$ is stable under the Casimir operator.*

Now, what is imperative for us is the following. If the representation ρ from Proposition 4.10 is irreducible, then by the Schur's lemma, we have that $\rho(\mathcal{C}) = \lambda \mathbb{1}$ for some constant λ. In other words, in the case of irreducible representations, Proposition 4.10 is tantamount to the eigenvalue problem for the Casimir operator. Luckily, the quantity $\rho(\mathcal{C})$ is computable by Freudenthal's formula. To state this latter, we write $\langle\, ,\, \rangle$ for the inner product on the dual space of the Cartan subalgebra of $\mathfrak{g}$, induced by the Killing form sign changed.

Proposition 4.12. *(The Freudenthal Formula)*
For any irreducible representation (V, ρ) of G over $\mathbb{C}$ with highest weight λ_ρ, we have

$$\rho(\mathcal{C}) = -4\pi^2 \langle \lambda_\rho + 2\delta_G, \lambda_\rho \rangle id_V, \tag{4.16}$$

where δ_G denotes the half sum of all positive roots of $\mathfrak{g}$

Although the reader may find the proof for this latter proposition in most books on representation theory, we should like to recommend [20]. Notice that here λ_ρ denotes the highest weight of the representation ρ and not an eigenvalue of Δ, for which we write λ^p. Notice also that the positive roots of any given semisimple Lie algebra $\mathfrak{g}$ are well known, and thus the value of δ_G. So, by dint of Theorem 4.6, it only remains to calculate the highest weights λ_ρ. Before we elaborate on this calculation, however, we shall have to digress to explain the solution to Problem 2.

4.3.2 The solution to Problem 2

To completely solve Problem 1 in the case of $M = \mathbb{S}^n$, we must settle two things. First, find the irreducible representations of $SO(n+1)$ in $C^\infty(\Lambda^p \mathbb{S}^n)$, which resolves Problem 2. Second, calculate the highest weights λ_ρ and obtain the spectrum of the Hodge-Laplacian by the Freudenthal formula. We shall now do the former and leave the latter for the immediate sequel. The reader might find it necessary to recall the discussion from Section 4.1, for the following is its natural continuation.

We have seen in Subsection 3.1.2 that the n- sphere is the homogeneous manifold $SO(n+1)/SO(n)$. Thus, we henceforth adopt the following notation. The Lie groups in the symmetric pair (G, K) are

$$G = SO(n+1) \quad \text{and} \quad K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathcal{M}_{n+1}(\mathbb{R}) \middle| A \in SO(n) \right\},$$

and their Lie algebras are respectively

$$\mathfrak{g} = so(n+1) = \left\{ X \in \mathcal{M}_{n+1}(\mathbb{R}) \mid X^\top + X = 0 \right\}$$

and

$$\mathfrak{k} = so(n) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \in \mathcal{M}_{n+1}(\mathbb{R}) \middle| Y^\top + Y = 0 \right\}.$$

Then the quotient $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$ consists of matrices of the form

$$\begin{pmatrix} 0 & -x_1 & \cdots & -x_n \\ x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix},$$

with $x_1, \cdots, x_n \in \mathbb{R}$, and we have the K-module isomorphism $\mathfrak{m}^* \cong \mathfrak{m}$.

Consider now the unit sphere $\mathbb{S}^n$ with the usual metric g and embedded in $\mathbb{R}^{n+1}$. Following the above notation, we have that $\mathbb{S}^n \cong G/K$ and $TS^n \cong G \times_K \mathfrak{m}$. Since the Killing form B on $\mathfrak{g} = so(n+1)$ is given by

$$B(X, Y) = (n-1) tr X \cdot Y \qquad (X, Y \in \mathfrak{g}),$$

the metric on $\mathbb{S}^n$ induced from the Killing form B is $2(n-1)g$. Furthermore, the Cartan subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ is known to be consisting of the matrices of the following type

$$2\pi \begin{pmatrix} (0) & & & & & & \\ & 0 & -\lambda_1 & & & & \\ & \lambda_1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & & 0 & -\lambda_m \\ & & & & & \lambda_m & 0 \end{pmatrix}.$$

Notice that the entries $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ are so indexed that $n = 2m-1$ for an odd n and $n = 2m$ for an even n. It is worth observing that when n is even, $\mathfrak{f}$ is also a Cartan subalgebra of $\mathfrak{k}$. For and odd n, however, a Cartan subalgebra of $\mathfrak{k}$ is the subspace $\mathfrak{f}_1$ consisting of elements of $\mathfrak{f}$ with $\lambda_1 = 0$. It will be convenient to think $\lambda_1, \cdots, \lambda_m$ as linear functionals on $\mathfrak{f}$ that are ordered linearly in $\mathfrak{f}^*$ as $\lambda_1 > \cdots > \lambda_m > 0$.

It is well-known from the representation theory of Lie groups that any dominant integral form Λ of the group $G = SO(n+1)$ with respect to the Cartan algebra $\mathfrak{f}$ is uniquely expressed as

$$\Lambda = k_1\lambda_1 + \cdots + k_m\lambda_m \,,$$

where the integers $k_1, \cdots, k_m$ satisfy the ordering

$$\begin{cases} k_1 \geq \cdots \geq k_{m-1} \geq |k_m| & (n = 2m-1), \\ k_1 \geq \cdots \geq k_{m-1} \geq k_m \geq 0 & (n = 2m). \end{cases} \tag{4.17}$$

When n is odd one sets

$$z_j = \lambda_{j+1} \,|_{\mathfrak{f}_1} \qquad (j = 1, \cdots, m-1),$$

where the right hand side means the restriction of the functional λ_{j+1} on $\mathfrak{f}$ to the subspace $\mathfrak{f}_1$, and $z_1, \cdots, z_{m-1}$ are ordered as $z_1 > \cdots > z_{m-1}$.

With the above notation in mind, it is not difficult to perceive that the solution to Problem 2 is an immediate consequence of the following, more general, fact from representation theory.

Proposition 4.13. *Let (V, ρ) be an irreducible G-module over $\mathbb{C}$ with the highest weight $\lambda_\rho = k_1\lambda_1 + \cdots + k_m\lambda_m$, where $k_1, \cdots k_m$ satisfy (4.17). Then, as a K-module, V decomposes into ireducible K-submodules as follows:*

(i) In case $n = 2m$,

$$V = \sum V_{k'_1\lambda_1 + \cdots + k'_m\lambda_m},$$

where the summation runs over all integers $k'_1, \cdots, k'_m$ such that

$$k_1 \geq k'_1 \geq k_2 \geq k'_2 \geq \cdots \geq k'_{m-1} \geq k_m \geq |k'_m|,$$

and $V_{k_1'\lambda_1+\cdots+k_m'\lambda_m}$ *denotes the irreducible* K*-submodule of* V *with the highest weight* $k_1\lambda_1 + \cdots + k_m\lambda_m$.
(ii) In case $n = 2m-1$,

$$V = \sum V_{k_1' z_1+\cdots+k_{m-1}' z_{m-1}},$$

where the summation runs over all integers $k_1', \cdots, k_m'$ *such that*

$$k_1 \geq k_1' \geq k_2 \geq k_2' \geq \cdots \geq k_{m-1}' \geq k_m \geq |k_m'|,$$

and the meaning of $V_{k_1' z_1+\cdots+k_{m-1}' z_{m-1}}$ *is similar to the above.*

We shall not dwell in the proof of this latter fact, for it is well beyond the scope of this paper. The reader may wish to consult [4, 23].

4.3.3 The computation of the highest weights

It now only remains to compute the highest weights λ_ρ to completely resolve Problem 1 for $M = \mathbb{S}^n$. Recall that $K = SO(n)$ and observe that the K-module $\mathfrak{m}$ is isomorphic to $\mathbb{R}^n$ with the standard representation of $SO(n)$. With Proposition 4.13 in mind, we must consider the following two cases.

Case I. Suppose that $n = 2m-1$. Then, it is known that, $\mathbb{C}\otimes_{\mathbb{R}}\Lambda^p\mathbb{R}^n$ is an irreducible $SO(n)$-module with the highest weight $\lambda_1 + \cdots + \lambda_p$ for each $p \leq m-1$. For $0 \leq p \leq n$, we have the $SO(n)$-module isomorphism $\mathbb{C}\otimes_{\mathbb{R}}\Lambda^p\mathbb{R}^n \cong \mathbb{C}\otimes_{\mathbb{R}}\Lambda^{n-p}\mathbb{R}^n$. To perceive the geometric interpretation of the latter isomorphism, the reader may wish to recall the definition of a G-isomorphism (4.3).

Case II. Suppose that $n = 2m$. Similalry to the former case, $\mathbb{C}\otimes_{\mathbb{R}}\Lambda^p\mathbb{R}^n$ is an irreducible $SO(n)$-module with the highest weight $\lambda_1 + \cdots + \lambda_p$ for each $p < m$, and $\mathbb{C}\otimes_{\mathbb{R}}\Lambda^m\mathbb{R}^n$ splits into two irreducible submodules with highest weights $\lambda_1+\cdots+\lambda_{m-1}-\lambda_m$ and $\lambda_1+\cdots+\lambda_{m-1}+\lambda_m$, respectively. For $p > m$, we again have the isomorphism $\mathbb{C}\otimes_{\mathbb{R}}\Lambda^p\mathbb{R}^n \cong \mathbb{C}\otimes_{\mathbb{R}}\Lambda^{n-p}\mathbb{R}^n$ as $SO(n)$-modules.

It is known from representation theory that the fundamental weights of the Lie algebra $\mathfrak{g} = so(n+1)$ are as follows

$$
\begin{aligned}
&\Lambda_0 = 0, \\
&\Lambda_j = \lambda_1 + \cdots + \lambda_j && (j = 1, 2, \cdots, m-2), \\
&\Lambda_{m-1} = \begin{cases} \lambda_1 + \cdots + \lambda_{m-1} & (n = 2m), \\ \frac{1}{2}(\lambda_1 + \cdots + \lambda_{m-1} - \lambda_m) & (n = 2m-1), \end{cases} \\
&\Lambda_m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_m).
\end{aligned}
$$

Furthermore, we can express uniquely every dominant integral form of the Lie group $G = SO(n+1)$ as a linear combination with non-negative integer coefficients of the fundamental weights Λ_j. Precisely speaking, a dominant integral form is a linear combination of $\Lambda_1, \cdots, \Lambda_{m-1}, 2\Lambda_m$ for $n = 2m$, and a linear combination of $\Lambda_1, \cdots, \Lambda_{m-2}, 2\Lambda_{m-1}, \Lambda_{m-1} + \Lambda_m, 2\Lambda_m$ for $n = 2m-1$. Notice that, with this observation, we have settled the computation of the highest weights, and it only remains to deal with the multiplicities of the irreducible representations. We already know from the Frobenius reciprocity law, recall Proposition 3.44, that the multiplicity μ_ρ of an irreducible representation (V, ρ) of G in $C^\infty(\Lambda^p \mathbb{S}^n)$ is equal to $\dim_{\mathbb{C}} \mathrm{Hom}_K(V, \Lambda^p \mathfrak{m}^*)$. One then computes this latter quantity utilising Schur's lemma, the K-irreducible decomposition of $\Lambda^p \mathfrak{m}^*$, and Proposition 4.13.

Two final observations before stating the Ikeda and Taniguchi theorem are the following. First, since $\mathbb{S}^n$ is an orientable manifold, we can consider without losing generality that $p \leq \dfrac{n}{2}$. One can also justify this latter consideration by the duality (4.3). Second, $\langle \cdot, \cdot \rangle$ stands for the inner product on $\mathfrak{k}^*$ induced from $\dfrac{-1}{2(n-1)} B(\cdot, \cdot)$. With these observations in mind, we arrive at the following beautiful theorem.

Theorem 4.14. *(Ikeda and Taniguchi' 1978) Let Δ be the Hodge-Laplacian acting on the sections $C^\infty(\Lambda^p \mathbb{S}^n)$ of the p-exterior bundle over the n-dimensional sphere $\mathbb{S}^n$. Supposing that $p \leq \dfrac{n}{2}$ the following hold true.*

(A) The highest weights λ_ρ *of the irreducible representations* ρ *intervening in* $C^\infty(\Lambda^p \mathbb{S}^n)$*, which is,* $\rho \in \mathcal{J}_G$*, are as follows:*

(A1) In the case $n = 2m$*,*

$$\lambda_\rho = \begin{cases} k\Lambda_1 + \Lambda_p,\ k\Lambda_1 + \Lambda_{p+1} & (0 \leq p \leq m-2), \\ k\Lambda_1 + \Lambda_{m-1},\ k\Lambda_1 + 2\Lambda_m & (p = m-1), \\ k\Lambda_1 + 2\Lambda_m & (p = m), \end{cases}$$

where k *runs over all non-negative integers.*

(A2) In the case $n = 2m-1$*,*

$$\lambda_\rho = \begin{cases} k\Lambda_1 + \Lambda_p,\ k\Lambda_1 + \Lambda_{p+1} & (0 \leq p \leq m-3), \\ k\Lambda_1 + \Lambda_{m-2},\ k\Lambda_1 + \Lambda_{m-1} + \Lambda_m & (p = m-2), \\ k\Lambda_1 + 2\Lambda_{m-1},\ k\Lambda_1 + \Lambda_{m-1} + \Lambda_m,\ k\Lambda_1 + 2\Lambda_m & (p = m-1), \end{cases}$$

Furthermore, the multiplicities μ_ρ *of the above* ρ *in* $C^\infty(\Lambda^p \mathbb{S}^n)$ *are equal to one except for the cases* $n = 2m$ *and* $p = m$*, when* $\mu_\rho = 2$*.*

(B) On an $SO(n+1)$*-irreducible submodule of differential forms on* $\mathbb{S}^n$ *with the highest weight* λ_ρ*, the Hodge-Laplacian has an eigenvalue given by* $4\pi^2\langle \lambda_\rho + 2\delta_G, \lambda_\rho \rangle$[9]*.*

We should like to finish this subsection by saying that the method described above remains essentially the same when computing the spectrum of Δ acting on the sections of the exterior p-bundle on $M = \mathbb{CP}^n$. In this case, however, Kälher geometry will be naturally required.

5 Epilogue

As this article is approaching its end, we would like to sincerely thank the reader for investing a part of their precious time to read and ponder the ideas presented herein. We genuinely hope that they might have found the lines above to be enjoyable mathematics and both stimulating and

[9] The values of $4\pi^2\langle \lambda_\rho + 2\delta_G, \lambda_\rho \rangle$ are given in Table (5.1) in the appendix.

inspiring for further studies. Notwithstanding all our good intentions and conscientious effort to write a sufficiently self-contained text, we should humbly admit that we are well aware that there remain many instances in this paper where the narrative could be improved and delivered better. Hopefully, some readers might care to venture into undertaking such a task. We should take one last liberty to briefly summarise what we did throughout and share our intuition, as naive, wrong, and vague as it may be, about possible further studies of these matters.

We see no better way to recapitulate the method described in this paper than to recall the diagram introduced in Section 2.

$$\begin{array}{ccc} \Lambda^p M & \xrightarrow{(\cong)} & G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}} \\ & {\scriptstyle (\Delta)} \searrow \quad \swarrow {\scriptstyle (\tilde{\Delta}=-\mathcal{C})} & \\ & M \cong G/K & \end{array}$$

It all began with the standard definition of the Hodge-Laplacian Δ acting on the sections $C^\infty(\Lambda^p M)$ of the exterior p-bundle over the manifold M. We have seen that the latter space, among the many structures it admits, is a G-module. Motivated by the difficulty of a direct assault calculation of the spectrum of Δ and in the vein of the latter fact, we sought to identify the G-module $C^\infty(\Lambda^p M)$ with another G-module that would permit a possible use of representation theory. It was feasible to make such an identification since the manifold $M = G/K$ was a Riemannian homogeneous space. More precisely, we perceived the identification $C^\infty(\Lambda^p M) \cong C^\infty\Big(G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}\Big)$ of G-modules. By the power of this latter identification, by considering the Riemannian geometry of the base manifold $M = G/K$ as well as by using some basic facts from Lie theory, we naturally defined a type of a Hodge-Laplacian operator $\tilde{\Delta}$ acting on the G-module $C^\infty\Big(G \times_K \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*\mathbb{C}}\Big)$ whose spectrum coincided with *spec*(Δ). Indeed, we proved in Theorem 4.6 that $\tilde{\Delta} = -\mathcal{C}$, which meant that to compute the spectrum of the Hodge-Laplacian Δ is to calculate the one of the Casimir operator $\mathcal{C}$. By so doing, we had our original geometrical problem translated into an algebraic one. Thus, having blended some of the ideas developed in Section 3 and having invoked some more advanced tools from representation theory, we culminated this paper by attempting a concise explanation of the truth of Theorem 4.14.

At the time of writing, in July 2022, and as far as we are concerned, the last explicit calculations of the spectrum of the Laplacian date back to some forty years ago. Ergo, without any shadow of a doubt, the full knowledge of the spectrum of the Laplacian remains somehow elusive. For this reason, it might be intriguing to look at the etymology of the word spectrum. Interestingly enough, the word spectrum allegedly originated in Latin in the early 1600s and bore the meaning of *an appearance, an image, an apparition, a spectre*, and is a derivative of the word **specere** which means *to look at, to view.* We know not to what extent the spectrum of the Laplacian operator is a spectre or an apparition, but we hope that new explicit computations in this direction will show up in the future. Driven by this hope, we dare to close this writing by posing the following questions.

Question 1. Consider the quotients $\mathbf{Gr}(r,n) = O(n)/O(r) \times O(n-r))$ and $\widetilde{\mathbf{Gr}}(r,n) = O(n)/O(r) \times O(n-r))$. These are the *Grassmannian* and the *oriented Grassmannian*, respectively. Compute the spectrum of the Hodge-Laplacian acting on the p-forms defined on the latter two manifolds.

Question 2. Compute the spectrum of the Hodge-Laplacian acting on p-forms on the *hyperbolic space* $\mathbb{H}^n$.

Question 3. Compute the spectrum of the Hodge-Laplacian acting on p-forms on the *anti-de Sitter space*, i.e., $AdS_{n+1} = O(2,n)/O(1,n)$.

Question 4. Consider $M = G/K$ with G being any of the exceptional Lie groups G_2, F_4, E_6, E_7, E_8. Fix G, consider an appropriate closed subgroup K of G and compute the spectrum of the Hodge-Laplacian acting on p-forms on $M = G/K$.

Question 5. Does it make sense to define a *Schrödinger-type* operator $S = \Delta + V$, where Δ is the Hodge-Laplacian and V a suitable potential p-form? If so, compute the spectrum of S on $\mathbb{S}^n$, $\mathbb{CP}^n$ and all the manifolds from Questions $1, 2, 3$, and 4. Could there be any relationship between $spec(\Delta)$ and $spec(S)$?

Question 6. The *Dirac operator* is a first-order differential operator D acting on a vector bundle over a Riemannian manifold satisfying the equation $D^2 = \Delta$, where Δ is the (Hodge)-Laplacian. Calculate $spec(D)$ on $\mathbb{S}^n$, $\mathbb{CP}^n$ and all the manifolds from Questions $1, 2, 3$, and 4.

We do not know whether other people posed some or all of these questions elsewhere. Nor do we know which of them have full or partial answers. Nonetheless, should the following and the last question make even a little sense for at least one reader, this paper has attained its goal.

Question 7. A question left wide open to the reader's curiosity, creativity and imagination.

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Appendix

Table 5.1: Spectra of the Hodge-Laplacian on spheres $\mathbb{S}^n$

	λ_ρ	$4\pi^2\langle\lambda_\rho + 2\delta_G, \lambda_\rho\rangle$
	$k\Lambda_1$	$k(k+n-1)$
case $n = 2m$	$k\Lambda_1 + \Lambda_p$ $(1 \leq p \leq m-1)$	$(k+p)(k+n+1-p)$
	$k\Lambda_1 + 2\Lambda_m$	$(k+m)(k+m+1)$
	$k\Lambda_1$	$k(k+n-1)$
	$k\Lambda_1 + \Lambda_p$ $(1 \leq p \leq m-2)$	$(k+p)(k+n+1-p)$
case $n = 2m-1$	$k\Lambda_1 + 2\Lambda_{m-1}$	$(k+m)^2$
	$k\Lambda_1 + \Lambda_{m-1} + \Lambda_m$	$(k+m-1)(k+m+1)$
	$k\Lambda_1 + 2\Lambda_m$	$(k+m)^2$

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