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On the nullity of homogeneous Riemannian manifolds

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. This is an expository article about the nullity of homogeneous Riemannian manifolds. It is based on recent results obtained in collaboration with A. J. Di Scala and F. Vittone [4, 5].

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1 Introduction

The nullity of a Riemannian manifold was defined by Chern and Kuiper [2] in the fifties. Let M be a Riemannian manifold. The *nullity* of the curvature tensor R at $p \in M$ is

$$\nu_p := \{v \in T_pM : R_{v,x} = 0, \forall x \in T_pM\}$$

or, equivalently, due to the identities of the curvature tensor,

$$\nu_p := \{v \in T_pM : R_{x,y}v = 0, \forall x, y \in T_pM\}$$

The nullity ν defines an autoparallel distribution in the open and dense subset Ω of M where the dimension $\dim(\nu_q)$ is locally constant.

Its extrinsic counterpart, the *relative nullity*, i.e. the nullity of the second fundamental form, has been extensively studied by many authors and it is always contained, for Euclidean submanifolds, in the nullity.

We study the *nullity* of a locally irreducible homogeneous Riemannian manifolds M. Namely, we develop, by geometric means, a general *structure theory* for homogeneous spaces in relation to the nullity. If M has a non-trivial nullity then the index of symmetry of M is non-trivial.

The nullity foliation is never a homogeneous foliation (i.e. given by the orbits of a subgroup of isometries). Moreover, no Killing field $X \neq 0$ can be always tangent to the nullity distribution (see Proposition 3.19 of [4]).

From the structure results it follows that the nullity is trivial if $M = G/H$ with G reductive, in particular, if M is compact. These results also help us in finding the first examples of homogeneous spaces with non-trivial nullity. In fact, we construct in [4] examples of conullity 3 in any dimension. Moreover, conullity 3 implies that G is solvable and H is trivial.

The main purpose of this survey is not only to show our main results, but also to draw the attention to some interesting and nice geometric techniques that not are usually considered in homogeneous geometry.

The author is grateful with the referee for may useful comments that led to a substantial improvement of this survey.

2 Killing fields and Kostant connection

Let $(M, \langle\, , \,\rangle)$ be a (connected) complete Riemannian manifold with Levi-Civita connection ∇. A vector field X of M is called a *Killing field* if

$$v \mapsto \nabla_v X$$

is a skew-symmetric endomorphisms of T_pM, for all $p \in M$.

Such a condition is called the *Killing equation* and reflects the fact that the flow ϕ_t of X is by isometries, i.e., preserves the metric tensor.

Much less known is the so-called *affine Killing equation* that reflects the fact ϕ_t is by affine transformations, i.e., preserves the Levi-Civita connection ∇,

$$\nabla_u B = R_{u,X} \qquad \textit{(affine Killing equation).} \qquad , \tag{2.1}$$

where $Bw = \nabla_w X$. Equivalently, the above equation can be written as

$$\nabla^2_{u,v} X = R_{u,X_p} v$$

(see equation 2.1.1 of [4]).

The Killing equation and the affine Killing equation motivates the introduction of the *Kostant connection* $\tilde{\nabla}$ on the so-called *canonical vector bundle* (see [3] and the references therein),

$$E := TM \oplus \Lambda^2(TM),$$

where $\Lambda^2(T_pM) \simeq \mathfrak{so}(T_pM)$.

$$\tilde{\nabla}_u(Z, B) := (\nabla_u Z - Bu, \nabla_u B - R_{u,Z_p}).$$

The Killing fields of M are *naturally identified* with the parallel sections of E. Namely, (X, B) *is a parallel section of E if and only if X is a Killing field of M and $B = \nabla X$.*

The Kostant connection allows to determine the initial conditions of a Killing field X at any $q \in M$ if one knows, at a fixed p, the initial conditions

$$(X)^p = (X_p, (\nabla X)_p).$$

In fact, we must compute the parallel transport, in the Kostant connection, of $(X)^p$ along any curve from p to q (in particular, we may use a geodesic).

The affine Killing equation $\nabla^2_{u,v} X = R_{u,X_p} v$ is very helpful for computing the initial conditions of the Lie bracket of any two Killing fields in terms of the initial conditions of the fields. Namely, If $(X)^p = (v, B)$, $(X')^p = (v', B')$, then

$$([X, X'])^p = (B'v - Bv', R_{v,v'} - [B, B']).$$

(see Lemma 2.4 of [8]).

So, one has the following useful formula for determining the curvature tensor in terms of Killing fields X, Y:

$$R_{X_p,Y_p} = (\nabla[X, Y])_p + [(\nabla X)_p, (\nabla Y)_p].$$

The well-known *Koszul formula* gives the Levi-Civita connection ∇ in terms of brackets, and scalar products, of vector fields. Since the Lie derivative of the metric tensor, along any Killing field, is zero, one has the following simple expression, for the Levi-Civita connection:

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle,$$

$X, Y, Z \in \mathcal{K}(M)$ (the vector space of Killing fields). This formula is very useful for deciding when a Killing field Y is a *transvection* at $p \in M$. Namely, whether $(\nabla Y)_p = 0$.

If w belongs to the Lie algebra $\tilde{\mathfrak{g}}$ of $I(M)$, then $\hat{w}$ is a Killing field where

$$\hat{w}_q = \frac{\mathrm{d}}{\mathrm{d}t}|_0 \mathrm{Exp}(tw)q = w.q.$$

Moreover, any Killing field can be written in this form. The flow ϕ_t associated to $\hat{w}$ is given by

$$\phi_t(q) = \mathrm{Exp}(tw)q.$$

The map $w \mapsto \hat{w}$, from $\tilde{\mathfrak{g}}$ into $\mathcal{K}(M)$ is a Lie algebra anti-isomorphism, since a Killing field is naturally related with a right-invariant vector field of $I(M)$.

3 Parallel transport along integral curves of Killing fields

Let $X \in \mathcal{K}(M)$ with associated flow ϕ_t, let $p \in M$ and $c(t) = \phi_t(p)$. If τ_t. denotes the parallel transport along $c(t)$, then

$$\tau_t^{-1} \circ \mathrm{d}\phi_t : T_pM \to T_pM$$

is a 1-parameter subgroup of linear isometries. Moreover,

$$\tau_t^{-1} \circ \mathrm{d}_p\phi_t = \mathrm{e}^{t(\nabla X)_p} \tag{3.1}$$

(see formula 2.2.1 of [4]).

General fact: *for any h in the closure*

$$c\ell\{\mathrm{e}^{t(\nabla X)_p} : t \in \mathbb{R}\} \subset \mathrm{SO}(T_pM),$$

there exists a sequence $\{t_k\} \to +\infty$ such that $\mathrm{e}^{t_k(\nabla X)_p} \to h$ (in particular, for $h = Id$). That is, the orbits $\mathrm{e}^{t(\nabla X)_p}u$ are quasi-periodic (see Section 2.1 of [4]). In fact, this is a general result about dense geodesics in a torus acting on a manifold.

4 Holonomy of homogeneous spaces

Let $M = G/H$ be a homogeneous Riemannian manifold. Let $\mathcal{K}^G(M)$ be the Killing fields induced by G. That is,

$$\mathcal{K}^G(M) := \{\hat{w} : w \in \mathfrak{g} = \mathrm{Lie}(G)\}$$

Let $\mathfrak{hol}(p)$ be the holonomy Lie algebra of M at p. The following results are due to B. Kostant [6]:

- The *algebraic span of* $\{(\nabla Z)_p : Z \in \mathcal{K}^G(M)\}$ contains $\mathfrak{hol}(p)$ and is contained in the normalizer in $\mathfrak{so}(T_pM)$ of that algebra.

- If M is locally irreducible then $\mathfrak{hol}(p)$ coincides with the *algebraic span of* $\{(\nabla Z)_p : Z \in \mathcal{K}^G(M)\}$

A useful criterion in deciding whether M is reducible: *If a non-trivial subspace of T_pM is left invariant by $\{(\nabla Z)_p : Z \in \mathcal{K}^G(M)\}$, then M splits.*

5 The index of symmetry

The general bibliography for this section is [1].

A field $X \in \mathcal{K}(M)$ is called a *transvection* at $q \in M$ if $(\nabla X)_q = 0$.

In this case $\gamma(t) := \phi_t(q)$ is a geodesic and $\mathrm{d}_q\phi_t$ gives the parallel transport along $\gamma(t)$.

The *Cartan subspace* at q is

$$\mathfrak{p}^q := \{X \in \mathcal{K}(M) : X \text{ is a transvection at } q\},$$

the *symmetric isotropy algebra* at q is

$$\mathfrak{k}^q := [\mathfrak{p}^q, \mathfrak{p}^q],$$

One has that

$$\tilde{\mathfrak{g}}^q := \mathfrak{k}^q \oplus \mathfrak{p}^q$$

is an involutive Lie algebra. The *symmetric subspace* at q is

$$\mathfrak{s}_q := \mathfrak{p}^q.q.$$

Since M is homogeneous the dimension of this subspace does not depend on q and thus $q \mapsto \mathfrak{s}_q$ defines a distribution, the so-called *distribution of symmetry,* which is autoparallel. The totally geodesic integral manifolds of $\mathfrak{s}$ are called the *leaves of symmetry* which are globally symmetric spaces. The *index of symmetry* is defined as the dimension of the distribution of symmetry.

6 The nullity of homogeneous spaces

Let $M = G/H$ be a homogeneous Riemannian manifold. Then the nullity distribution ν, being a geometric object, is invariant under any isometry $g \in I(M)$, i.e., $g_*(\nu) = \nu$. Note that if Z is a Killing field, then $Z_{\gamma_v(t)}$ is a Jacobi field along any geodesic $\gamma_v(t)$, where $v \in T_pM$. If $\gamma_v(t)$ is tangent to the nullity, then, by the Jacobi equation,

$$Z_{\gamma_v(t)} = \tau_t(Z_p) + t\,\tau_t(\nabla_v Z),$$

where τ_t denotes the parallel transport along $\gamma_v(t)$.

One has that ∇Z must be parallel along $\gamma_v(t)$. In fact, by the affine Killing equation,

$$\nabla_{\gamma_v'(t)}(\nabla Z) = R_{\gamma_v'(t), Z_{\gamma_v(t)}} = 0.$$

So, we have

$$\begin{cases} Z_{\gamma_v(t)} = \tau_t(Z_p) + t\,\tau_t(\nabla_v Z), \\ \nabla_{\gamma_v'(t)}(\nabla Z) = 0. \end{cases} \tag{6.1}$$

This implies the following important fact: *If Z is not parallel along a geodesic γ_v in the nullity, then it grows linearly. Moreover, ∇Z is bounded along γ_v, since it is parallel.*

Let $\gamma_v(t) = \phi_t(p)$ be a homogeneous geodesic that lies in the nullity, where ϕ_t is the flow associated to $X \in \mathcal{K}^G(M)$. Let Z be a Killing field of M that is not bounded, or equivalently non-parallel, along $\gamma_v(t)$. Then, by bringing back Z to p by means of the flow ϕ_{-t}, after dividing by t and accumulating in a sequence so that (3.1) tends to the identity, we obtain a transvection Y at p in the direction $Y_p = \nabla_v Z \neq 0$. Let us be more precise with this argument: let us consider one parameter family $Z^t = (\phi_{-t})_*(Z)$ of Killing fields, where $Z^t_q = \mathrm{d}\phi_{-t}(Z_{\phi_t(q)})$ and $q \in M$. The initial conditions of Z^t at p are given by

$$\begin{aligned} \left(Z^t\right)^p &= \left(\mathrm{d}\phi_{-t}(Z_{\gamma_v(t)}), (\mathrm{d}_p\phi_t)^{-1} C^t \mathrm{d}_p\phi_t\right) \\ &= \left((\mathrm{d}_p\phi_t)^{-1}((Z_{\gamma_v(t)}), (\mathrm{d}_p\phi_t)^{-1} C^t \mathrm{d}_p\phi_t\right) \\ &= \left((\mathrm{d}_p\phi_t)^{-1}(\tau_t(Z_p) + t\tau_t(\nabla_v Z)), (\mathrm{d}_p\phi_t)^{-1} C^t \mathrm{d}_p\phi_t\right) \end{aligned}$$

where C^t is the skew-symmetric endomorphism of $T_{\gamma_v(t)}M$ given by $(\nabla Z)_{\gamma_v(t)} = \tau_t \circ (\nabla Z)_p \circ \tau_{-t}$ (see 6.1). Observe, from 3.1, that

$$(\mathrm{d}_p\phi_t)^{-1}\tau_t = \mathrm{e}^{-t(\nabla X)_p}.$$

Hence

$$\big(Z^t\big)^p = \big(\mathrm{e}^{-t(\nabla X)_p}(v) + t\mathrm{e}^{-t(\nabla X)_p}(\nabla_v Z), \mathrm{e}^{-t(\nabla X)_p} C^0 \mathrm{e}^{t(\nabla X)_p}\big).$$

Take a sequence, as in last paragraph of Section 3, $\{t_k\} \to +\infty$ such that $\mathrm{e}^{t_k(\nabla X)_p} \to Id$. Then, taking into account that the second initial condition of Z^t is bounded, independently of t, $\frac{1}{t_k} Z^{t_k}$ converges to a Killing field Y with initial conditions at p given by

$$Y^p = (\nabla_v Z, 0).$$

Such a transvection Y is called *adapted* to $v \in \nu_p$ and it is not in general tangent to the nullity. Since any integral manifold N of ν is homogeneous and flat, then for any $q \in N$ the directions of homogeneous geodesics starting at q span $T_q N$ (see the last paragraph before Lemma 2.11 of [4]). This implies that for any Killing field Z of M induced by G and for any $w \in T_p N$ there exists a transvection at p with initial condition $\nabla_w Z$. Hence we have proved Proposition 3.6 of [4] which is the first part of (2) in Theorem 8.1 (for the definition of the distribution $\hat{\nu}$ see Section 7).

There exist $Z \in \mathcal{K}^G(M)$ and $v \in T_p M$ such that $\nabla_v Z \notin \nu_p$ and so there exists a transvection at p which is not tangent to the nullity at p (see Section 7). In particular, the index of symmetry of M must be positive.

7 Geometric distributions associated to ν

Let us define some geometric distributions associated to the nullity distribution ν. First of all we define the adapted distribution $\hat{\nu}$ by :

$$\hat{\nu}_p = \{\nabla_{\nu_p} Z, \text{ where } Z \in \mathcal{K}(M)^G\},$$

that can be defined for any G-invariant distribution.

The so-called *osculating distribution* of order k, associated to ν, is defined by

$$\nu_q^{(1)} = \nu_q + \text{span}\,\{\nabla_w X : X \in C^\infty(\nu), w \in T_q M\},$$

$$\nu^{(k)} = (\nu^{(k-1)})^{(1)}$$

. Since ν if G-invariant one has that

$$\nu^{(1)} = \nu + \hat{\nu} \tag{7.1}$$

In fact, let U be a field that lies in ν and let $Z \in \mathcal{K}^G(M)$. Then $\nabla_{Z_p} U = \nabla_{U_p} Z + [Z, U]_p$. Since the flow of Z preserves ν one has that $[Z, U]_p \in \nu_p$. From this it follows (7.1).

Then, if ν is not a parallel distribution, $\hat{\nu}$ is not contained in ν (for instance, if M is locally irreducible). This proves the assertion in the last paragraph of Section 6

Remark 7.1. If $M = G/K$ is compact, then any Killing field of M is bounded. Then $\hat{\nu} \subset \nu$ and so $\nu^{(1)} = \nu$ and thus ν is a parallel distribution. This implies that the nullity distribution coincides with the local Euclidean de Rham factor of M.

By making use of the same arguments one has that

$$\nu^{(k)} = \nu^{(k-1)} + (\nu^{(k-1)})\hat{}.$$

In our geometric construction it also appears another natural $I(M)$-invariant distribution $\mathcal{U}$, the so-called *bounded distribution*, obtained by adding to ν the directions of the Killing fields whose normal component are bounded on a given leaf of ν. The distribution $\mathcal{U}$ is a G-invariant integrable distribution and ν is parallel in the directions of $\mathcal{U}$. Moreover, it is the largest G-invariant distribution with this property and it does not depend on the presentation group G. The integral manifold of $\mathcal{U}$ by p is given by $\mathfrak{u}^p.p$, where $\mathfrak{u}^p$ is the so-called bounded algebra at p (see Section 5 of [4]).

$$\mathfrak{u}^p := \{Z \in \mathcal{K}^G(M) \,:\, \nabla_{\nu_p} Z \subset \nu_p\}.$$

Equivalently, $Z \in u^p$ if and only if the projection of $Z_{|N(p)}$ to $\nu^\perp$ is bounded, where $N(p)$ is the integral manifold of ν by p (see Section 5 of [4]).

8 Main results

Let us first state the results in [4] whose main ingredients have been introduced and sketched in the previous sections.

Theorem 8.1 ([4]). *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor. Assume that the nullity distribution ν is non-trivial and let k be its codimension. Let $\nu^{(1)}, \nu^{(2)}$, $\hat{\nu}$ and $\mathcal{U}$, be the osculating, of order 1 and 2, the adapted and the bounded G-invariant distributions associated to ν, respectively. Then*

(1) *$\nu^{(1)} = \nu + \hat{\nu}$ is autoparallel and flat, and $\mathcal{U}$ is integrable. Moreover, we have the following inclusions (and so, $k \geq 3$):*

$$\{0\} \subsetneq \nu \subsetneq \nu^{(1)} \subsetneq \nu^{(2)} \subset \mathcal{U} \subsetneq TM.$$

Moreover, the integral manifolds of ν and $\nu^{(1)}$ are simply connected (and so isometric to a Euclidean space).

(2) *For any $p \in M$, $v \in \hat{\nu}_p$, there exists a transvection $Y \in \mathcal{K}^G(M)$ at p, with $Y_p = v$, and the Jacobi operator $R_{\cdot,v}v$ is null. Moreover, $[Y, [Y, \mathcal{K}(M)]] = 0$ and Y does not belong to the center of $\mathcal{K}^G(M)$. In particular, there exists such a transvection with $Y_p \notin \nu_p$.*

(3) *The representation of the isotropy H on $\nu_p^\perp$ is faithful ($p = [e]$). Moreover, $\dim H \leq \frac{1}{2}(k-2)(k-3)$ (if the equality holds, then $H \simeq \mathrm{SO}(k-2)$). In particular, if $k = 3$, $H = \{e\}$ (if G is connected).*

(4) *If $k = 3$, then G is solvable. Moreover, there exist irreducible examples in any dimension where G is not unimodular, and so M does not admit a finite volume quotient.*

By making use of the previous result, we found some obstructions for the existence of non-trivial nullity. In fact, the existence of transvections of order 2 imposes general restrictions on the presentation group G. Namely,

Proposition 8.2 ([4]). *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor. Then the nullity distribution is trivial, with any G-invariant metric, in any of the following cases:*

(a) *If the Lie algebra of G is reductive (in particular, if M is compact).*
(b) *If the Lie algebra $\mathfrak{g}$ of G is 2-step nilpotent.*

We have also the following result:

Theorem 8.3 ([4]). *Let $M = G/H$ be a simply connected homogeneous Riemannian manifold without a Euclidean de Rham factor, where $G = I(M)^o$. Assume that M has a non-trivial nullity distribution ν. Then*

(1) *Any integral manifold $N(q)$ of the nullity distribution ν is a closed embedded submanifold of M (or equivalently, the Lie subgroup E^q of G that leaves $N(q)$ invariant is closed).*

(2) *M is the total space of a Euclidean affine bundle over the quotient $B = G/E^p$ of M by the leaves of nullity with standard fibre $N(p) \simeq \mathbb{R}^\mu$ ($p = [e]$).*

(3) *$\nu^\perp$ defines an affine metric connection, in the sense of Ehresmann, on the affine bundle $M \to B$. Moreover, the holonomy group associated to $\nu^\perp$ is transitive (or equivalently, $\nu^\perp$ is completely non-integrable).*

The complete proofs of the above results exceeds the scope of this survey and can be found in [4].

The nullity distribution of a homogenous Riemannian manifold is far from being homogeneous (i.e. given by the tangent spaces of the orbits of a group of isometries). Let us state Proposition 3.19 of [4]:

Proposition 8.4 ([4]). *Let M be a homogeneous Riemannian manifold without Euclidean (local) de Rham factor. Assume that the nullity distribution ν of M is non-trivial. Then there exists no Killing field $X \neq 0$ of M such that X lies in ν.*

Proof. We will prove this result in the case that M is simply connected and irreducible (the general case follows in a standard way from it). Let us assume, for simplicity, that M is simply connected and irreducible. Let $X \neq 0$ be a Killing field of M that lies in the nullity distribution. From the affine Killing equation 2.1 we have that ∇X is a parallel skew-symmetric $(1,1)$-tensor of M. If $\ker(\nabla X) = TM$, then X is a parallel field and so M splits off a line. A contradiction. Then for any $p \in M$ $(\nabla X)_p$ has a complex eigenvalue $\lambda \notin \mathbb{R}$. Let $\{0\} \neq \mathbb{V} \subset T_pM$ be the $(\nabla X)_p$-invariant subspace associated to λ and $\bar{\lambda}$. Since ∇X is a parallel tensor, then $\mathbb{V}$ extends to a parallel distribution of M. Since M is irreducible, then $\mathbb{V} = T_pM$. In this case, eventually by rescaling X, we have that $J = \nabla X$ is a Kähler structure on M and so M is a Kähler (homogeneous) manifold. Then the field $\xi = JX$ lies in the nullity distribution ν. From the fact that $\nabla^2 X = 0$ and $\nabla J = 0$ we obtain that

$$\nabla^2 \xi = 0.$$

Then ξ satisfies the affine Killing equation (2.1) $\nabla^2_{u,v}\xi = R_{u,\xi}v$. If ϕ_t is the flow associated to ξ, for any given t, ϕ_t is a homothetic transformation of M associated to the constant e^{ta}, where $A = aId$ is the symmetric part of $\nabla\xi$ (cf. [KN], Lemma 1, p. 242). In fact, this symmetric part A, from the affine Killing equation, must be parallel. Since M is irreducible, A has only one (constant) eigenvalue. In our particular case $\nabla\xi = J\nabla X = -Id$, and so $a = -1$. But, in a homogeneous non-flat irreducible space, any homothetic transformation is an isometry (this is a general fact for a complete Riemannian manifolds (see [KN][Theorem 3.6, p. 242]). Then $a = 0$ that is a contradiction.

□

Observe that in Theorem 8.3 the action of G on G/E^p, regarded as the quotient of M by the leaves of the nullity foliation, is almost effective. In fact, let $X \in \mathcal{K}^G(M)$ that projects trivially to the quotient G/E^p. Then X must be tangent to the nullity distribution and so $X = 0$. Moreover, the projection to the quotient $\pi : M \to B$ is never a Riemannian submersion. In fact, let $Z \in \mathcal{K}^G(M)$ such that it does not belong to the bounded algebra $\mathfrak{u}^p$. Equivalently, the projection Z' of $Z_{|N(p)}$ to $\nu^\perp$ is not bounded (see last paragraph of Section 7). Such a Killing fileld Z must exit since $\mathcal{U}$ is a proper distribution of M (see part (1) of Theorem 8.1). But the projection $Z^\perp$ of Z to $\nu^\perp$ is a projectable horizontal field of M. If π is a Riemannian submersion, then the length of $Z^\perp$ must be constant along any integral manifold of ν. Then $Z' = (Z^\perp)_{|N(p)}$ has constant length. A contradiction that proves that π is not a Riemannian submersion.

9 Further developments

Let $M = G/H$ be a simply connected irreducible homogeneous Riemannian with a non-trivial nullity distribution ν. In a recent result [5] we obtained that for any $v \in \nu_p$ there exists a transvection at p with initial condition v (possibly, not in $\mathfrak{g}$). Moreover, such transvections generate an abelian ideal $\mathfrak{j}$ of the full isometry algebra that contains the adapted transvections at p. Observe that being $\mathfrak{j}$ an ideal, constructed from the geometry, it does not depend on $p \in M$. This result gives a general picture of how these transvections lie in the full isometry algebra. We state without proof the following theorem that will appear in a forthcoming article.

Theorem 9.1 ([5]). *Let $M = G/H$ be a simply connected irreducible homogeneous Riemannian manifold with a non-trivial nullity distribution ν. Then*

(1) *The adapted transvections at p generate an abelian ideal $\mathfrak{i}$ of the Killing fields $\mathcal{K}^G(M)$ induced by G (and does not depend on $p \in M$).*

(2) *The distribution $\mathcal{D}$ of M given by $\mathcal{D}_p = \mathfrak{i}.p + \nu_p$ does not depend on the presentation group G. Moreover, $\mathcal{D}$ is integrable with intrisically*

flat integral manifolds.

(3) *For any $v \in \nu_p(M)$ there exists a transvection with initial condition v (possibly, not in $\mathcal{K}^G(M)$). Moreover, the ideal $\mathfrak{j}$ of $\mathcal{K}(M)$ generated by such transvections at p is abelian and contains $\mathfrak{i}$ (and does not depend on $p \in M$). Furthermore, $\mathcal{D}_p = \mathfrak{j}.p$.*

(4) *$\nu^{(2)} \subset \mathcal{D}$. Moreover, ν and $\nu^{(1)}$ are parallel along $\mathcal{D}$. Moreover, ν and $\nu^{(1)}$ are parallel along any orbit of the closure $\bar{A}$ of the normal abelian Lie group A associated to $\mathfrak{j}$.*

(5) *The projection to the quotient $\bar{A}\backslash M$ of M by the orbits of $\bar{A}$ is a G-invariant Riemannian submersion with intrinsically flat fibers.*

The above result helps in finding new examples of homogeneous Riemannian manifolds with non-trivial nullity.

Proposition 9.2 ([5])**.** *Let K be a simply connected compact simple Lie group and let $\rho : K \to SO_n$ be an irreducible orthogonal representation. Let $\mathbb{V}_0$ be a non-trivial vector subspace of $\mathbb{R}^n$ such that* $\dim(\mathbb{V}_0)(1+\dim(K)) < n$. *Then there exists a left invariant metric $\langle\, , \rangle$ on $G = \mathbb{R}^n \rtimes_\rho K$ such that $M = (G, \langle\, , \rangle)$ is an irreducible Riemannian manifold and the nullity distribution ν of M has dimension at least* $\dim(\mathbb{V}_0)$.

The above proposition answers the question of whether there exists irreducible examples with non-trivial topology and thus not admitting a transitive solvable group in contrast with the case of conullity 3 (see part (4) of Theorem 8.1). Moreover, there are such examples with arbitrary large dimension.

Remark 9.3 ([5])**.** Let us assume that $M = G/H$ is simply connected with conullity 3. By part (3) of Theorem 8.1 one has that H is trivial and so G is the identity component of the full group of isometries of M. Then M is a Lie group with a left invariant metric. By making use of the fact that the transvections in the direction of ν_p generate an abelian ideal $\mathfrak{j}$ of $\mathfrak{g}$ that contains the adapted transvections at p it is not hard to to prove that $G = \mathbb{R}^n \rtimes \mathbb{R}$ (semidirect product). This simplifies the proof given in [4], that involved many cases, of the fact that $\mathfrak{g}$ is solvable. Examples in any dimension are given in Section 9 of [4].

Remark 9.4 ([5])**.** The adapted transvections at p, as well as the transvections in the direction of ν_p lie in an abelian ideal of $\mathcal{K}^G(M)$. If $Z \neq 0$ is any of such transvections, then it does not lie in the centre of $\mathcal{K}^G(M)$. Otherwise, Z would be a transvection at any point and thus a parallel field. Hence M would split off a line. Since Z does not belong to the the centre of $\mathcal{K}^G(M)$, we conclude that Z has order two, i.e. $\mathrm{ad}_Z^2 = 0$ (see part (2) of Theorem 8.1).

Open questions. Assume that $M = G/H$ is locally irreducible with a non-trivial nullity, where G acts almost effectively by isometries.

– *Can M be Kähler?*

– *Is the identity component of H trivial?* (as in the case of conullity 3, see part (3) of Theorem 8.1).

– *Characterize the irreducible homogeneous spaces with conullity* 3.

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