Matemática
Contemporânea

Vol. 51, 180–220
http://doi.org/10.21711/231766362022/rmc519

On the convergence rate of Galerkin approximations for the magnetohydrodynamic type equations

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The first author was partially supported by Grant PGC2018-098308-B-I00 (by MCIN/AEI/ 10.13039/501100011033 and by ERDF A way of making Europe), US-1381261 (I+D+i FEDER Andalucía 2014-2020) project and Grant P20_01120 (PAIDI/JUNTA/FEDER, UE), e-mail: angeles@us.es

The second author was partially supported by MATH-AMSUD project 21-MATH-03 (CTMicrAAPDEs), CAPES-PRINT 88887.311962/2018-00 (Brazil), Project UTA-Mayor, 4753-20, Universidad de Tarapacá (Chile), e-mail: mmedar@academicos.uta.cl

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Abstract. The motion of incompressible electrical conducting fluids can be modeled by magnetohydrodynamics equations, which consider the Navier-Stokes equations coupled with Maxwell's equations. For the classical Navier-Stokes system, there exists an extensively study of the convergence rate for the Galerkin approximations. In this work, we extend the estimates rates of spectral Galerkin approximations for Navier–Stokes equations to the magnetohydrodynamic equations. We prove optimal error estimates in the $L^2(\Omega)$ and $H^1(\Omega)$–norms and obtain a result similar to that one of Rautmann for the $H^2(\Omega)$–norm for the Navier–Stokes case. In this sense, we reach basically the same level of knowledge as in the case of the classical Navier-Stokes.

Keywords: estimates rates, optimal error estimates, magnetohydrodynamic type equations, spectral Galerkin method.

2020 Mathematics Subject Classification: 35Q30, 35Q61, 76W05.

1 Introduction

In several situations, the motion of incompressible electrical conducting fluids can be modeled by the so-called equations of magnetohydrodynamics (in reduced form, MHD equations), which can be described as the coupling of the Navier-Stokes equations and the Maxwell's equations. In order to describe these equations, we consider a bounded domain $\Omega \subset \mathbb{R}^3$, $T > 0$, denoted $Q_T \equiv \Omega \times (0,T)$ and $S_T \equiv \partial\Omega \times (0,T)$. In the case where there is free motion of heavy ions, not directly due to the electric field (see [16], [27], [28]), these equations can be reduced to the form:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{\eta}{\rho_m}\Delta\mathbf{u} + \dfrac{1}{\rho_m}\nabla\left(p^* + \dfrac{\mu}{2}\mathbf{h}^2\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\dfrac{\mu}{\rho_m}\mathbf{h}\cdot\nabla\mathbf{h} = \mathbf{f} \quad \text{in } Q_T, \\ \dfrac{\partial \mathbf{h}}{\partial t} - \dfrac{1}{\mu\sigma}\Delta\mathbf{h} + (\mathbf{u}\cdot\nabla)\mathbf{h} - (\mathbf{h}\cdot\nabla)\mathbf{u} + \nabla q = 0 \quad \text{in } Q_T, \\ \qquad\qquad\qquad\qquad\qquad\qquad \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \quad \text{in } Q_T, \end{cases} \tag{1.1}$$

together with the following boundary and initial conditions:

$$\begin{cases} \mathbf{u} = \mathbf{0}, & \mathbf{h} = \mathbf{0} \quad \text{on } S_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), & \mathbf{h}(x,0) = \mathbf{h}_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

Here, $\mathbf{u}$ and $\mathbf{h}$ are unknown velocity and magnetic field, respectively, p^* is an unknown hydrostatic pressure, q is an unknown function related to the heavy ions (in such way that the density of electric current, $\mathbf{j}_0$, generated by this motion satisfies the relation $\operatorname{rot} \mathbf{j}_0 = -\sigma \nabla q$), ρ_m is the density of mass of the fluid (assumed to be a positive constant), $\mu > 0$ is a constant magnetic permeability of the medium, $\sigma > 0$ is a constant electric conductivity, $\eta > 0$ is a constant viscosity of the fluid and $\mathbf{f}$ is a given external force field.

The initial value problem associated to the system (1.1)–(1.2) has been studied by several authors. Lassner [13], by using the semigroup results of Kato and Fujita [8], proved the existence and uniqueness of strong solutions, local in time for any data and global in time for small data. Boldrini and Rojas–Medar [24] studied the existence of weak solutions and the reproductive property using the Galerkin method, and Boldrini and Rojas–Medar [4], [25] improved theses results to local solutions for any data, and global strong solutions for small data (and uniqueness) by using the spectral Galerkin method. Damázio and Rojas–Medar [7] studied the regularity of weak solutions, Notte–Cuello and Rojas–Medar [14] used an iterative approach to show the existence and uniqueness of the strong solutions. As usual for the Navier-Stokes-type systems, in $2D$-domains the strong solution is globally obtained without restrictions over the data. In the $3D$-case, the strong solution is local in time for any data, and global in time assuming smallness for the data. The initial value problem in time dependent domains was studied by Rojas–Medar and Beltrán–Barrios [22], and by Berselli and Ferreira [2]. Zhao [31] studied the problem in unbounded domain with boundary uniformly of C^3 class.

In the stationary case, Chizhonkov, [5], assuming small data, proved the existence and uniqueness of a solution for the stationary problem in

two-dimensional and three-dimensional domains. He proposed a second-order approximations, and the resulting nonlinear algebraic system was solved via an efficient iteration process which converges under the small data assumption.

On the other hand, concerning the classical Navier–Stokes system there exist an extensively study of the convergence rate for the Galerkin approximations. The first work in this way was given by Rautmann in [17], where he proved the optimal convergence in the $H^1(\Omega)$–norm, but the optimal convergence in the $L^2(\Omega)$–norm was left as an open problem (see [17], pp. 438). This question was answered by Salvi [26]. Applying the same method and assuming the uniform boundedness in time of the $L^2(\Omega)$-norm of the gradient of the velocity and the exponential stability in the $H^1(\Omega)$–norm of the solution, Heywood [10] was able to derive an optimal uniform in time error estimates for the velocity in the $H^1(\Omega)$–norm. Also, without explicitly assuming $H^1(\Omega)$–exponential stability (this being in general difficult to verify), Boldrini and Rojas–Medar [23] proved an optimal uniform in time error estimate for the spectral Galerkin approximations in the $H^1(\Omega)$ and $L^2(\Omega)$–norms, under the requirement that the external force field has a mild form of decay.

The study of the convergence rate in the $\|\cdot\|_{\boldsymbol{H}^2(\Omega)}$–norm is difficult, since estimates of higher order spatial derivates of the solutions are required together with a compatibility condition to be satisfied by the initial value of the solution. This condition, being a nonlinear and nonlocal one, will be virtually uncheckable for given data (see Heywood and Rannacher [11], see Rautmann [20]). Related to this condition, the work of Rautmann [20] give an answer to the question *"how smooth a Navier–Stokes solution can be at time $t = 0$ without any compatibility condition?"*. Making use of this result, Rautmann ([18], [19]) proved the convergence rate in the $\|\cdot\|_{\boldsymbol{H}^2(\Omega)}$–norm of the spectral Galerkin approximation of the solution without any compatibility condition. These results were extended to the finite element discretization of the Navier–Stokes equations by Bause [1], and to other systems of fluid mechanics by Boldrini et al. (see [3]), and

Climent-Ezquerra et al. (see [6]).

In this work, we extend the estimates rates of spectral Galerkin approximations for the Navier-Stokes system to the magnetohydrodynamic equations (1.1)–(1.2). We prove optimal error estimates in the $L^2(\Omega)$ and $H^1(\Omega)$–norms and obtain a result similar to the Rautmann in [19] for $H^2(\Omega)$–norms. We reach in this way basically the same level of knowledge as in the case of the classical Navier-Stokes.

Finally, the paper is organized as follows. In Section 2 the basic notation is stated and the main known results are given. In Section 3, optimal error estimates in the L^2 and H^1-norms are stablished, which are necessary in order to prove our main result. In Section 4, the main result is proved, namely the point-wise convergence rate in the H^2–norm of the error without compatibility conditions on the initial data. The main novelties of this work are summarized in Section 5

2 Preliminaries

2.1 Function Spaces

Functions spaces and basic notation which we will use throughout this paper are introduced as follows: The functions are either $\mathbb{R}$ or $\mathbb{R}^3$, and, as a usual simplification, sometimes we will not distinguish them in our notation; the difference will be clear from the context. Vector functions will be written in bold letters. $L^2(\Omega)$–product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The H^s norm is denoted by $\|\cdot\|_{\boldsymbol{H}^s}$. Here $H^s = W^{s,2}(\Omega)$ ($s \geq 0$, $s \in \mathbb{R}$) are the usual Sobolev spaces. H_0^1 denotes the closure of $C_0^\infty(\Omega)$ in the H^1–norm. Let

$$\mathbf{C}_{0,\sigma}^\infty(\Omega) := \{\boldsymbol{v} \in (C_0^\infty(\Omega))^3 : \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \Omega\},$$

$$\boldsymbol{V} = \left\{\text{closure of } \mathbf{C}_{0,\sigma}^\infty(\Omega) \text{ in } \boldsymbol{H}_0^1(\Omega)\right\},$$

$$\boldsymbol{H} = \left\{\text{closure of } \mathbf{C}_{0,\sigma}^\infty(\Omega) \text{ in } \boldsymbol{L}^2(\Omega)\right\}$$

and

$$\boldsymbol{V}^* = \{\text{topological dual of } \boldsymbol{V}\}\,.$$

In order to give an operator interpretation of problem (1.1)–(1.2), we shall introduce the well known Helmholtz and Weyl decomposition. The Hilbert space $\boldsymbol{L}^2(\Omega)$ admits the Helmholtz and Weyl decomposition (cf. [29]):

$$\boldsymbol{L}^2 = \boldsymbol{H} \oplus \boldsymbol{H}^{\perp},$$

where $\oplus$ denotes direct sum and:

$$\boldsymbol{H}^{\perp} = \{\nabla\pi \,:\, \pi \in H^{1,}(\Omega)\}.$$

Let P be the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{H}$. Then the operator $A : \boldsymbol{H} \to \boldsymbol{H}$ given by $A = -P\Delta$ with domain $D(A) = \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ is called the Stokes operator. It is well known that A is a positive self-adjoint operator and is characterized by the following relation:

$$(A\boldsymbol{w}, \boldsymbol{v}) = (\nabla\boldsymbol{w}, \nabla\boldsymbol{v}) \text{ for all } \boldsymbol{w} \in D(A), \boldsymbol{v} \in \boldsymbol{V}.$$

From now on, we also denote the inner product in $\boldsymbol{H}$ by the $\boldsymbol{L}^2$–inner product $(\cdot,\cdot)$. The general L^p-norm will be denoted by $\|\cdot\|_{L^p(\Omega)}$; to make easier the notation, in the case $p = 2$ we simply denote the L^2–norm by $\|\cdot\|$. We shall denote by $\boldsymbol{w}^k(x)$ and λ_k the eigenfunctions and the eigenvalues of the Stokes operator. It is well known (see [29]) that $\boldsymbol{w}^k(x)$ are orthogonal in the inner products $(\cdot,\cdot)$, $(\nabla\cdot,\nabla\cdot)$ and $(A\cdot, A\cdot)$ and complete in the spaces $\boldsymbol{H}$, $\boldsymbol{V}$ and $\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, respectively. For each $k \in \mathbb{N}$, we denote by P_k the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{V}_k = \text{span}[\boldsymbol{w}^1(x), \ldots, \boldsymbol{w}^k(x)]$.

2.2 Definitions and problems associated to (1.1)-(1.2).

Throughout this work, we will deal with the following notion of strong solution for (1.1)-(1.2).

Definition 2.1. Let $\mathbf{u}_0, \mathbf{h}_0 \in \mathbf{V}$ and $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$. By a strong solution of the problem (1.1)–(1.2), we mean a pair of vector-valued functions

$(\mathbf{u},\mathbf{h})$ such that $\mathbf{u},\mathbf{h} \in L^\infty(0,T;\mathbf{V}) \cap \mathbf{L}^2(0,T;D(A))$ and that satisfies (1.1)–(1.2).

As a first step to set up and prove the main results of this work, and using the properties of the operator P, we can reformulate the problem (1.1)–(1.2), as follows: find $(\mathbf{u},\mathbf{h})$ in suitable spaces, satisfying:

$$\begin{cases} (\mathbf{u}_t,\mathbf{v}) + (\nabla\mathbf{u},\nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v}) - ((\mathbf{h}\cdot\nabla)\mathbf{h},\mathbf{v}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (\mathbf{f},\mathbf{v}), \quad \forall\,\mathbf{v}\in\mathbf{V}, \\ (\mathbf{h}_t,\mathbf{z}) + (\nabla\mathbf{h},\nabla\mathbf{z}) + ((\mathbf{u}\cdot\nabla)\mathbf{h},\mathbf{z}) \\ \qquad\qquad\qquad\qquad -((\mathbf{h}\cdot\nabla)\mathbf{u},\mathbf{z}) = 0, \quad \forall\mathbf{z}\in\mathbf{V}, \\ \mathbf{u}(\boldsymbol{x},0) = \mathbf{u}_0(\boldsymbol{x}), \qquad \mathbf{h}(\boldsymbol{x},0) = \mathbf{h}_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega. \end{cases} \tag{2.1}$$

Observe that, because we do not focus on the dependence of the error on the η,μ,σ or ρ_m, then we consider them all equal to 1.

In order to establish the results concerning estimates for spectral Galerkin approximation, we need to fix some problems. The *spectral Galerkin approximations* for $(\mathbf{u},\mathbf{h})$ are defined for each $k\in\mathbb{N}$ as the solution $(\mathbf{u}^k,\mathbf{h}^k)\in C^2([0,T];\mathbf{V}_k)\times C^2([0,T];\mathbf{V}_k)$ of:

$$\begin{cases} (\mathbf{u}_t^k,\mathbf{v}) + (\nabla\mathbf{u}^k,\nabla\mathbf{v}) + ((\mathbf{u}^k\cdot\nabla)\mathbf{u}^k,\mathbf{v}) \\ \qquad\qquad -((\mathbf{h}^k\cdot\nabla)\mathbf{h}^k,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\,\mathbf{v}\in\mathbf{V}_k, \\ (\mathbf{h}_t^k,\mathbf{z}) + (\nabla\mathbf{h}^k,\nabla\mathbf{z}) + ((\mathbf{u}^k\cdot\nabla)\mathbf{h}^k,\mathbf{z}) \\ \qquad\qquad -((\mathbf{h}^k\cdot\nabla)\mathbf{u}^k,\mathbf{z}) = 0, \quad \forall\mathbf{z}\in\mathbf{V}_k, \\ \mathbf{u}(\boldsymbol{x},0) = P_k\mathbf{u}_0(\boldsymbol{x}), \quad \mathbf{h}(x,0) = P_k\mathbf{h}_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega. \end{cases} \tag{2.2}$$

Recall that the eigenfunctions expansion of $\mathbf{u}$ and $\mathbf{h}$ can be written, respectively, as:

$$\mathbf{u}(x,t) = \sum_{i=1}^{\infty} a_i(t)\mathbf{w}^i(x) \quad \text{and} \quad \mathbf{h}(x,t) = \sum_{i=1}^{\infty} c_i(t)\mathbf{w}^i(x), \tag{2.3}$$

where $\mathbf{w}^i$ are the eigenfunctions of the Stokes operator. The partial sums of the series for $\mathbf{u}$ and $\mathbf{h}$ will also appear in our study, whose expression

are given, respectively, by:

$$\begin{cases} \mathbf{v}^k(t) &= P_k\mathbf{u}(t) = \displaystyle\sum_{i=1}^{k} a_i(t)\mathbf{w}^i(x) \\ \\ \mathbf{b}^k(t) &= P_k\mathbf{h}(t) = \displaystyle\sum_{i=1}^{k} c_i(t)\mathbf{w}^i(x). \end{cases} \tag{2.4}$$

2.3 Known results

By using the spectral Galerkin approximations (2.2), Rojas-Medar and Boldrini ([4], [25]) proved the following results:

Theorem 2.2. *Assume the following conditions for the initial data* $\mathbf{u}_0$, $\mathbf{h}_0$, *and the external force* $\mathbf{f}$ *of* (1.1)-(1.2)*:*

$$\mathbf{u}_0,\, \mathbf{h}_0 \in \mathbf{V}, \quad \mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega)) \tag{2.5}$$

Then, on a (possibly small) time interval $[0,T_1]$, $0 < T_1 \leq T$, *problem* (1.1)-(1.2) *has a unique strong solution* $(\mathbf{u},\mathbf{h})$. *This solution belongs* $C([0,T_1];\mathbf{V}) \times C([0,T_1];\mathbf{V})$. *Moreover, there exist* C^0*-functions* $F(t)$ *and* $G(t)$ *such that for any* $t \in [0,T_1]$, *there hold:*

$$\begin{aligned} \|\nabla\mathbf{u}(t)\|^2 + \|\nabla\mathbf{h}(t)\|^2 + \int_0^t (\|A\mathbf{u}(s)\|^2 + \|A\mathbf{h}(s)\|^2)ds &\leq F(t), \\ \int_0^t (\|\mathbf{u}_t(s)\|^2 + \|\mathbf{h}_t(s)\|^2)ds &\leq G(t). \end{aligned}$$

Moreover, the same kind of estimates holds uniformly in $n \in \mathbb{N}$ *for the Galerkin approximations* $(\mathbf{u}^n, \mathbf{h}^n)$.

Theorem 2.3. *Assume* (2.5) *and*

$$\mathbf{u}_0,\, \mathbf{h}_0 \in D(A), \quad \mathbf{f}_t \in L^2(0,T;\mathbf{L}^2(\Omega)). \tag{2.6}$$

Then:

$$\begin{aligned}
\|\mathbf{u}_t(t)\|^2 + \|\mathbf{h}_t(t)\|^2 + \int_0^t (\|\nabla \mathbf{u}_t(s)\|^2 + \|\nabla \mathbf{h}_t(s)\|^2) ds &\leq H_0(t), \\
\|A\mathbf{u}(t)\|^2 + \|A\mathbf{h}(t)\|^2 &\leq H_1(t), \\
\int_0^t (\|\mathbf{u}_{tt}(s)\|^2_{V^\star} + \|\mathbf{h}_{tt}(s)\|^2_{V^\star}) ds &\leq H_2(t),
\end{aligned}$$

for any $t \in [0, T_1]$*, where* $H_i(t)$*,* $i = 0, 1, 2$ *are continuous functions* $t \in [0, T_1]$*. Therefore:*

$$\mathbf{u}(t), \mathbf{h}(t) \in C^1([0, T_1]; \mathbf{V}) \cap C([0, T_1]; D(A)).$$

Moreover, the same kind of estimates holds uniformly in n *for the Galerkin approximations* $(\mathbf{u}^n, \mathbf{h}^n)$*.*

The idea of the proof is taking first the solutions of (2.2), and prove the convergence of such $(\boldsymbol{u}^n, \boldsymbol{h}^n)$ to the solutions $(\boldsymbol{u}, \boldsymbol{h})$ of (2.1).

Relating to the Stokes operator and the orthogonal projector P_k, the following lemma can be found in the Rautmann's paper [17].

Lemma 2.4. *Consider* $P_k : \boldsymbol{L}^2(\Omega) \to \boldsymbol{V}_k$ *the operator defined in (2.4)*$_1$ *and* λ_{k+1} *the* $(k+1)$*-eigenvalue of the Stokes operator.*

i) If $\mathbf{w} \in \mathbf{V}$*, then there holds:*

$$\|\mathbf{w} - P_k \mathbf{w}\|^2 \leq \frac{1}{\lambda_{k+1}} \|\nabla \mathbf{w}\|^2.$$

ii) Also, if $\mathbf{w} \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$*, we have:*

$$\|\mathbf{w} - P_k \mathbf{w}\|^2 \leq \frac{1}{\lambda_{k+1}^2} \|A\mathbf{w}\|^2, \qquad \|\nabla \mathbf{w} - \nabla P_k \mathbf{w}\|^2 \leq \frac{1}{\lambda_{k+1}} \|A\mathbf{w}\|^2.$$

Some of the classical Sobolev interpolation inequalities, considered in this manuscript, can be found in the following result:

Lemma 2.5. *Consider A the Stokes operator defined in Subsection 2.1. The following estimates are true:*

i) $\|\mathbf{v}\|_{\mathbf{L}^\infty(\Omega)} \leq C\|A\mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{V} \cap \mathbf{H}^2(\Omega),$

ii) $\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \leq C\|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{V},$

iii) $\|\mathbf{v}\|_{\mathbf{L}^3(\Omega)} \leq C\|\mathbf{v}\|^{1/2}\|\nabla \mathbf{v}\|^{1/2}, \quad \forall \mathbf{v} \in \mathbf{V}$

iv) $\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \leq C\|\mathbf{v}\|^{1/4}\|\nabla \mathbf{v}\|^{3/4}, \quad \forall \mathbf{v} \in \mathbf{V}.$

3 First convergence rates

Consider the eigenfunctions expansion of $\boldsymbol{u}$ and $\boldsymbol{h}$, respectively,

$$\boldsymbol{u}(x,t) = \sum_{i=1}^{\infty} a_i(t)\boldsymbol{w}^i(x),$$

$$\boldsymbol{h}(x,t) = \sum_{i=1}^{\infty} c_i(t)\boldsymbol{w}^i(x),$$

and the kth partial sums of the series for $\boldsymbol{u}$ and $\boldsymbol{h}$, respectively,

$$\boldsymbol{v}^k(t) = P_k\boldsymbol{u}(t) = \sum_{i=1}^{k} a_i(t)\boldsymbol{w}^i(x),$$

$$\boldsymbol{b}^k(t) = P_k\boldsymbol{h}(t) = \sum_{i=1}^{k} c_i(t)\boldsymbol{w}^i(x)$$

Then, the following expressions of the error can be defined:

$$\boldsymbol{e}^k(t) = \boldsymbol{u}(t) - \boldsymbol{v}^k(t), \;\; \tilde{\boldsymbol{e}}^k(t) = \boldsymbol{h}(t) - \boldsymbol{b}^k(t), \tag{3.1}$$

$$\boldsymbol{E}^k(t) = \boldsymbol{v}^k(t) - \boldsymbol{u}^k(t), \;\; \tilde{\boldsymbol{E}}^k(t) = \boldsymbol{b}^k(t) - \boldsymbol{h}^k(t), \tag{3.2}$$

where $\boldsymbol{u}^k$ and $\boldsymbol{h}^k$ are the kth Galerkin approximations of $\boldsymbol{u}$ and $\boldsymbol{h}$ solutions of (2.2), respectively.

The purpose of this section is to obtain some "first maesures" of the distance between the solutions of (2.1) and (2.2), that we split as:

$$\boldsymbol{u}(t) - \boldsymbol{u}^k(t) = \boldsymbol{e}^k(t) + \boldsymbol{E}^k(t), \quad \text{and} \quad \boldsymbol{h}(t) - \boldsymbol{h}^k(t) = \tilde{\boldsymbol{e}}^k(t) + \tilde{\boldsymbol{E}}^k(t).$$

These variables $\boldsymbol{E}^k$ and $\tilde{\boldsymbol{E}}^k$ satisfy:

$$\begin{cases} (\boldsymbol{E}_t^k, \boldsymbol{v}) + (\nabla \boldsymbol{E}^k, \nabla \boldsymbol{v}) + ((\boldsymbol{e}^k \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) + ((\boldsymbol{E}^k \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) \\ \qquad +((\boldsymbol{u}^k \cdot \nabla)\boldsymbol{e}^k, \boldsymbol{v}) + ((\boldsymbol{u}^k \cdot \nabla)\boldsymbol{E}^k, \boldsymbol{v}) - ((\tilde{\boldsymbol{e}}^k \cdot \nabla)\boldsymbol{h}, \boldsymbol{v}) \\ \qquad -((\tilde{\boldsymbol{E}}^k \cdot \nabla)\boldsymbol{h}, \boldsymbol{v}) - ((\boldsymbol{h}^k \cdot \nabla)\tilde{\boldsymbol{e}}^k, \boldsymbol{v}) - ((\boldsymbol{h}^k \cdot \nabla)\tilde{\boldsymbol{E}}^k, \boldsymbol{v}) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}_k, \\ (\tilde{\boldsymbol{E}}_t^k, \boldsymbol{z}) + (\nabla \tilde{\boldsymbol{E}}^k, \nabla \boldsymbol{z}) + ((\boldsymbol{e}^k \cdot \nabla)\boldsymbol{h}, \boldsymbol{z}) + ((\boldsymbol{E}^k \cdot \nabla)\boldsymbol{h}, \boldsymbol{z}) \\ \qquad +((\boldsymbol{u}^k \cdot \nabla)\tilde{\boldsymbol{e}}^k, \boldsymbol{z}) + ((\boldsymbol{u}^k \cdot \nabla)\tilde{\boldsymbol{E}}^k, \boldsymbol{z}) - ((\tilde{\boldsymbol{e}}^k \cdot \nabla)\boldsymbol{u}, \boldsymbol{z}) \\ \qquad -((\tilde{\boldsymbol{E}}^k \cdot \nabla)\boldsymbol{u}, \boldsymbol{z}) - ((\boldsymbol{h}^k \cdot \nabla)\boldsymbol{e}^k, \boldsymbol{z}) - ((\boldsymbol{h}^k \cdot \nabla)\boldsymbol{E}^k, \boldsymbol{z}) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{z} \in \boldsymbol{V}_k, \\ \boldsymbol{E}^k(x,0) = \tilde{\boldsymbol{E}}^k(x,0) = 0, \qquad x \in \Omega. \end{cases} \tag{3.3}$$

The following estimates for the L^2-norm of $\boldsymbol{E}^k$ and $\tilde{\boldsymbol{E}}^k$ can be obtained:

Lemma 3.1.
Let $\boldsymbol{E}^k$ and $\tilde{\boldsymbol{E}}^k$ be the solutions of system (3.3).

- *Under the assumptions of Theorem 2.2, the following estimate holds:*

$$\|\boldsymbol{E}^k(t)\|^2 + \|\tilde{\boldsymbol{E}}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}}. \tag{3.4}$$

- *Under the assumptions of Theorem 2.3, the following estimate holds:*

$$\|\boldsymbol{E}^k(t)\|^2 + \|\tilde{\boldsymbol{E}}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}^2}. \tag{3.5}$$

Proof. Setting $\boldsymbol{v} = \boldsymbol{E}^k$ in (3.3), we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{E}^k\|^2 &+ \|\nabla \boldsymbol{E}^k\|^2 = -((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}^k) - ((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}^k)\\ &- ((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{e}^k,\boldsymbol{E}^k) + ((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}^k)\\ &+ ((\tilde{\boldsymbol{E}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}^k) + ((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{e}}^k,\boldsymbol{E}^k)\\ &+ ((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{E}}^k,\boldsymbol{E}^k) = \sum_{i=1}^{7} I_i.\end{aligned} \tag{3.6}$$

Analogously, setting $\boldsymbol{z} = \tilde{\boldsymbol{E}}^k$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{\boldsymbol{E}}^k\|^2 &+ \|\nabla \tilde{\boldsymbol{E}}^k\|^2 = -((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{h},\tilde{\boldsymbol{E}}^k) - ((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{h},\tilde{\boldsymbol{E}}^k)\\ &- ((\boldsymbol{u}^k\cdot\nabla)\tilde{\boldsymbol{e}}^k,\tilde{\boldsymbol{E}}^k) + ((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{u},\tilde{\boldsymbol{E}}^k)\\ &+ ((\tilde{\boldsymbol{E}}^k\cdot\nabla)\boldsymbol{u},\tilde{\boldsymbol{E}}^k) + ((\boldsymbol{h}^k\cdot\nabla)\boldsymbol{e}^k,\tilde{\boldsymbol{E}}^k)\\ &+ ((\boldsymbol{h}^k\cdot\nabla)\boldsymbol{E}^k,\tilde{\boldsymbol{E}}^k).\end{aligned} \tag{3.7}$$

Now, we estimate the right-hand of (3.6) and (3.7) as follows:

$$\begin{aligned}|I_1| &= |-((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}^k)| = |((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{E}^k,\boldsymbol{u})|\\ &\le \|\boldsymbol{e}^k\|\|\nabla\boldsymbol{E}^k\|\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \le C_\epsilon\|A\boldsymbol{u}\|^2\|\boldsymbol{e}^k\|^2 + \epsilon\|\nabla\boldsymbol{E}^k\|^2,\end{aligned} \tag{3.8}$$

where we have used the fact that $\nabla\cdot\boldsymbol{e}^k = 0$ and the estimates of the Lemma 2.5. Also:

$$\begin{aligned}|I_2| &= |-((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}^k)| \le \|\boldsymbol{E}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{u}\|\|\boldsymbol{E}^k\|_{\boldsymbol{L}^6(\Omega)}\\ &\le C\|\boldsymbol{E}^k\|^{1/2}\|\nabla\boldsymbol{E}^k\|^{3/2}\|\nabla\boldsymbol{u}\|\\ &\le C_\epsilon\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{E}^k\|^2 + \epsilon\|\nabla\boldsymbol{E}^k\|^2.\end{aligned} \tag{3.9}$$

Again, using that $\nabla\cdot\boldsymbol{u}^k = 0$,

$$\begin{aligned}|I_3| &= |-((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{e}^k,\boldsymbol{E}^k)| = |((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{E}^k,\boldsymbol{e}^k)|\\ &\le \|\boldsymbol{u}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{E}^k\|\|\boldsymbol{e}^k\| \le C\|A\boldsymbol{u}^k\|\|\boldsymbol{e}^k\|\|\nabla\boldsymbol{E}^k\|\\ &\le C_\epsilon\|A\boldsymbol{u}^k\|^2\|\boldsymbol{e}^k\|^2 + \epsilon\|\nabla\boldsymbol{E}^k\|^2.\end{aligned} \tag{3.10}$$

$$\begin{aligned}|I_4| &= |((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}^k)| \le \|\tilde{\boldsymbol{e}}^k\|\|\nabla\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{E}^k\|_{\boldsymbol{L}^3(\Omega)}\\ &\le C_\epsilon\|A\boldsymbol{h}\|^2\|\tilde{\boldsymbol{e}}^k\|^2 + \epsilon\|\nabla\boldsymbol{E}^k\|^2.\end{aligned} \tag{3.11}$$

$$\begin{aligned} |I_5| &= |((\tilde{\boldsymbol{E}}^k \cdot \nabla)\boldsymbol{h}, \boldsymbol{E}^k)| \leq \|\tilde{\boldsymbol{E}}^k\| \|\nabla \boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)} \|\boldsymbol{E}^k\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq C_\epsilon \|A\boldsymbol{h}\|^2 \|\tilde{\boldsymbol{E}}^k\|^2 + \epsilon \|\nabla \boldsymbol{E}^k\|^2. \end{aligned} \tag{3.12}$$

$$\begin{aligned} |I_6| &= |((\boldsymbol{h}^k \cdot \nabla)\tilde{\boldsymbol{e}}^k, \boldsymbol{E}^k)| = |((\boldsymbol{h}^k \cdot \nabla)\boldsymbol{E}^k, \tilde{\boldsymbol{e}}^k)| \\ &\leq C \|\boldsymbol{h}^k\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla \boldsymbol{E}^k\| \|\tilde{\boldsymbol{e}}^k\| \\ &\leq C_\epsilon \|A\boldsymbol{h}^k\|^2 \|\tilde{\boldsymbol{e}}^k\|^2 + \epsilon \|\nabla \boldsymbol{E}^k\|^2. \end{aligned} \tag{3.13}$$

$$\begin{aligned} |I_7| &= |((\boldsymbol{h}^k \cdot \nabla)\boldsymbol{E}^k, \tilde{\boldsymbol{E}}^k)| \leq C \|\boldsymbol{h}^k\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla \boldsymbol{E}^k\| \|\tilde{\boldsymbol{E}}^k\| \\ &\leq C_\epsilon \|A\boldsymbol{h}^k\|^2 \|\tilde{\boldsymbol{E}}^k\|^2 + \epsilon \|\nabla \boldsymbol{E}^k\|^2. \end{aligned} \tag{3.14}$$

The equality (3.6) together with estimates (3.8)–(3.13) implies the differential inequality:

$$\begin{aligned} \frac{d}{dt}\|\boldsymbol{E}^k\|^2 &+ \|\nabla \boldsymbol{E}^k\|^2 \leq C\|\nabla \boldsymbol{u}\|^4 \|\boldsymbol{E}^k\|^2 + C\|\boldsymbol{e}^k\|^2 \|A\boldsymbol{u}^k\|^2 \\ &+ C\left(\|A\boldsymbol{h}\|^2 + \|A\boldsymbol{h}^k\|^2\right)\left(\|\tilde{\boldsymbol{E}}^k\|^2 + \|\tilde{\boldsymbol{e}}^k\|^2\right). \end{aligned} \tag{3.15}$$

Analogously, we have:

$$\begin{aligned} \frac{d}{dt}\|\tilde{\boldsymbol{E}}^k\|^2 &+ \|\nabla \tilde{\boldsymbol{E}}^k\|^2 \leq C\|\tilde{\boldsymbol{e}}^k\|^2 (\|A\boldsymbol{u}^k\|^2 + \|A\boldsymbol{u}\|^2 + \|A\boldsymbol{h}^k\|^2) \\ &+ C\|\boldsymbol{e}^k\|^2 \|A\boldsymbol{h}\|^2 + C\|A\boldsymbol{u}\|^2 \|\tilde{\boldsymbol{E}}^k\|^2 \\ &+ C(\|\nabla \boldsymbol{h}\|^4 + \|A\boldsymbol{h}^k\|^2)\|\boldsymbol{E}^k\|^2. \end{aligned} \tag{3.16}$$

Adding inequalities (3.15) and (3.16), we obtain:

$$\begin{aligned} &\frac{d}{dt}(\|\boldsymbol{E}^k\|^2 + \|\tilde{\boldsymbol{E}}^k\|^2) + \|\nabla \boldsymbol{E}^k\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k\|^2 \\ &\quad \leq \left[\|\boldsymbol{e}^k\|^2 + \|\tilde{\boldsymbol{e}}^k\|^2\right]\phi_1(t) + \left[\|\boldsymbol{E}^k\|^2 + \|\tilde{\boldsymbol{E}}^k\|^2\right]\phi_2(t), \end{aligned} \tag{3.17}$$

where

$$\begin{cases} \phi_1(t) = \|A\boldsymbol{u}(t)\|^2 + \|A\boldsymbol{h}(t)\|^2 + \|A\boldsymbol{u}^k(t)\|^2 + \|A\boldsymbol{h}^k(t)\|^2, \\ \phi_2(t) = \|\nabla \boldsymbol{u}(t)\|^4 + \|\nabla \boldsymbol{h}(t)\|^4 + \|A\boldsymbol{u}(t)\|^2 + \|A\boldsymbol{h}(t)\|^2 + \|A\boldsymbol{h}^k(t)\|^2 \end{cases}$$

By using Gronwall's Lemma (using that $\|\boldsymbol{E}^k(0)\| = \|\tilde{\boldsymbol{E}}^k(0)\| = 0$), we obtain:

$$\|\boldsymbol{E}^k(t)\|^2 + \|\tilde{\boldsymbol{E}}^k(t)\|^2 \leq \exp\left(\int_0^t \phi_2(s)\,ds\right)\int_0^t \left(\|\boldsymbol{e}^k(s)\|^2 + \|\tilde{\boldsymbol{e}}^k(s)\|^2\right)\phi_1(s)ds$$

Since $\boldsymbol{u}_0, \boldsymbol{h}_0 \in \boldsymbol{V}$, from Theorem 2.2 we have that $\phi_1(\cdot), \phi_2(\cdot) \in L^1(0,T)$. Taking into account (3.1), we know that

$$\boldsymbol{u} - P_k\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{v}^k = \boldsymbol{e}^k, \qquad \boldsymbol{h}(t) - P_k\boldsymbol{h}(t) = \boldsymbol{h}(t) - \boldsymbol{b}^k(t) = \tilde{\boldsymbol{e}}^k(t)$$

and $\boldsymbol{e}^k$ and $\tilde{\boldsymbol{e}}^k$ satisfy Lemma 2.4, *i*). Therefore, we obtain the estimate (3.4). Also, since $\boldsymbol{u}_0, \boldsymbol{h}_0 \in D(A)$ from Theorem 2.3, we have $\phi_1(\cdot), \phi_2(\cdot) \in L^\infty(0,T)$ and applying Lemma 2.4 *ii*), we obtain the estimate (3.5). This completes the proof of the lemma. □

Corollary 3.2. *The following estimates for* $\|\nabla \boldsymbol{E}^k\|$ *and* $\|\nabla \tilde{\boldsymbol{E}}^k\|$ *hold:*

- *Under the assumptions of Theorem 2.2 then:*

$$\int_0^t (\|\nabla \boldsymbol{E}^k(s)\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k(s)\|^2 ds \leq \frac{C}{\lambda_{k+1}}. \tag{3.18}$$

- *Under the assumptions of Theorem 2.3 then:*

$$\int_0^t (\|\nabla \boldsymbol{E}^k(s)\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k(s)\|^2) ds \leq \frac{C}{\lambda_{k+1}^2}.$$

Proof. Integrating from 0 to t inequality (3.17), recalling that $\|\boldsymbol{E}^k(0)\| = \|\tilde{\boldsymbol{E}}(0)\| = 0$, we obtain:

$$\begin{aligned} &\|\boldsymbol{E}^k(t)\|^2 + \|\tilde{\boldsymbol{E}}^k(t)\|^2 + \int_0^t (\|\nabla \boldsymbol{E}^k(s)\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k(s)\|^2) ds \\ &\leq C \int_0^t \left[\|\boldsymbol{e}^k(s)\|^2 + \|\tilde{\boldsymbol{e}}^k(s)\|^2\right] \phi_1(s) ds \\ &+ C \int_0^t \left[\|\boldsymbol{E}^k(s)\|^2 + \|\tilde{\boldsymbol{E}}^k(s)\|^2\right] \phi_2(s) ds. \end{aligned} \tag{3.19}$$

The estimates are obtained from (3.19) and Lemma 3.1. □

Lemma 3.3. *Under the hypotheses of Theorem 2.3, we have that there exists a constant* $C > 0$ *such that:*

$$\|\nabla \boldsymbol{E}^k(t)\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}}. \tag{3.20}$$

$$\int_0^t \left(\|A\boldsymbol{E}^k(s)\|^2 + \|A\tilde{\boldsymbol{E}}^k(s)\|^2\right) ds \leq \frac{C}{\lambda_{k+1}}. \tag{3.21}$$

Proof. Setting $\boldsymbol{v} = A\boldsymbol{E}^k$ in (3.3), we have:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{E}^k\|^2 &+ \|A\boldsymbol{E}^k\|^2 \\
&= -((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{u}, A\boldsymbol{E}^k) - ((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{u}, A\boldsymbol{E}^k) \\
&- ((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{e}^k, A\boldsymbol{E}^k) - ((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{E}^k, A\boldsymbol{E}^k) \\
&+ ((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{h}, A\boldsymbol{E}^k) + ((\tilde{\boldsymbol{E}}^k\cdot\nabla)\boldsymbol{h}, A\boldsymbol{E}^k) \\
&+ ((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{e}}^k, A\boldsymbol{E}^k) + ((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{E}}^k, A\boldsymbol{E}^k) \\
&= \sum_{i=1}^{8} J_i.
\end{aligned} \tag{3.22}$$

We estimate as follows the rigth–hand side of (3.22):

$$\begin{aligned}
|J_1| &= |((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{u}, A\boldsymbol{E}^k)| \le \|\boldsymbol{e}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|A\boldsymbol{E}^k\| \\
&\le C_\epsilon\|\nabla\boldsymbol{e}^k\|^2\|A\boldsymbol{u}\|^2 + \epsilon\|A\boldsymbol{E}^k\|^2,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
|J_2| &= |((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{u}, A\boldsymbol{E}^k| \le \|\boldsymbol{E}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|A\boldsymbol{E}^k\| \\
&\le C_\epsilon\|\nabla\boldsymbol{E}^k\|^2\|A\boldsymbol{u}\|^2 + \epsilon\|A\boldsymbol{E}^k\|^2,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
|J_3| &= |((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{e}^k, A\boldsymbol{E}^k)| \le \|\boldsymbol{u}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{e}^k\|\|A\boldsymbol{E}^k\| \\
&\le C_\epsilon\|\nabla\boldsymbol{e}^k\|^2\|A\boldsymbol{u}^k\|^2 + \epsilon\|A\boldsymbol{E}^k\|^2,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
|J_4| &= |((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{E}^k, A\boldsymbol{E}^k)| \le \|\boldsymbol{u}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{E}^k\|\|A\boldsymbol{E}^k\| \\
&\le C_\epsilon\|\nabla\boldsymbol{E}^k\|^2\|A\boldsymbol{u}^k\|^2 + \epsilon\|A\boldsymbol{E}^k\|^2,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
|J_5| &= |((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{h}, A\boldsymbol{E}^k)| \le \|\tilde{\boldsymbol{e}}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)}\|A\boldsymbol{E}^k\| \\
&\le C_\epsilon\|\nabla\tilde{\boldsymbol{e}}^k\|^2\|A\boldsymbol{h}\|^2 + \epsilon\|A\boldsymbol{E}^k\|^2,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
|J_6| &= |((\tilde{\boldsymbol{E}}^k\cdot\nabla)\boldsymbol{h}, A\boldsymbol{E}^k)| \le \|\tilde{\boldsymbol{E}}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{h}\|\|A\boldsymbol{E}^k\| \\
&\le C_\epsilon\|\nabla\tilde{\boldsymbol{E}}^k\|^2\|A\boldsymbol{h}\|^2 + \epsilon\|A\boldsymbol{E}^k\|^2.
\end{aligned} \tag{3.28}$$

$$\begin{aligned} |J_7| &= |((\boldsymbol{h}^k \cdot \nabla)\tilde{\boldsymbol{e}}^k, A\boldsymbol{E}^k)| \leq \|\boldsymbol{h}^k\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla \tilde{\boldsymbol{e}}^k\| \|A\boldsymbol{E}^k\| \\ &\leq C_\epsilon \|\nabla \tilde{\boldsymbol{e}}^k\|^2 \|A\boldsymbol{h}^k\|^2 + \epsilon \|A\boldsymbol{E}^k\|^2, \end{aligned} \tag{3.29}$$

$$\begin{aligned} |J_8| &= |((\boldsymbol{h}^k \cdot \nabla)\tilde{\boldsymbol{E}}^k, A\boldsymbol{E}^k)| \leq \|\boldsymbol{h}^k\|_{\boldsymbol{L}^\infty(\Omega)} \|\nabla \tilde{\boldsymbol{E}}^k\| \|A\boldsymbol{E}^k\| \\ &\leq C_\epsilon \|\nabla \tilde{\boldsymbol{E}}^k\|^2 \|A\boldsymbol{h}^k\|^2 + \epsilon \|A\boldsymbol{E}^k\|^2, \end{aligned} \tag{3.30}$$

Inequalities (3.23)–(3.30) together with equality (3.22) imply:

$$\begin{aligned} \frac{d}{dt}\|\nabla \boldsymbol{E}^k\|^2 + \|A\boldsymbol{E}^k\|^2 &\leq C\left(\|\nabla \boldsymbol{e}^k\|^2 + \|\nabla \boldsymbol{E}^k\|^2\right)\psi_1(t) \\ &+ C\left(\|\nabla \tilde{\boldsymbol{e}}^k\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k\|^2\right)\psi_2(t), \end{aligned} \tag{3.31}$$

where

$$\begin{cases} \psi_1(t) &= \|A\boldsymbol{u}(t)\|^2 + \|A\boldsymbol{u}^k(t)\|^2, \\ \psi_2(t) &= \|A\boldsymbol{h}(t)\|^2 + \|A\boldsymbol{h}^k(t)\|^2. \end{cases} \tag{3.32}$$

By using Theorem 2.3, we have $\psi_i(t) \leq M$, for all $i = 1, 2, 3$.

Analogously, we obtain:

$$\begin{aligned} \frac{d}{dt}\|\nabla \tilde{\boldsymbol{E}}^k\|^2 + \|A\tilde{\boldsymbol{E}}^k\|^2 &\leq C\left(\|\nabla \boldsymbol{e}^k\|^2 + \|\nabla \boldsymbol{E}^k\|^2\right)\psi_2(t) \\ &+ C\left(\|\nabla \tilde{\boldsymbol{e}}^k\|^2 + \|\nabla \tilde{\boldsymbol{E}}^k\|^2\right)\psi_1(t), \end{aligned} \tag{3.33}$$

From (3.31) and (3.33), using Lemma 2.4 i) and Corollary 3.2, we obtain (3.20). Integrating (3.33) in time and using (3.20) and Lemma 2.4 i), we obtain (3.21). □

Remark 3.4. Under the hypotheses of Theorem 2.2, estimate (3.20) would also be true using Gronwall's Lemma instead of Corollary 3.2. Indeed, starting from (3.31) and using Lemma 2.4 i), we obtain:

$$\begin{aligned} \frac{d}{dt}\|\nabla \boldsymbol{E}^k\|^2 + \|A\boldsymbol{E}^k\|^2 &\leq C\left(\frac{1}{\lambda_{k+1}} + \|\nabla \boldsymbol{E}^k\|^2\right)\psi_1(t) \\ &+ C\left(\frac{1}{\lambda_{k+1}} + \|\nabla \tilde{\boldsymbol{E}}^k\|^2\right)\psi_2(t), \end{aligned} \tag{3.34}$$

A similar procedure for (3.33) leads to

$$\begin{aligned}\frac{d}{dt}\|\nabla\tilde{\boldsymbol{E}}^k\|^2+\|A\tilde{\boldsymbol{E}}^k\|^2 &\leq C\left(\frac{1}{\lambda_{k+1}}+\|\nabla\boldsymbol{E}^k\|^2\right)\psi_1(t)\\ &+ C\left(\frac{1}{\lambda_{k+1}}+\|\nabla\tilde{\boldsymbol{E}}^k\|^2\right)\psi_2(t),\end{aligned} \tag{3.35}$$

Adding (3.34) to (3.35), we obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\boldsymbol{E}^k\|^2+\|\nabla\tilde{\boldsymbol{E}}^k\|^2\right)+\left(\|A\boldsymbol{E}^k\|^2+\|A\tilde{\boldsymbol{E}}^k\|^2\right)\\ &\leq C\left(\|\nabla\boldsymbol{E}^k\|^2+\|\nabla\tilde{\boldsymbol{E}}^k\|^2\right)(\psi_1(t)+\psi_2(t))\\ &+\frac{C}{\lambda_{k+1}}(\psi_1(t)+\psi_2(t)).\end{aligned} \tag{3.36}$$

Hypotheses on Theorem 2.2 only give that $\psi_1,\psi_2\in L^1(0,T)$. Using Gronwall's Lemma, estimate (3.20) is obtained. Estimated (3.21) can also be obtained integrating (3.36) in time and using (3.20).

Corollary 3.5. *Assume the hypotheses of Lemma 3.3, there exists a positive constant $C>0$ such that:*

$$\int_0^t(\|\boldsymbol{E}_t^k(s)\|^2+\|\tilde{\boldsymbol{E}}_t^k(s)\|^2)ds\leq\frac{C}{\lambda_{k+1}}.$$

Proof. Setting $\boldsymbol{v}=\boldsymbol{E}_t^k$ in $(3.3)_1$, and taking into account that:

$$(A\boldsymbol{E}^k,\boldsymbol{E}_t^k)=-(\Delta\boldsymbol{E}^k,\boldsymbol{E}_t^k)=(\nabla\boldsymbol{E}^k,\nabla\boldsymbol{E}_t^k)=\frac{d}{dt}\|\nabla\boldsymbol{E}^k\|^2,$$

we obtain:

$$\begin{aligned}\|\boldsymbol{E}_t^k\|^2 \;+\; &\frac{d}{dt}\|\nabla\boldsymbol{E}^k\|^2=-((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}_t^k)-((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}_t^k)\\ &-((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{e}^k,\boldsymbol{E}_t^k)-((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{E}^k,\boldsymbol{E}_t^k)\\ &+((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}_t^k)+((\tilde{\boldsymbol{E}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}_t^k)\\ &+((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{e}}^k,\boldsymbol{E}_t^k)+((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{E}}^k,\boldsymbol{E}_t^k)=\sum_{i=1}^8K_i.\end{aligned} \tag{3.37}$$

Now, we estimate the right-hand side of (3.37) as follows:

$$\begin{aligned}
|K_1| &= |-((\boldsymbol{e}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}_t^k)| \le \|\boldsymbol{e}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{u}\|^2\|\nabla\boldsymbol{e}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

$$\begin{aligned}
|K_2| &= |-((\boldsymbol{E}^k\cdot\nabla)\boldsymbol{u},\boldsymbol{E}_t^k)| \le \|\boldsymbol{E}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{u}\|^2\|\nabla\boldsymbol{E}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

$$\begin{aligned}
|K_3| &= |-((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{e}^k,\boldsymbol{E}_t^k)| \le \|\boldsymbol{u}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{e}^k\|\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{u}^k\|^2\|\nabla\boldsymbol{e}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

$$\begin{aligned}
|K_4| &= |-((\boldsymbol{u}^k\cdot\nabla)\boldsymbol{E}^k,\boldsymbol{E}_t^k)| \le \|\boldsymbol{u}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{E}^k\|\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{u}^k\|^2\|\nabla\boldsymbol{E}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

$$\begin{aligned}
|K_5| &= |((\tilde{\boldsymbol{e}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}_t^k)| \le \|\tilde{\boldsymbol{e}}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{h}\|^2\|\nabla\tilde{\boldsymbol{e}}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

$$\begin{aligned}
|K_6| &= |((\tilde{\boldsymbol{E}}^k\cdot\nabla)\boldsymbol{h},\boldsymbol{E}_t^k)| \le \|\tilde{\boldsymbol{E}}^k\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{h}\|^2\|\nabla\tilde{\boldsymbol{E}}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

$$\begin{aligned}
|K_7| &= |((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{e}}^k,\boldsymbol{E}_t^k)| \le \|\boldsymbol{h}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\tilde{\boldsymbol{e}}^k\|\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{h}^k\|^2\|\nabla\tilde{\boldsymbol{e}}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2,
\end{aligned}$$

and

$$\begin{aligned}
|K_8| &= |((\boldsymbol{h}^k\cdot\nabla)\tilde{\boldsymbol{E}}^k,\boldsymbol{E}_t^k)| \le \|\boldsymbol{h}^k\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\tilde{\boldsymbol{E}}^k\|\|\boldsymbol{E}_t^k\| \\
&\le C_\epsilon\|A\boldsymbol{h}^k\|^2\|\nabla\tilde{\boldsymbol{E}}^k\|^2+\epsilon\|\boldsymbol{E}_t^k\|^2.
\end{aligned}$$

Taking $\epsilon>0$ small enough, the above estimates together with (3.37) imply:

$$\begin{aligned}
\|\boldsymbol{E}_t^k\|^2+\frac{d}{dt}\|\nabla\boldsymbol{E}^k\|^2 &\le C\|\nabla\boldsymbol{E}^k\|^2\psi_1(t)+C\|\nabla\boldsymbol{e}^k\|^2\psi_1(t) \\
&+ C\|\nabla\tilde{\boldsymbol{E}}^k\|^2\psi_2(t)+C\|\nabla\tilde{\boldsymbol{e}}^k\|^2\psi_2(t)
\end{aligned} \tag{3.38}$$

where ψ_1 and ψ_2 were defined in (3.32) and are L^1-functions in time thanks to Theorem 2.2.

Setting $\boldsymbol{z} = \tilde{\boldsymbol{E}}_t^k$ in $(3.3)_2$ and reasoning as before, we obtain:

$$\begin{aligned} \|\tilde{\boldsymbol{E}}_t^k\|^2 + \frac{d}{dt}\|\nabla \tilde{\boldsymbol{E}}^k\|^2 &\leq C\|\nabla \tilde{\boldsymbol{E}}^k\|^2\psi_1(t) + C\|\nabla \tilde{\boldsymbol{e}}^k\|^2\psi_1(t) \\ &+ C\|\nabla \boldsymbol{E}^k\|^2\psi_2(t) + C\|\nabla \boldsymbol{e}^k\|^2\psi_2(t) \end{aligned} \tag{3.39}$$

Adding (3.38) to (3.39), using Lemma 2.4 and Gronwall Lemma, we obtain:

$$\int_0^t \left(\|\boldsymbol{E}_t^k(s)\|^2 + \|\tilde{\boldsymbol{E}}_t^k(s)\|^2\right) ds \leq \frac{C}{\lambda_{k+1}}.$$

□

Our first result on error estimates read as follows:

Theorem 3.6. *The following error estimates hold:*

- *Under the assumptions of Theorem 2.2 the approximations* $(\boldsymbol{u}^k, \boldsymbol{h}^k)$ *satisfy:*

$$\begin{aligned} &\|\boldsymbol{u}(t) - \boldsymbol{u}^k(t)\|^2 + \|\boldsymbol{h}(t) - \boldsymbol{h}^k(t)\|^2 \\ &+ \int_0^t \left(\|\nabla \boldsymbol{u}(s) - \nabla \boldsymbol{u}^k(s)\|^2 + \|\nabla \boldsymbol{h}(s) - \nabla \boldsymbol{h}^k(s)\|^2\right) ds \\ &\qquad\qquad \leq \frac{C}{\lambda_{k+1}}. \end{aligned} \tag{3.40}$$

- *Under the assumptions of Theorem 2.3 the approximations* $(\boldsymbol{u}^k, \boldsymbol{h}^k)$ *satisfy:*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^k(t)\|^2 + \|\boldsymbol{h}(t) - \boldsymbol{h}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}^2}. \tag{3.41}$$

Proof. We recall that:

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^k(t)\|^2 \leq C(\|\boldsymbol{e}^k(t)\|^2 + \|\boldsymbol{E}^k(t)\|^2).$$

From Lemma 2.4 i), we obtain:

$$\|\boldsymbol{e}^k(t)\|^2 \leq \frac{1}{\lambda_{k+1}}\|\nabla \boldsymbol{u}(t)\|^2.$$

Since $\boldsymbol{u}_0 \in \boldsymbol{V}$, we have $\|\nabla \boldsymbol{u}(t)\|^2 \leq M$, consequently:

$$\|\boldsymbol{e}^k(t)\|^2 \leq \frac{M}{\lambda_{k+1}}. \tag{3.42}$$

On the other hand, since $\boldsymbol{u}_0 \in \boldsymbol{V}$ from Lemma 3.1 (inequality (3.4)), we get:

$$\|\boldsymbol{E}^k(t)\|^2 + \|\tilde{\boldsymbol{E}}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}}. \tag{3.43}$$

Also, we have:

$$\int_0^t \|\nabla \boldsymbol{u}(s) - \nabla \boldsymbol{u}^k(s)\|^2 ds \leq C \int_0^t (\|\nabla \boldsymbol{e}^k(s)\|^2 + \|\nabla \boldsymbol{E}^k(s)\|^2) ds.$$

Thus, from Lemma 2.4 i), we obtain:

$$\int_0^t \|\nabla \boldsymbol{e}^k(s)\|^2 ds \leq \frac{1}{\lambda_{k+1}} \int_0^t \|A\boldsymbol{u}(s)\|^2 ds.$$

Since $\boldsymbol{u}_0 \in \boldsymbol{V}$, we have $\displaystyle\int_0^t \|A\boldsymbol{u}(s)\|^2 ds \leq M_1$, consequently:

$$\int_0^t \|\nabla \boldsymbol{e}^k(t)\|^2 \leq \frac{M_1}{\lambda_{k+1}}. \tag{3.44}$$

On the other hand, since $\boldsymbol{u}_0 \in \boldsymbol{V}$ from Corollary 3.2 (inequality (3.18)), we get:

$$\int_0^t \|\nabla \boldsymbol{E}^k(s)\|^2 ds \leq \frac{C}{\lambda_{k+1}}. \tag{3.45}$$

Therefore, (3.42)–(3.45) imply the estimate (3.40), for velocity $\boldsymbol{u}$, analogously, we prove the estimate for magnetic field $\boldsymbol{h}$.

Now, we prove the estimate (3.41). In this case, we will do it for the case of the magnetic field, the case of velocity is similar.

We recall that:

$$\|\boldsymbol{h}(t) - \boldsymbol{h}^k(t)\|^2 \leq C(\|\tilde{\boldsymbol{e}}^k(t)\|^2 + \|\tilde{\boldsymbol{E}}^k(t)\|^2).$$

From Lemma 2.4 ii), we obtain:

$$\|\tilde{\boldsymbol{e}}^k(t)\|^2 \leq \frac{1}{\lambda_{k+1}^2} \|A\boldsymbol{h}(t)\|^2.$$

Since $\boldsymbol{h}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, we have $\|A\boldsymbol{h}(t)\|^2 \leq M$, consequently:

$$\|\tilde{\boldsymbol{e}}^k(t)\|^2 \leq \frac{M}{\lambda_{k+1}^2}. \tag{3.46}$$

By other hand, since $\boldsymbol{h}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ from Lemma 3.1 (inequality (3.5)), we get:

$$\|\tilde{\boldsymbol{E}}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}^2}. \tag{3.47}$$

Therefore, (3.46) and (3.47) imply estimate (3.41). This complete the proof. □

Theorem 3.7. *Under the hypotheses of Theorem 2.3, we have that there exists a constant $C > 0$ such that:*

$$\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^k(t)\|^2 + \|\nabla \boldsymbol{h}(t) - \nabla \boldsymbol{h}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}}. \tag{3.48}$$

Proof. We recall that:

$$\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}^k(t)\|^2 \leq C(\|\nabla \boldsymbol{e}^k(t)\|^2 + \|\nabla \boldsymbol{E}^k(t)\|^2).$$

From Lemma 2.4 i), we obtain:

$$\|\nabla \boldsymbol{e}^k(t)\|^2 \leq \frac{1}{\lambda_{k+1}} \|A\boldsymbol{u}(t)\|^2.$$

Since $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, we have $\|A\boldsymbol{u}(t)\|^2 \leq M$, consequently:

$$\|\nabla \boldsymbol{e}^k(t)\|^2 \leq \frac{M}{\lambda_{k+1}}. \tag{3.49}$$

By the other hand, since $\boldsymbol{u}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ from Lemma 3.3, we get:

$$\|\nabla \boldsymbol{E}^k(t)\|^2 \leq \frac{C}{\lambda_{k+1}}. \tag{3.50}$$

Therefore, (3.49) and (3.50) imply the estimate (3.48) for the velocity. The error estimate for the magnetic field appearing in (3.48) is similar. □

Corollary 3.8. *Under the hypothesis of Theorem 2.3, there exists a positive constant $C > 0$ such that:*

$$\int_0^t (\|\boldsymbol{u}_t(s) - \boldsymbol{u}_t^k(s)\|^2 + \|\boldsymbol{h}_t(s) - \boldsymbol{h}_t^k(s)\|^2)\, ds \leq \frac{C}{\lambda_{k+1}} \tag{3.51}$$

and, if $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H}^1(\Omega))$ then:

$$\int_0^t (\|A\boldsymbol{u}(s) - A\boldsymbol{u}^k(s)\|^2 + \|A\boldsymbol{h}(s) - A\boldsymbol{h}^k(s)\|^2)\, ds \leq \frac{C}{\lambda_{k+1}}. \tag{3.52}$$

Proof. We have:

$$\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\|^2 \leq C(\|\boldsymbol{e}_t^k(t)\|^2 + \|\boldsymbol{E}_t^k(t)\|^2). \tag{3.53}$$

From Lemma 2.4, we obtain:

$$\|\boldsymbol{e}_t^k(t)\|^2 \leq \frac{1}{\lambda_{k+1}} \|\nabla \boldsymbol{u}_t(t)\|^2,$$

and consequently:

$$\int_0^t \|\boldsymbol{e}_t^k(s)\|^2 ds \leq \frac{1}{\lambda_{k+1}} \int_0^t \|\nabla \boldsymbol{u}_t(s)\|^2 ds.$$

From Theorem 2.3, we have $\displaystyle\int_0^t \|\nabla \boldsymbol{u}_t(s)\|^2 ds \leq M$, thus:

$$\int_0^t \|\boldsymbol{e}_t^k(s)\|^2 ds \leq \frac{M}{\lambda_{k+1}}.$$

Finally, from Corollary 3.5 we have the same kind of estimate for $\boldsymbol{E}_t^k$. Putting both estimates for $\boldsymbol{e}_t^k$ and $\boldsymbol{E}_t^k$ in (3.53), inequality (3.51) for velocity is obtained. The case of the magnetic field is similar.

Now, we prove (3.52). We have:

$$\|A\boldsymbol{u}(t) - A\boldsymbol{u}^k(t)\|^2 \leq C(\|A\boldsymbol{e}^k(t)\|^2 + \|A\boldsymbol{E}^k(t)\|^2),$$

and consequently:

$$\int_0^t \|A\boldsymbol{u}(s) - A\boldsymbol{u}^k(s)\|^2 ds \leq C \int_0^t (\|A\boldsymbol{e}^k(s)\|^2 ds + \|A\boldsymbol{E}^k(s)\|^2) ds.$$

We observe that:

$$A\boldsymbol{e}^k = (P-P_k)\boldsymbol{f} - (P-P_k)\boldsymbol{u}_t + (P-P_k)((\boldsymbol{h}\cdot\nabla)\boldsymbol{h}) - (P-P_k)((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}).$$

Thus:

$$\begin{aligned}\int_0^t \|A\boldsymbol{e}^k(s)\|^2 ds &\leq C\left(\int_0^t \|(P-P_k)\boldsymbol{f}(s)\|^2\, ds\right.\\ &+ \int_0^t \|(P-P_k)\boldsymbol{u}_t(s)\|^2\, ds\\ &+ \int_0^t \|(P-P_k)((\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s))\|^2 ds\\ &+ \left.\int_0^t \|(P-P_k)((\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{u}(s))\|^2 ds\right),\end{aligned} \tag{3.54}$$

Also for all $\boldsymbol{g}\in\boldsymbol{H}^1(\Omega)$, we have that:

$$\|(P-P_k)\boldsymbol{g}\|^2 \leq \frac{1}{\lambda_{k+1}}\|\boldsymbol{g}\|^2_{\boldsymbol{H}^1(\Omega)}, \tag{3.55}$$

Indeed, note that we have used that $(P-P_k)\boldsymbol{g} = (P-P_k)(P\boldsymbol{g})$ and $P\boldsymbol{g}\in\boldsymbol{V}$ for any $\boldsymbol{g}\in\boldsymbol{H}^1(\Omega)$. Moreover, as $P:\boldsymbol{H}^1(\Omega)\to\boldsymbol{V}$ is a linear and continuous operator (see [30]), then $\|P\boldsymbol{g}\|^2_{\boldsymbol{H}^1(\Omega)} \leq \|\boldsymbol{g}\|^2_{\boldsymbol{H}^1(\Omega)}$.

Applying (3.55) for $\boldsymbol{g}=\boldsymbol{f}$, since $\boldsymbol{f}\in L^2(0,T;\boldsymbol{H}^1(\Omega))$, we have:

$$\int_0^t \|(P-P_k)\boldsymbol{f}(s)\|^2 \leq \frac{1}{\lambda_{k+1}}\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \leq \frac{C}{\lambda_{k+1}}.$$

And due to $\boldsymbol{u}_t\in L^2([0,T];\boldsymbol{V})$ (see Theorem 2.3 and Lemma 2.4 *i*) for $\boldsymbol{w}=\boldsymbol{u}_t$), we deduce:

$$\int_0^t \|(P-P_k)\boldsymbol{u}_t\|^2 ds \leq \frac{C}{\lambda_{k+1}},$$

In the same way,

$$\int_0^t \|(P-P_k)((\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s))\|^2 ds \leq \frac{1}{\lambda_{k+1}}\int_0^t \|((\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s))\|^2_{\boldsymbol{H}^1(\Omega)} ds,$$

but:

$$\|((\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s))\|_{\boldsymbol{H}^1(\Omega)} \leq C\|A\boldsymbol{h}(s)\|^2.$$

As a consequence of using Theorem 2.3, we have:

$$\int_0^t \|(P-P_k)((\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s))\|^2 ds \leq \frac{C}{\lambda_{k+1}},$$

and, similarly,

$$\int_0^t \|(P-P_k)((\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{u}(s))\|^2 ds \leq \frac{C}{\lambda_{k+1}}.$$

Therefore:

$$\int_0^t \|A\boldsymbol{e}^k(s)\|^2 de \leq \frac{C}{\lambda_{k+1}}. \tag{3.56}$$

Finally, from (3.56) and (3.21) in Lemma 3.3, we have the desired result. □

4 H^2–error estimates for the velocity and the magnetic field

We recall that, using (3.1)-(3.2), we have been using the following decompositions of the errors:

$$\boldsymbol{u}(t)-\boldsymbol{u}^k(t)=\boldsymbol{e}^k(t)+\boldsymbol{E}^k(t), \quad \text{and} \quad \boldsymbol{h}(t)-\boldsymbol{h}^k(t)=\tilde{\boldsymbol{e}}^k(t)+\tilde{\boldsymbol{E}}^k(t).$$

where $\boldsymbol{u}^k$ and $\boldsymbol{h}^k$ are the kth Galerkin approximations of $\boldsymbol{u}$ and $\boldsymbol{h}$, respectively.

With these notations, in order to estimate $A\boldsymbol{u}-A\boldsymbol{u}^k$ and $A\boldsymbol{h}-A\boldsymbol{h}^k$, we need to estimate $A\boldsymbol{e}^k$ and $A\boldsymbol{E}^k$ and $A\tilde{\boldsymbol{e}}^k$ and $A\tilde{\boldsymbol{E}}^k$. As a first step, we estimate $A^\alpha\boldsymbol{e}^k$ and $A^\alpha\boldsymbol{E}^k$ and $A^\alpha\tilde{\boldsymbol{e}}^k$ and $A^\alpha\tilde{\boldsymbol{E}}^k$ for $\alpha\in[0,1)$.

4.1 Some properties of the Stokes operator

In this subsection, we remember some results that will be essential to achieve our goals. The fractional powers A^α with domain of definition

$D(A^\alpha) \subset \boldsymbol{H}$ are defined for any real α by means of the spectral representation of A (see for instance [9]). For $\alpha < \beta$ the imbedding $D(A^\beta) \hookrightarrow D(A^\alpha)$ is compact and $D(A^\beta)$ is dense in $D(A^\alpha)$. Therefore, A is a sectorial operator and A is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}$. On $D(A^\alpha)$, the operator A^α commute, with e^{-tA}. Moreover, the following properties are satisfied (see [8]):

$$\|A^\alpha e^{-t\mathcal{A}}\| \leq t^{-\alpha} \quad \text{with} \quad t > 0,\ 0 \leq \alpha \leq \boldsymbol{e}. \tag{4.1}$$

$$\begin{aligned} \|A^{\alpha+\beta} e^{-tA}\boldsymbol{v}\| = \|A^\alpha e^{-tA} A^\beta v\| \leq t^{-\alpha}\|A^\beta \boldsymbol{v}\| \quad &\text{with} \quad \boldsymbol{v} \in D(A^\beta), \\ &t > 0,\ 0 \leq \alpha \leq \boldsymbol{e}. \end{aligned} \tag{4.2}$$

$$\|(e^{-tA} - I)\boldsymbol{v}\| \leq \frac{t^\sigma}{\sigma}\|A^\sigma \boldsymbol{v}\| \ \text{ with } \ v \in D(A^\sigma),\ t > 0,\ 0 < \sigma < 1. \tag{4.3}$$

$$\|(e^{-tA} - I)\boldsymbol{v}\| \longrightarrow 0^+ \ \text{ when } t \to 0, \quad v \in \boldsymbol{H}. \tag{4.4}$$

The following three theorems will be used in the arguments of the next sections.

Lemma 4.1. *(See [21], Corollary 3.4)*
Assume $\boldsymbol{u}_i \in D(A^{3/4+\eta})$ for some $\eta > 0$ and $\boldsymbol{v}_i \in D(A)$ for $i = 1, 2$. Then:

$$\begin{aligned} \|A^\zeta P(\boldsymbol{u}_2 \nabla \boldsymbol{v}_2 - \boldsymbol{u}_1 \nabla \boldsymbol{v}_1)\| \ &\leq \ C\|A^{3/4+\eta}(\boldsymbol{u}_2 - \boldsymbol{u}_1)\|\|A\boldsymbol{v}_2\| \\ &+ \ C\|A^{3/4+\eta}\boldsymbol{u}_1\|\|A(\boldsymbol{v}_2 - \boldsymbol{v}_1)\| \end{aligned}$$

holds for all $\zeta \in [0, 1/4)$, the constant C depending only on η and ζ.

Lemma 4.2. *(See [15], Lemma 6.5)*
Let T, α, and β be positive constants and let r be a constant with $0 < r < 1$. Then, any continuous positive function f defined for $t \in [0, T]$, satisfying:

$$f(t) \leq \alpha + \beta \int_0^t (t - s)^{-r} f(s) ds,$$

verifies:

$$f(t) \leq C\alpha e^{C\beta^{1/(1-r)}t},$$

with a positive constant C which depends only on r.

Lemma 4.3. *(See [12], pag 38)*
Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ defining the norm $|\cdot|_{\mathcal{H}}$. Let A^* be a symmetric operator which has the complete orthonormal system of eigenfunctions (e_i^*) corresponding to the sequence (λ_i^*) of eigenvalues $0<\lambda_1^*\leq\lambda_2^*\leq\ldots\leq\lambda_i^*\to\infty$ with $i\to\infty$. Then, the error estimate:

$$|f-\sum_{i=1}^{k}\langle f,e_i^*\rangle e_i^*|_{\mathcal{H}}\leq(\lambda_{k+1}^*)^{-1}|A^*f|_{\mathcal{H}}$$

holds for any $f\in D(A^*)$.

Now, we prove some regularity results for the solution obtained in Theorems 2.2 and 2.3, which are necessary to obtain our results . Firstly, observe that we can write the following representation of the solution obtained in Theorem 2.2 as follows:

$$\begin{aligned}
\boldsymbol{u}(t)&=e^{-At}\boldsymbol{u}_0\\
&\quad+\int_0^t e^{-(t-s)A}P\Big[\boldsymbol{f}-(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{u}(s)+(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s)\Big]\,ds,\\
\boldsymbol{h}(t)&=e^{-At}\boldsymbol{h}_0\\
&\quad+\int_0^t e^{-(t-s)A}\,P\Big[-(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{h}(s)+(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)\Big]\,ds.
\end{aligned}\tag{4.5}$$

Theorem 4.4. *Suppose that $\boldsymbol{f}\in C([0,T],\boldsymbol{H}^1(\Omega))$ and $\boldsymbol{u}_0,\boldsymbol{h}_0\in D(A^{1+\epsilon})$, then the solution $(\boldsymbol{u},\boldsymbol{h})$ of (1.1) and (1.2) satisfies for $0\leq\epsilon<1/4$,*

$$\boldsymbol{u},\,\boldsymbol{h}\in C([0,T];D(A^{1+\epsilon}))\cap C^1([0,T];D(A^{\epsilon})).$$

Proof. The proof is similar to that Bause [1] (Theorem 3.11). In fact, it is exactly equal in the case of the velocity $\boldsymbol{u}(t)$. For the treatment of $\boldsymbol{h}(t)$, we sketch the main ideas following [1]:

Step 1: If $0\leq\epsilon<1/4$, then $D(A^{1+\epsilon})=\boldsymbol{H}^{2+2\epsilon}(\Omega)\cap\boldsymbol{H}_0^1(\Omega)$.

Step 2: If $0\leq\epsilon<1/4$, then $(\boldsymbol{u}\cdot\nabla)\boldsymbol{h}\in C([0,T];D(A^{\epsilon}))$ and $(\boldsymbol{h}\cdot\nabla)\boldsymbol{u}\in C([0,T];D(A^{\epsilon}))$.

Indeed, $\boldsymbol{u} \in D(A)$ and $\boldsymbol{h} \in D(A)$ imply $(\boldsymbol{u}\cdot\nabla)\boldsymbol{h} \in \boldsymbol{H}^1(\Omega)$, using the Lemmata 4.1, 4.2 and 4.3 we have:

$$\begin{aligned}
\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{h}\|_{\boldsymbol{H}^1(\Omega)} &\leq \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{h}\| + \|(\nabla\boldsymbol{u}\cdot\nabla)\boldsymbol{h}\| + \|(\boldsymbol{u}\cdot\nabla)\nabla\boldsymbol{h}\| \\
&\leq C\Big(\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{h}\| + \|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla\boldsymbol{h}\|_{\boldsymbol{L}^4(\Omega)} \\
&+ \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\boldsymbol{h}\|_{\boldsymbol{H}^2(\Omega)}\Big) \\
&\leq C\|A\boldsymbol{u}\|\|A\boldsymbol{h}\|.
\end{aligned}$$

Moreover, since $\boldsymbol{H}^1(\Omega) \to \boldsymbol{H}^{2\epsilon}(\Omega)$ is continuous, we get that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{h} \in D(A^\epsilon)$, $0 \leq \epsilon < 1/4$.

With respect to the continuity of term $(\boldsymbol{u}\cdot\nabla\boldsymbol{h})(t)$, observe that:

$$\begin{aligned}
&\|A^\epsilon(\boldsymbol{u}\cdot\nabla\boldsymbol{h})(t) - A^\epsilon(\boldsymbol{u}\cdot\nabla\boldsymbol{h})(s)\| \\
&\quad\leq C\Big(\|(\boldsymbol{u}(t)-\boldsymbol{u}(s))\cdot\nabla\boldsymbol{h}(t)\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{u}(s)\cdot\nabla(\boldsymbol{h}(t)-\boldsymbol{h}(s))\|_{\boldsymbol{H}^1(\Omega)}\Big) \\
&\quad\leq C\Big(\|A\boldsymbol{h}(t)\|\|A(\boldsymbol{u}(t)-\boldsymbol{u}(s))\| + \|A\boldsymbol{u}(s)\|\|A(\boldsymbol{h}(t)-\boldsymbol{h}(s))\|\Big).
\end{aligned}$$

Note that hypotheses for the data guarantee that Theorem 2.3 holds and thus $\boldsymbol{u}$, $\boldsymbol{h} \in C([0,T];D(A))$. Then, statement of Step 2 for $(\boldsymbol{u}\cdot\nabla)\boldsymbol{h}$ is true. The result for $(\boldsymbol{h}\cdot\nabla)\boldsymbol{u}$ is analogous.

Step 3: If $0 \leq \epsilon < 1/4$ then $\boldsymbol{h} \in C([0,T];D(A^{1+\epsilon}))$.

By applying the operator $A^{1+\epsilon}$ at both sides of the integral equation of $\boldsymbol{h}(t)$ (4.5), we have:

$$\begin{aligned}
A^{1+\epsilon}\boldsymbol{h}(t) &= e^{-tA}A^{1+\epsilon}\boldsymbol{h}_0 \\
&+ \int_0^t A^\beta e^{-(t-s)A}A^\sigma\Big[-(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{h}(s) + (\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)\Big]\,ds
\end{aligned} \tag{4.6}$$

where $\beta \in (0,1)$, $\sigma \in (0,1/4)$ such that $\beta+\sigma = 1+\epsilon$. From (4.1), we obtain:

$$\begin{aligned}
&\|A^\beta e^{-(t-s)A}A^\sigma(-(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{h}(s) + (\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s))\| \\
&\quad\leq \frac{1}{(t-s)^\beta}\|A^\sigma(-\boldsymbol{u}(s)\cdot\nabla\boldsymbol{h}(s) + (\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s))\|
\end{aligned} \tag{4.7}$$

Using that $\|e^{-tA}A^{1+\epsilon}\boldsymbol{h}_0\| \leq \|A^{1+\epsilon}\boldsymbol{h}_0\| \leq \|\boldsymbol{h}_0\|_{\boldsymbol{H}^{2+2\epsilon}}$ and Step 2, we deduce that $\boldsymbol{h}(t) \in D(A^{1+\epsilon})$ for $t \in [0,T]$.

To prove the continuity, the analysis is similar to that one made in Step 2: we analyze the L^2-bounds for $A^{1+\epsilon}\boldsymbol{h}(t+\tau) - A^{1+\epsilon}\boldsymbol{h}(t)$, $\tau \in \mathbb{R}$ getting, for example, for $\tau > 0$. Starting from (4.6), and using (4.7), (4.2) and (4.3):

$$\begin{aligned}
&\|A^{1+\epsilon}\boldsymbol{h}(t+\tau) - A^{1+\epsilon}\boldsymbol{h}(t)\| \leq \|(e^{-\tau A} - I)e^{-tA}A^{1+\epsilon}\boldsymbol{h}_0\| \\
&\quad + \int_0^t (t-s)^{-\beta}\frac{\tau^\delta}{\delta}(\|(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{h}(s)\|_{\boldsymbol{H}^{2(\delta+\sigma)}} + \|(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)\|_{\boldsymbol{H}^{2(\delta+\sigma)}})ds \\
&\quad + \int_t^{t+\tau} (t+\tau-s)^{-\beta}(\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{h}(s)\|_{\boldsymbol{H}^{2\sigma}} + \|(\boldsymbol{h}\cdot\nabla)\boldsymbol{u}(s)\|_{\boldsymbol{H}^{2\sigma}})ds \\
&\quad \leq \|(e^{-\tau A} - I)e^{-tA}A^{1+\epsilon}\boldsymbol{h}_0\| + Ct^{1-\beta}\tau^\delta + C\tau^{1-\beta},
\end{aligned}$$

here $\delta > 0$ is such that $\delta + \sigma < 1/4$ and σ as above. The case $\tau < 0$ can be dealt with in a similar way. Taking in to account (4.4), it follows that $\boldsymbol{h} \in C([0,T]; D(A^{1+\epsilon}))$.

Step 4: By using the evolution equations (1.1), we obtain that $\partial_t\boldsymbol{u}$, $\partial_t\boldsymbol{h} \in C([0,T]; D(A^\epsilon))$. □

4.2 Estimates in $D(A^\alpha)$ for $0 \leq \alpha < 1$.

Lemma 4.5. *Consider the initial data $(\boldsymbol{u}_0, \boldsymbol{h}_0) \in D(A) \times D(A)$ and let α be such that $0 \leq \alpha < 1$. Under the hypotheses of Theorem 2.3, the estimates:*

$$\begin{aligned}
\|A^\alpha\boldsymbol{e}^k(t)\| &= \|A^\alpha\boldsymbol{u}(t) - A^\alpha\boldsymbol{v}^k(t)\| \leq \frac{C(\alpha+\epsilon)}{\lambda_{k+1}^\epsilon}, \\
\|A^\alpha\tilde{\boldsymbol{e}}^k(t)\| &= \|A^\alpha\boldsymbol{h}(t) - A^\alpha\boldsymbol{b}^k(t)\| \leq \frac{C(\alpha+\epsilon)}{\lambda_{k+1}^\epsilon},
\end{aligned}$$

hold for any $\epsilon > 0$ such that $0 < \alpha + \epsilon < 1$.

Proof. Observe that the operator A^ϵ commute with P_k for any $\epsilon \in (0,1)$. Since A^ϵ is again positive definite symmetric operator in $\boldsymbol{H}$, having the eigenvalues λ_k^ϵ and the eigenfunctions $\boldsymbol{w}_k$, $k = 1, 2, ...$, we can apply Lemma 4.3 for $\boldsymbol{f} = A^\alpha \boldsymbol{u}$, $\mathcal{A}^* = A^\epsilon$ to obtain the above estimates. □

Applying the projection operator P_k to the equation (1.1) and taking into account that this operator commutes with ∂_t and A, we obtain the system of evolution equations:

$$\begin{cases} (\partial_t + A)P_k\boldsymbol{u} &= P_k[\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{h}\cdot\nabla)\boldsymbol{h}] \\ (\partial_t + A)P_k\boldsymbol{h} &= P_k[(\boldsymbol{h}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{h}\cdot\nabla)\boldsymbol{u}] \\ P_k\boldsymbol{u}(0) &= P_k\boldsymbol{u}_0 \\ P_k\boldsymbol{h}(0) &= P_k\boldsymbol{h}_0 \end{cases}$$

Noting that the projection P_k being orthogonal in $\boldsymbol{H}$, the estimate in Fujita-Kato ([8], Lemma 1.2) holds with P_k instead P. This is due to the properties of the exponential function and projectors P and P_k (see [18, 19] for a precise definition and properties of such operators). This $P_k\boldsymbol{u}$ is a unique solution of the integral equation:

$$\begin{aligned} \boldsymbol{v}^k(t) = P_k\boldsymbol{u}(t) &= e^{-tA}P_k\boldsymbol{u}_0 \\ &+ \int_0^t e^{-(t-s)A}P_k\Big[\boldsymbol{f} - (\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{u}(s) + (\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s)\Big]ds. \end{aligned} \tag{4.8}$$

Analogously, we have:

$$\begin{aligned} \boldsymbol{b}^k(t) = P_k\boldsymbol{h}(t) &= e^{-tA}P_k\boldsymbol{h}_0 \\ &+ \int_0^t e^{-(t-s)A}P_k\Big[(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s) - (\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)\Big]\,ds. \end{aligned} \tag{4.9}$$

In order to obtain estimates for $A^\alpha(\boldsymbol{u} - \boldsymbol{u}^k)$ and $A^\alpha(\boldsymbol{h} - \boldsymbol{h}^k)$, and after getting the results of Lemma 4.5, we focus on the estimates of $A^\alpha\boldsymbol{E}^k$ and $A^\alpha\tilde{\boldsymbol{E}}^k$. This is what is made in the proof of the following result:

Theorem 4.6. *Consider the initial data* $(\boldsymbol{u}_0, \boldsymbol{h}_0) \in D(A) \times D(A)$. *Let* α *be such that* $0 \leq \alpha < 1$, *under the hypotheses of Theorem 2.3, the estimates:*

$$\begin{aligned} \|A^\alpha \boldsymbol{u}(t) - A^\alpha \boldsymbol{u}^k(t)\| &\leq \frac{C(\alpha+\epsilon)}{\lambda_{k+1}^\epsilon} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}, \\ \|A^\alpha \boldsymbol{h}(t) - A^\alpha \boldsymbol{h}^k(t)\| &\leq \frac{C(\alpha+\epsilon)}{\lambda_{k+1}^\epsilon} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2} \end{aligned}$$

hold for any $\epsilon > 0$ *such that* $0 < \alpha + \epsilon < 1$.

Proof. Note that from integral equations, we can write:

$$\begin{aligned} \boldsymbol{E}^k(t) = &\int_0^t e^{-(t-s)A}[-P_k(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{u}(s) + P_k(\boldsymbol{u}^k(s)\cdot\nabla)\boldsymbol{u}^k(s)]ds \\ &- \int_0^t e^{-(t-s)A}[-P_k(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{h}(s) + P_k(\boldsymbol{h}^k(s)\cdot\nabla)\boldsymbol{h}^k(s)]ds. \end{aligned} \tag{4.10}$$

Consequently,

$$\begin{aligned} \|A^\alpha \boldsymbol{E}^k(t)\| \leq &\int_0^t \|A^\alpha e^{-(t-s)A} P_k(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^k(s)\cdot\nabla\boldsymbol{u}^k(s))\|ds \\ + &\int_0^t \|A^\alpha e^{-(t-s)A} P_k(\boldsymbol{h}(s)\cdot\nabla\boldsymbol{h}(s) - \boldsymbol{h}^k(s)\cdot\nabla\boldsymbol{h}^k(s))\|ds \\ \leq &\int_0^t \|A^\alpha e^{-(t-s)A}\|\|P_k(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^k(s)\cdot\nabla\boldsymbol{u}^k(s))\|ds \\ + &\int_0^t \|A^\alpha e^{-(t-s)A}\|\|P_k(\boldsymbol{h}(s)\cdot\nabla\boldsymbol{h}(s) - \boldsymbol{h}^k(s)\cdot\nabla\boldsymbol{h}^k(s))\|ds \\ = &\; L_1 + L_2. \end{aligned}$$

By using (4.1), we have for all $0 < \alpha \leq e$ and $t > 0$,

$$\|A^\alpha e^{-(t-s)A}\| \leq \frac{1}{(t-s)^\alpha}. \tag{4.11}$$

Observe, moreover, that using Theorem 2.3 and Theorem 3.7:

$$\begin{aligned}
&\|P_k((\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{u}(s)-(\boldsymbol{u}^k(s)\cdot\nabla)\boldsymbol{u}^k(s))\| \\
&\quad\le \|(\boldsymbol{u}(s)-\boldsymbol{u}^k(s))\cdot\nabla\boldsymbol{u}(s)\|+\|\boldsymbol{u}^k(s)\cdot\nabla(\boldsymbol{u}(s)-\boldsymbol{u}^k(s))\| \\
&\quad\le \|\boldsymbol{u}(s)-\boldsymbol{u}^k(s)\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{u}(s)\|_{\boldsymbol{L}^6(\Omega)} \\
&\qquad +\|\boldsymbol{u}(s)\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla(\boldsymbol{u}(s)-\boldsymbol{u}^k(s)\| \\
&\quad\le C\|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}^k(s)\|\|A\boldsymbol{u}(s)\| \\
&\qquad +\|A\boldsymbol{u}^k(s)\|\|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}^k(s)\| \\
&\quad\le C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}.
\end{aligned} \tag{4.12}$$

Consequently, combining (4.11) and (4.12), for all $0<\alpha<1$, we obtain:

$$L_1 \le C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}\frac{T^{1-\alpha}}{1-\alpha}.$$

The bound for L_2 can be obtained similarly. Thus, for all $0<\alpha<1$, we have:

$$\|A^\alpha \boldsymbol{E}^k(t)\| \le C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2},$$

where C depend on $T,\alpha,\partial\Omega$.

The error estimate for the magnetic filed $\|A^\alpha\tilde{\boldsymbol{E}}^k\|$ follows the same argument, pointing out that $\boldsymbol{b}^k=P_k\boldsymbol{h}$ is the unique solution of the integral equation:

$$\boldsymbol{b}^k(t)=e^{-tA}P_k\boldsymbol{h}_0+\int_0^t e^{-(t-s)A}P_k[(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)-(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)]ds,$$

and that $\tilde{\boldsymbol{E}}^k$ can be written as:

$$\begin{aligned}
\tilde{\boldsymbol{E}}^k(t)=&\int_0^t e^{-(t-s)A}[-P_k(\boldsymbol{u}(s)\cdot\nabla)\boldsymbol{h}(s)+P_k(\boldsymbol{u}^k(s)\cdot\nabla)\boldsymbol{h}^k(s)]ds \\
&+\int_0^t e^{-(t-s)A}[-P_k(\boldsymbol{h}(s)\cdot\nabla)\boldsymbol{u}(s)+P_k(\boldsymbol{h}^k(s)\cdot\nabla)\boldsymbol{u}^k(s)]ds,
\end{aligned} \tag{4.13}$$

The previous estimates together with Lemma 4.5 imply that:

$$\begin{aligned}
\|A^\alpha\boldsymbol{u}(t)-A^\alpha\boldsymbol{u}^k(t)\| &\le \|A^\alpha\boldsymbol{e}^k(t)\|+\|A^\alpha\boldsymbol{E}^k(t)\| \\
&\le \frac{C(\alpha+\epsilon)}{\lambda_{k+1}^\epsilon}+C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}
\end{aligned}$$

Estimate for $\|A^\alpha \boldsymbol{h}(t) - A^\alpha \boldsymbol{h}^k(t)\|$ is immediate. □

4.3 Estimates in $D(A)$

Once estimates for $A^\alpha(\boldsymbol{u} - \boldsymbol{u}^k)$ and $A^\alpha(\boldsymbol{h} - \boldsymbol{h}^k)$, for $0 \leq \alpha < 1$, have been obtained, we focus on the obtention of estimates for $A(\boldsymbol{u} - \boldsymbol{u}^k)$ and $A(\boldsymbol{h} - \boldsymbol{h}^k)$. As made in Subsection 4.2, first we do estimates for $A\boldsymbol{e}^k$ and $A\tilde{\boldsymbol{e}}^k$, and later estimates for $A\boldsymbol{E}^k$ and $A\tilde{\boldsymbol{E}}^k$.

Lemma 4.7. *Under the hypotheses of Theorem 4.4 the following estimates are true:*

$$\|A\boldsymbol{e}^k(t)\| = \|A\boldsymbol{u}(t) - A\boldsymbol{v}^k(t)\| \leq \frac{C}{\lambda_{k+1}^\epsilon}, \tag{4.14}$$

$$\|A\tilde{\boldsymbol{e}}^k(t)\| = \|A\boldsymbol{h}(t) - A\boldsymbol{b}^k(t)\| \leq \frac{C}{\lambda_{k+1}^\epsilon}, \tag{4.15}$$

for $0 \leq \epsilon < 1/4$.

Proof. By using Lemma 4.3 for $\boldsymbol{f} = A\boldsymbol{u}$ and $\mathcal{A} = A^\epsilon$, we obtain:

$$\|A\boldsymbol{u}(t) - A\boldsymbol{v}^k(t)\| = \|A(I - P_k)\boldsymbol{u}(t)\| = \|(I - P_k)A\boldsymbol{u}(t)\| \leq \frac{1}{\lambda_{k+1}^\epsilon}\|A^{1+\epsilon}\boldsymbol{u}(t)\|.$$

The hypothesis of Theorem 4.4 guarantee the boundedness of $\|A^{1+\epsilon}\boldsymbol{u}(t)\|$ that implies (4.14). The proof (4.15) is analogous. □

Lemma 4.8. *Under the conditions of Theorem 4.4, the following estimates are satisfied for* $0 < \epsilon < 1/4$*:*

$$\|A\boldsymbol{E}^k(t)\| \leq \frac{C(\frac{3}{4} + \epsilon)}{\lambda_{k+1}^\epsilon} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2} \tag{4.16}$$

$$\|A\tilde{\boldsymbol{E}}^k(t)\| \leq \frac{C(\frac{3}{4} + \epsilon)}{\lambda_{k+1}^\epsilon} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}. \tag{4.17}$$

Proof. From (4.10), we have for any $\epsilon \in (0, 1/4)$:

$$\begin{aligned}
\|A\boldsymbol{E}^k(t)\| &\leq \int_0^t \|A^{1-\epsilon}e^{-(t-s)A}A^\epsilon P_k(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^k(s)\cdot\nabla\boldsymbol{u}^k(s))\|ds \\
&+ \int_0^t \|A^{1-\epsilon}e^{-(t-s)A}A^\epsilon P_k(\boldsymbol{h}(s)\cdot\nabla\boldsymbol{h}(s) - \boldsymbol{h}^k(s)\cdot\nabla\boldsymbol{h}^k(s))\|ds \\
&\leq \int_0^t \|A^{1-\epsilon}e^{-(t-s)A}\|\|A^\epsilon P_k(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^k(s)\cdot\nabla\boldsymbol{u}^k(s))\|ds \\
&+ \int_0^t \|A^{1-\epsilon}e^{-(t-s)A}\|\|A^\epsilon P_k(\boldsymbol{h}(s)\cdot\nabla\boldsymbol{h}(s) - \boldsymbol{h}^k(s)\cdot\nabla\boldsymbol{h}^k(s))\|ds.
\end{aligned} \tag{4.18}$$

By using Lemma 4.1 with $\zeta = \eta = \epsilon$, we obtain:

$$\begin{aligned}
&\|A^\epsilon(P_k[(\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}(s) - \boldsymbol{u}^k(s)\cdot\nabla\boldsymbol{u}^k(s))]\| \\
&\quad\leq C\|A^{3/4+\epsilon}(\boldsymbol{u}(s) - \boldsymbol{u}^k(s))\|\|A\boldsymbol{u}(s)\| \\
&\quad+C\|A^{3/4+\epsilon}\boldsymbol{u}^k(s)\|\|A(\boldsymbol{u}(s) - \boldsymbol{u}^k(s))\|.
\end{aligned} \tag{4.19}$$

Analogously, we have:

$$\begin{aligned}
&\|A^\epsilon(P_k[(\boldsymbol{h}(s)\cdot\nabla\boldsymbol{h}(s) - \boldsymbol{h}^k(s)\cdot\nabla\boldsymbol{h}^k(s))]\| \\
&\quad\leq C\|A^{3/4+\epsilon}(\boldsymbol{h}(s) - \boldsymbol{h}^k(s))\|\|A\boldsymbol{h}(s)\| \\
&\quad+C\|A^{3/4+\epsilon}\boldsymbol{h}^k(s)\|\|A(\boldsymbol{h}(s) - \boldsymbol{h}^k(s))\|.
\end{aligned} \tag{4.20}$$

Thanks to Theorem 4.4, $\|A\boldsymbol{h}(s)\|$ and $\|A^{3/4+\epsilon}\boldsymbol{h}^k(s)\|$ are bounded for $s \in [0, t]$. From (4.18), by using (4.19), (4.20) and (4.11), we get:

$$\begin{aligned}
\|A\boldsymbol{E}^k(t)\| &\leq \int_0^t \frac{C}{(t-s)^{1-\epsilon}}(\|A^{3/4+\epsilon}(\boldsymbol{u}(s) - \boldsymbol{u}^k(s))\| + \|A(\boldsymbol{u}(s) - \boldsymbol{u}^k(s))\|)ds \\
&+ \int_0^t \frac{1}{(t-s)^{1-\epsilon}}C(\|A^{3/4+\epsilon}(\boldsymbol{h}(s) - \boldsymbol{h}^k(s))\| + \|A(\boldsymbol{h}(s) - \boldsymbol{h}^k(s))\|)ds.
\end{aligned}$$

From Theorem 3.7 and Theorem 4.6, we obtain:

$$\begin{aligned}\|A\boldsymbol{E}^k(t)\| \leq & \left[\frac{C(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \left(\frac{1}{\lambda_{k+1}}\right)^{1/2}\right]\int_0^t \frac{1}{(t-s)^{1-\epsilon}}ds \\ & + \int_0^t \frac{1}{(t-s)^{1-\epsilon}}(\|A\boldsymbol{e}^k(s)\| + \|A\boldsymbol{E}^k(s)\|)ds \\ & + \int_0^t \frac{1}{(t-s)^{1-\epsilon}}(\|A\tilde{\boldsymbol{e}}^k(s)\| + \|A\tilde{\boldsymbol{E}}^k(s)\|)ds.\end{aligned}$$

By using Lemma 4.7 and $\displaystyle\int_0^t \frac{1}{(t-s)^{1-\epsilon}}ds = \frac{t^{\epsilon}}{\epsilon}$, we have:

$$\int_0^t \frac{1}{(t-s)^{1-\epsilon}}\left(\|A\boldsymbol{e}^k(s)\| + \|A\tilde{\boldsymbol{e}}^k(s)\|\right)ds \leq \frac{C}{\lambda_{k+1}^{\epsilon}}.$$

Therefore, we obtain:

$$\begin{aligned}\|A\boldsymbol{E}^k(t)\| \quad \leq \quad & \frac{C(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2} \\ + \quad & \int_0^t \frac{1}{(t-s)^{1-\epsilon}}\left(\|A\boldsymbol{E}^k(s)\| + \|A\tilde{\boldsymbol{E}}^k(s)\|\right)ds.\end{aligned} \tag{4.21}$$

In the same sense, starting by (4.13),

$$\begin{aligned}\|A\tilde{\boldsymbol{E}}^k(t)\| \quad \leq \quad & \frac{C(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2} \\ + \quad & \int_0^t \frac{1}{(t-s)^{1-\epsilon}}\left(\|A\boldsymbol{E}^k(s)\| + \|A\tilde{\boldsymbol{E}}^k(s)\|\right)ds.\end{aligned} \tag{4.22}$$

Adding (4.21) and (4.22), and according to Lemma 4.2, we obtain:

$$\|A\boldsymbol{E}^k(t)\| + \|A\tilde{\boldsymbol{E}}^k(t)\| \leq \frac{C(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2},$$

for a C depending on T, that, in particular, implies (4.16) and (4.17). □

Theorem 4.9. *Under the assumptions of Theorem 2.3, if moreover* $\boldsymbol{f} \in C([0,T], \boldsymbol{H}^1(\Omega))$ *and* $\boldsymbol{u}_0, \boldsymbol{h}_0 \in D(A^{1+\epsilon})$*, with* $\epsilon \in (0, \frac{1}{4})$*, then*

$$\|A\boldsymbol{u}(t) - A\boldsymbol{u}^k(t)\| + \|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\| \leq C\left[\frac{(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \left(\frac{1}{\lambda_{k+1}}\right)^{1/2}\right]$$
$$\|A\boldsymbol{h}(t) - A\boldsymbol{h}^k(t)\| + \|\boldsymbol{h}_t(t) - \boldsymbol{h}_t^k(t)\| \leq C\left[\frac{(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \left(\frac{1}{\lambda_{k+1}}\right)^{1/2}\right] \tag{4.23}$$

Proof. By applying Lemmata 4.7 and 4.8 to the following splitting:

$$\begin{aligned} \|A\boldsymbol{u}(t) - A\boldsymbol{u}^k(t)\| &\leq \|A\boldsymbol{e}^k(t)\| + \|A\boldsymbol{E}^k(t)\| \\ \|A\boldsymbol{h}(t) - A\boldsymbol{h}^k(t)\| &\leq \|A\tilde{\boldsymbol{e}}^k(t)\| + \|A\tilde{\boldsymbol{E}}^k(t)\|, \end{aligned}$$

we conclude:

$$\|A\boldsymbol{u}(t) - A\boldsymbol{u}^k(t)\| \leq C\left[\frac{(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \left(\frac{1}{\lambda_{k+1}}\right)^{1/2}\right] \tag{4.24}$$

and

$$\|A\boldsymbol{h}(t) - A\boldsymbol{h}^k(t)\| \leq C\left[\frac{(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + \left(\frac{1}{\lambda_{k+1}}\right)^{1/2}\right] \tag{4.25}$$

On the other hand, $\boldsymbol{u}_t - \boldsymbol{u}_t^k$ satisfies:

$$\begin{aligned} \boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t) = &-A(\boldsymbol{u}(t) - \boldsymbol{u}^k(t)) \\ &-(P((\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)) - P_k((\boldsymbol{u}^k(t)\cdot\nabla\boldsymbol{u}^k(t))) \\ &+(P((\boldsymbol{h}(t)\cdot\nabla)\boldsymbol{h}(t)) - P_k((\boldsymbol{h}^k(t)\cdot\nabla\boldsymbol{h}^k(t))) + P\boldsymbol{f} - P_k\boldsymbol{f}. \end{aligned}$$

Therefore:

$$\begin{aligned} \|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\| \leq &\|A(\boldsymbol{u}(t) - \boldsymbol{u}^k(t))\| \\ &+\|P[(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)] - P_k[(\boldsymbol{u}^k(t)\cdot\nabla)\boldsymbol{u}^k(t)]\| \\ &+\|P[(\boldsymbol{h}(t)\cdot\nabla)\boldsymbol{h}(t)] - P_k[(\boldsymbol{h}^k(t)\cdot\nabla)\boldsymbol{h}^k(t)]\| \\ &+\|P\boldsymbol{f} - P_k\boldsymbol{f}\| \end{aligned} \tag{4.26}$$

We bound the terms on the right side of (4.26) in the following way: the first one is bounded as in (4.23); the second one, by using Lemma 2.4

(applied to $\boldsymbol{f} = P_k(\boldsymbol{u}\cdot\nabla\boldsymbol{u})$) and (4.12) (where Theorem 2.3 and Theorem 3.7 have been used), we obtain:

$$\begin{aligned}
& \|P[(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)] - P_k[(\boldsymbol{u}^k(t)\cdot\nabla)\boldsymbol{u}^k(t)]\| \\
\leq\ & \|(I-P_k)(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)\| + \|P_k((\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t) - (\boldsymbol{u}^k(t)\cdot\nabla)\boldsymbol{u}^k(t))\| \\
\leq\ & \frac{C}{\lambda_{k+1}^{1/2}}\|(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)\|_{\boldsymbol{H}^1} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}
\end{aligned}$$

Using again Theorem 2.3 to bound $\|(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)\|_{\boldsymbol{H}^1}$, we get:

$$\|P[(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)] - P_k[(\boldsymbol{u}^k(t)\cdot\nabla)\boldsymbol{u}^k(t)]\| \leq C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}.$$

Analogously:

$$\|P[(\boldsymbol{h}(t)\cdot\nabla)\boldsymbol{h}(t)] - P_k[(\boldsymbol{h}^k(t)\cdot\nabla)\boldsymbol{h}^k(t)]\| \leq C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}.$$

And the last term can be written as:

$$\|(I-P_k)\boldsymbol{f}\| \leq C\frac{\|\boldsymbol{f}\|_{\boldsymbol{H}^1}}{\lambda_{k+1}^{1/2}}.$$

So that:

$$\|\boldsymbol{u}_t(t) - \boldsymbol{u}_t^k(t)\| \leq \frac{C(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}. \tag{4.27}$$

By proceeding in the same way for the magnetic field we obtain:

$$\|\boldsymbol{h}_t(t) - \boldsymbol{h}_t^k(t)\| \leq \frac{C(\frac{3}{4}+\epsilon)}{\lambda_{k+1}^{\epsilon}} + C\left(\frac{1}{\lambda_{k+1}}\right)^{1/2}. \tag{4.28}$$

Combining (4.24) with (4.27), and (4.25) with (4.28), estimates (4.23) are obtained. □

5 Conclusions

In this work, the Galerkin spectral method for MHD-system have been analyzed obtaining optimal convergence rates in the $L^\infty([0,T);L^2(\Omega))$ and

$L^\infty([0,T);H^1(\Omega))$-norms. Rates in the $L^\infty([0,T);H^2)$-norm have also been proved. The word "optimal" means that the difference between the strong solution of problem (1.1)-(1.2), denoted by $(\boldsymbol{u},\boldsymbol{h})$ and given by Theorem 2.3 (or Theorem 2.2), and the approximate solution of (2.2), denoted by $(\boldsymbol{u}^k,\boldsymbol{h}^k)$ and also provided by Theorem 2.3 (or Theorem 2.2), estimated in the norms cited above have the best possible rate measured in terms of the power of the inverse of the Stokes operator eigenvalues λ_{k+1}. In this sense, the same level of knowledge proved for classical Navier-Stokes equations has been aimed for MHD-system.

Acknowledgements

The first author author was partially supported by Grant PGC2018-098308-B-I00 (by MCIN/AEI/ 10.13039/501100011033 and by ERDF A way of making Europe), US-1381261 (I+D+i FEDER AndalucÃa 2014-2020) project and Grant P20_01120 (PAIDI/JUNTA/FEDER, UE). Second author was partially supported by MATH-AMSUD project 21-MATH-03 (CTMicrAAPDEs), CAPES-PRINT 88887.311962/2018-00 (Brazil), Project UTA-Mayor, 4753-20, Universidad de Tarapacá (Chile). Finally, the third author was partially supported by DIUFRO DI15-0021.

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