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Ground state solutions for Schrödinger-Born-Infeld equations

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Abstract. In this note we revise the arguments used to find ground states solutions for an elliptic system which envolves the Schrödinger equation coupled with the electrostatic equation of the Born-Infeld electromagnetic theory. The main difficulties are related to the second equation of the system which is nonlinear.

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1 Introduction

In this survey we present some results obtained in [2, 18] on an elliptic system of interest in physics.

In the last years we noted an increasing interest in studying systems of equations which model the interaction of the matter with electromagnetic fields. In particular many theories have been considered which consider matter equation, such as the the Klein-Gordon or Schrödinger equation

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coupled with equations of the electromagnetic field, such as the Maxwell equations, or even equations which better describe the electromagnetic field such as the Bopp-Podolsky or Born-Infeld equation. Actually, without entering in physical details here, the Maxwell and Bopp-Podolsky equations are approximations of the more sophisticated Born-Infeld theory introduced in [8, 9].

Roughly speaking, the coupling cited above of a matter field with the electromagnetic field models a physical situation where a charge particle interacts with the electromagnetic field generated by its motion in space. This interaction can be described rigorously by means of the Gauge Theories and consists, practically, in changing the differential operators which appear in the Lagrangian with the so-called "covariant derivatives". The search of standing waves solutions, or normal modes, in equilibrium with its electromagnetic field, in a purely electrostatic situation leads to an elliptic system of two coupled equations: the first one related to the matter field, and the second related to the electric potential which is different according to the theory of electromagnetic field which is considered.

We studied recently these kind of systems and proved existence and multiplicity of solutions in many situations. The interested reader may consult the pioneering paper [5] and the subsequent papers which combines the Klein-Gordon and the Maxwell equations [6, 12], the Klein-Gordon and the Born-Infeld equations [11, 28], the Schrödinger and the Maxwell equations [3, 4, 10, 14, 15, 19, 20, 26, 21, 23, 24], the Schrödinger and the Bopp-Podolsky equations [1, 13, 16, 25, 22].

In this survey we consider the Schrödinger equation coupled with the Born-Infeld equation of the electromagnetic field. This system has a lagrangian $\mathcal{L}_{BI}$, which describes the physical situation better then the lagrangian used in the Maxwell theory of the electromagnetic field $\mathcal{L}_M$. The price to pay is that the equation of the electrostatic potential is nonlinear, in contrast to the case of the Maxwell Theory where the Poisson equation appears.

Coming to our problem, the search of standing waves for the Schrödinger equation in equilibrium with its own electrostatic field in the Born-Infeld theory reduces to find solutions $u, \phi : \mathbb{R}^3 \to \mathbb{R}$ of the following system

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases} \tag{$\mathcal{SBI}$}$$

to which we will refer as Schrödinger-Born-Infeld system. Here $p > 1$ and the power like nonlinearity simulates the interaction between many particles.

A first look at the system suggests that a great attention has to be paid to the second equation. Indeed in contrast to the Maxwell or Bopp-Podolsky theory, here the operator is nonlinear and one has to choose the right space where ϕ has to be. A second difficulty which appears is the fact that we are working in the whole space where the compact embeddings of Sobolev spaces are not true. From a variational point of view this brings some difficulties in order to prove the compactness of Palais-Smale related to the energy functional. Nevertheless we are able to give some existence results, which depending on the values of p are based on two different approaches.

2 Statements of the results

To state correctly our results, some preliminaries are in order. For what concerns the space where look for the unknown u, the usual Sobolev space $H^1(\mathbb{R}^3)$ seems to be the correct one. On the other hand, the second equations forces us to restrict the space where we find the unknown ϕ. It happens that the right set has to be something of type

$$\mathcal{X} := \mathcal{D}^{1,2}(\mathbb{R}^3) \cap \{\phi \in C^{0,1}(\mathbb{R}^3) : \|\nabla\phi\|_\infty \leq 1\}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the L^2-norm of the gradient.

Hereafter we denote by $\|\cdot\|_q$ the norm in $L^q(\mathbb{R}^3)$, for $q \in [1,+\infty]$ and with $\|\cdot\|$ the usual norm in $H^1(\mathbb{R}^3)$.

The fact that $\mathcal{X}$ should be the right space where to work is corroborated by the following properties of $\mathcal{X}$ which can be seen in [7, Lemma 2.1].

Lemma 2.1. *The following assertions hold:*

1. *$\mathcal{X}$ is continuously embedded in $W^{1,p}(\mathbb{R}^3)$, for all $p \in [6,+\infty)$;*
2. *$\mathcal{X}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$;*
3. *if $\phi \in \mathcal{X}$, then $\lim_{|x|\to\infty} \phi(x) = 0$;*
4. *$\mathcal{X}$ is weakly closed;*
5. *if $(\phi_n)_n \subset \mathcal{X}$ is bounded, there exists $\bar\phi \in \mathcal{X}$ such that, up to a subsequence, $\phi_n \rightharpoonup \bar\phi$ weakly in $\mathcal{X}$ and uniformly on compact sets.*

In particular the above results state that $\mathcal{X}$ is the right space where to search for ϕ, either for the vanishing condition at infinity and for the Sobolev embedding. A reasonable definition of *weak solution* is the following. We say that a couple $(u,\phi) \in H^1(\mathbb{R}^3)\times\mathcal{X}$ is a weak solution of ($\mathcal{SBI}$) if

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} \nabla u\cdot\nabla v + uv + \phi uv = \int_{\mathbb{R}^3} |u|^{p-1}uv \\ \displaystyle\int_{\mathbb{R}^3} \frac{\nabla\phi\cdot\nabla\psi}{\sqrt{1-|\nabla\phi|^2}} = \int_{\mathbb{R}^3} u^2\psi. \end{cases}$$

for all $(v,\psi) \in C_c^\infty(\mathbb{R}^3)\times C_c^\infty(\mathbb{R}^3)$. Unfortunately, as we will se later, we are forced to work in a radial setting.

Let us introduce then the spaces of radial functions

$$H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radially symmetric}\}$$

and

$$\mathcal{X}_r = \{\phi \in \mathcal{X} \mid \phi \text{ is radially symmetric}\}.$$

The main theorems are the following. We state them separately since different methods have been used to prove the results.

Theorem 2.2. *For any $p \in (5/2, 5)$, problem ($\mathcal{SBI}$) possesses a radial ground state solution, namely a solution $(u, \phi) \in H_r^1(\mathbb{R}^3) \times \mathcal{X}_r$ minimizing the functional F among all the nontrivial radial solutions.*

A brief explanation has to be done here. We will find solutions by variational methods, hence as critical points of an energy functional F. The solution of ground state is by definition a critical point of F which is at the lower critical level.

Theorem 2.3. *The above result also holds for $p \in (2, 5/2]$.*

We point out that in both results the unknowns u and ϕ are classical solutions, namely of class $C^2(\mathbb{R}^3)$.

The remaining of the paper is organised as follows. In Section 3 we give some preliminaries results which are used to set the variational framework of the problem. In Section 4 we recall the arguments used to prove Theorem 2.2 and in Section 5 we just sketch how to obtain the result in Theorem 2.3. As we said before, the interested reader is referred to [2, 18] where all the details, quite technical, can be found.

3 Functional setting and preliminary results

Formally, the system ($\mathcal{SBI}$) comes variationally from the action functional F defined by

$$F(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + u^2\right) + \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1} - \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \sqrt{1-|\nabla\phi|^2}\right).$$

which is well defined for $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$. So we are reduced to find critical points of F and a first variational principle hold:

Proposition 3.1. *A pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$ is a weak solution of ($\mathcal{SBI}$) if and only if it is a critical point of F.*

The proof of this result is standard, although the fact that $\mathcal{X}$ is not a vector space creates a difficulty in particular when we need to make variations of the functional. Note that to this aim, the radial symmetry is not necessary.

A further difficulty is related to the fact that the functional is unbounded from above and from below. In fact it is easy to exhibit a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that $F(u_n, 0) \to +\infty$ and, fixed a function $u_0 \in H^1(\mathbb{R}^3) \setminus \{0\}$, a sequence $\{\phi_n\} \subset \mathcal{X}$ such that $F(u_0, \phi_n) \to -\infty$. However this last obstacle is overcome by a “classical substitution argument”.

This is done by solving the second equation of ($\mathcal{SBI}$), for any fixed $u \in H^1(\mathbb{R}^3)$. Let us start by considering the functional

$$E : H^1(\mathbb{R}^3) \times \mathcal{X} \to \mathbb{R}$$

defined as

$$E(u, \phi) = \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi|^2}\right) - \int_{\mathbb{R}^3} \phi u^2.$$

The following lemma states its main properties.

Lemma 3.2. *For any $u \in H^1(\mathbb{R}^3)$ fixed, there exists a unique $\phi_u \in \mathcal{X}$ such that the following properties hold:*

1. *ϕ_u is the unique minimizer of the functional $E(u, \cdot) : \mathcal{X} \to \mathbb{R}$ and $E(u, \phi_u) \leqslant 0$, namely*

$$\int_{\mathbb{R}^3} \phi_u u^2 \geqslant \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_u|^2}\right); \tag{3.1}$$

2. *$\phi_u \geqslant 0$ and $\phi_u = 0$ if and only if $u = 0$;*

3. *if ϕ is a weak solution of the second equation of system ($\mathcal{SBI}$), then $\phi = \phi_u$ and it satisfies the following equality*

$$\int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1 - |\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} \phi_u u^2.$$

Moreover, if $u \in H^1_r(\mathbb{R}^3)$, then $\phi_u \in \mathcal{X}_r$ is the unique weak solution of the second equation of system ($\mathcal{SBI}$).

We observe that we need to require the radial symmetry of $u \in H^1(\mathbb{R}^3)$ because, for a general u fixed, it is easy to show that there exists ϕ_u the unique minimizer of the functional $E(u,\cdot) : \mathcal{X} \to \mathbb{R}$ but we are not able to say that it is also a weak solution of the second equation of ($\mathcal{SBI}$) (see [7] for details). However, one can construct also in this case a reduced functional which is of class C^1. Then the restriction to the radial setting and its necessity appears in virtue of the last sentence of the previous lemma.

By Lemma 3.2, we can define the following one-variable functional on $H^1(\mathbb{R}^3)$ as

$$\begin{aligned} I(u) &= F(u,\phi_u) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2\right)+\frac{1}{2}\int_{\mathbb{R}^3}\phi_u u^2-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1} \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla\phi_u|^2}\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2\right)-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}-\frac{1}{2}E(u,\phi_u). \end{aligned}$$

The proof of this fact is straightforward, in fact it is not known if the correspondence $u \mapsto \phi_u$ is C^1 and then the result has to be proved by hands using the definition.

It is convenient of course to restrict I to $H^1_r(\mathbb{R}^3)$ in virtue of the last statement of Lemma 3.2. This choice is justified also by the fact that $H^1_r(\mathbb{R}^3)$ is a natural constraint for I in the sense that

$$I'(u)[v]=0 \quad \forall v \in H^1_r(\mathbb{R}^3) \Longrightarrow I'(u)[v]=0 \quad \forall v \in H^1(\mathbb{R}^3).$$

Indeed I is invariant under the action of $O(3)$ on $H^1(\mathbb{R}^3)$, that is

$$T_g : u \in H^1(\mathbb{R}^3) \mapsto u\circ g \in H^1(\mathbb{R}^3), \quad g \in O(3).$$

This can be seen by making use of Lemma 3.2, which gives

$$E(u, T_{g^{-1}}\phi_{T_g u}) = E(T_g u, \phi_{T_g u}) \leqslant E(T_g u, T_g \phi_u) = E(u,\phi_u)$$

and so $\phi_u = T_{g^{-1}}\phi_{T_g u}$ due to the uniqueness of the minimizer of $E(u,\cdot)$. Having $\phi_{T_g u} = T_g\phi_u$ the invariance of I is now evident. Hence we can apply the Palais Principle of Symmetric Criticality, implying that we can restrict ourselves to find critical points of I on $H^1_r(\mathbb{R}^3)$.

Standard computations give a second variational principle:

Proposition 3.3. *If $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{X}$ is a weak nontrivial solution of ($\mathcal{SBI}$), then $\phi = \phi_u$ and u is a critical point of I. On the other hand, if $u \in H^1_r(\mathbb{R}^3) \setminus \{0\}$ is a critical point of I, then (u,ϕ_u) is a weak nontrivial (radial) solution of ($\mathcal{SBI}$).*

Finally, u is a ground state of I if and only if (u,ϕ_u) is a ground state of F.

In other words, to find solutions of ($\mathcal{SBI}$) we are reduced to find critical points of I on $H^1_r(\mathbb{R}^3)$, and in particular the solution of minimal energy of the system is exactly the ground state of I.

Unfortunately, although we have a power nonlinearity, it is not available an expression of the functional under the rescaling

$$t \in (0,+\infty) \mapsto u_t := t^\alpha u(t^\beta\cdot) \in H^1_r(\mathbb{R}^3),$$

hence classical rescaling arguments used in other contexts (e.g. for the Schrödinger-Maxwell system) cannot be used, and this is another difficulty of the problem.

Another important fact is related to an identity that all the solutions of ($\mathcal{SBI}$) have to satisfy. This is known as a Pohozaev identity. It states the following: if (u,ϕ) is a solution of ($\mathcal{SBI}$) of class $C^2(\mathbb{R}^3)$, then the following identity is satisfied:

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 + 2\int_{\mathbb{R}^3}\frac{|\nabla\phi|^2}{\sqrt{1-|\nabla\phi|^2}} - \frac{3}{2}\int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla\phi|^2}\right) = \frac{3}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}.$$

The proof is made by multiplying the equation by $x\cdot\nabla u$, integrating on a ball B_R, making straightforward computations and passing to the limit as $R \to +\infty$. Formally it can be obtained by the equation

$$\frac{d}{dt} I(u(tx))\Big|_{t=1} = 0.$$

The details are left to the reader.

The solutions we find using the variational method (by obtaining critical points of I) are indeed classical, as standard boot-strap arguments show. The Pohozaev identity is then justified.

Let us pass now to describe the methods we use to find critical points of I. They are based on suitable modifications of Mountain Pass type arguments.

4 Sketch of proof of Theorem 2.2

The following technical lemma will be useful to study the geometry of the functional I. We think it is instructive to present also the proof.

Lemma 4.1. *Let q be in $[2, 3)$. Then there exist positive constants C and C' such that, for any $u \in H^1(\mathbb{R}^3)$, we have*

$$\|\nabla \phi_u\|_2^{\frac{q-1}{q}} \leqslant C\|u\|_{2(q^*)'} \leqslant C'\|u\|,$$

where q^* is the critical Sobolev exponent related to q and $(q^*)'$ is its conjugate exponent, namely

$$q^* = \frac{3q}{3-q} \quad \textit{and} \quad (q^*)' = \frac{3q}{4q-3}.$$

Proof. Since $\|\nabla \phi_u\|_\infty \leqslant 1$ and $q \in [2, 3)$ we have

$$\|\phi_u\|_{q^*} \leqslant C\|\nabla \phi_u\|_q = C\left(\int_{\mathbb{R}^3} |\nabla \phi_u|^2 |\nabla \phi_u|^{q-2}\right)^{1/q} \leqslant C\|\nabla \phi_u\|_2^{2/q},$$

so, by (3.1) and being $2(q^*)' \in [2, 6]$, it follows

$$\begin{aligned}\|\nabla \phi_u\|_2^2 &\leqslant C\int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla \phi_u|^2}\right) \leqslant C\int_{\mathbb{R}^3} \phi_u u^2 \\ &\leqslant C\|\phi_u\|_{q^*}\|u\|^2_{2(q^*)'} \leqslant C\|\nabla \phi_u\|_2^{2/q}\|u\|^2_{2(q^*)'}\end{aligned}$$

and we get the conclusion. ∎

When $p \in (5/2, 5)$ the proof of the main result involves the following monotonicity trick. See [17, 27].

Proposition 4.2. *Let $(X, \|\cdot\|)$ be a Banach space and $J \subset \mathbb{R}^+$ an interval. Consider a family of C^1 functionals I_λ on X defined by*

$$I_\lambda(u) = A(u) - \lambda B(u), \qquad \text{for } \lambda \in J,$$

with B non-negative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$ and such that $I_\lambda(0) = 0$. For any $\lambda \in J$, we set

$$\Gamma_\lambda := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\}.$$

Assume that for every $\lambda \in J$, the set Γ_λ is non-empty and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0.$$

Then for almost every $\lambda \in J$, there is a sequence $(v_n)_n \subset X$ such that

(i) *$\{v_n\}_n$ is bounded in X;*

(ii) *$I_\lambda(v_n) \to c_\lambda$, as $n \to +\infty$;*

(iii) *$I'_\lambda(v_n) \to 0$ in the dual space X^{-1} of X, as $n \to +\infty$.*

The importance of the previous result is that it permits to obtain a critical point of our functional I by means of "approximation". Indeed, instead of working with the original functional I we deal with a slight perturbation I_λ for which one has a bounded Palais-Smale sequence (whenever the above result applies), and after some computations, a critical point u_λ. Of course this critical point u_λ does not solve our equation, but just an approximated one, due to the presence of the parameter λ.

Then the idea is to send λ to 1 (since in this case we recover our original functional I) and control the behaviour of $\{u_\lambda\}$ in order to achieve a critical point of I. The advantage of this procedure is that our "approximating sequence" $\{u_\lambda\}$ satisfies an identity $I'_\lambda(u_\lambda) = 0$ which reveals very useful in the computations.

So, in our case $X = H^1_r(\mathbb{R}^3)$

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi_u u^2 - \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \sqrt{1 - |\nabla\phi_u|^2}\right),$$

$$B(u) = \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}.$$

and by (3.1), $A(u) \to +\infty$ as $\|u\| \to +\infty$.

Then we look for bounded Palais-Smale sequences of the following perturbed functionals

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2) + \frac{1}{2}\int_{\mathbb{R}^3}\phi_u u^2 \\ &- \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \sqrt{1 - |\nabla\phi_u|^2}\right) - \frac{\lambda}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}, \end{aligned}$$

for almost all λ near 1.

The fact that the above abstract Proposition 4.2 is applicable, is due to the following two results, in the first of which Lemma 4.1 has a main role.

Proposition 4.3. *For all $\lambda \in [1/2, 1]$, the set Γ_λ is not empty.*

Proof. Indeed fixed $\lambda \in [1/2, 1]$ and $u \in H^1_r(\mathbb{R}^3) \setminus \{0\}$, by Lemma 4.1 and for $q \in [2, 3)$, and standard inequalities we have

$$\begin{aligned} I_\lambda(u) &\leqslant \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}\phi_u u^2 - \frac{\lambda}{p+1}\|u\|^{p+1}_{p+1} \\ &\leqslant \frac{1}{2}\|u\|^2 + c\|\phi_u\|_6\|u\|^2_{\frac{12}{5}} - \frac{\lambda}{p+1}\|u\|^{p+1}_{p+1} \\ &\leqslant \frac{1}{2}\|u\|^2 + c\|\nabla\phi_u\|_2\|u\|^2 - \frac{\lambda}{p+1}\|u\|^{p+1}_{p+1} \\ &\leqslant \frac{1}{2}\|u\|^2 + c\|u\|^{\frac{3q-2}{q-1}} - \frac{\lambda}{p+1}\|u\|^{p+1}_{p+1}. \end{aligned}$$

Therefore, if $\lambda \in [1/2, 1]$ and $t > 0$, we infer that

$$I_\lambda(tu) \leqslant c_1 t^2 + c_2 t^{\frac{3q-2}{q-1}} - c_3\lambda t^{p+1}.$$

Since $p \in (5/2, 5)$, we can find $q \in [2, 3)$ such that $I_\lambda(tu) < 0$, for t sufficiently large. $\square$

Proposition 4.4. *For any $\lambda \in [1/2, 1]$, there exist $\alpha > 0$ and $\rho > 0$, sufficiently small, such that $I_\lambda(u) \geqslant \alpha$, for all $u \in H^1(\mathbb{R}^3)$, with $\|u\| = \rho$. As a consequence $c_\lambda \geqslant \alpha > 0$.*

Then a standard proof gives the following.

Proposition 4.5. *For almost every $\lambda \in J$, there exists $u_\lambda \in H^1_r(\mathbb{R}^3)$, $u_\lambda \neq 0$, such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$.*

In this way we have found a nontrivial solution u_λ of the following perturbed equation

$$-\Delta u + u + \phi_u u = \lambda |u|^{p-1} u \quad \text{in } \mathbb{R}^3 \tag{4.1}$$

for almost any value of λ near one.

As we said before, the next step is then to deduce the existence of a non-trivial critical point for I. This is done by using the important fact that u_λ satisfies the equation $I'_\lambda(u_\lambda) = 0$ and then, Nehari and Pohozaev identity are available

$$\frac{d}{dt} I_\lambda(tu(x))\Big|_{t=1} = 0 \quad \text{and} \quad \frac{d}{dt} I_\lambda(u(tx))\Big|_{t=1} = 0.$$

Indeed writing down and combining the above identities together and then passing to the limit as $\lambda \to 1$ we obtain a nontrivial critical point u^* of I.

At this stage we do not know if it is actually a ground state for I. But we define

$$\mathcal{S}_r := \left\{ u \in H^1_r(\mathbb{R}^3) \setminus \{0\} \mid I'(u) = 0 \right\} \neq \emptyset,$$
$$\sigma_r := \inf_{u \in \mathcal{S}_r} I(u).$$

The above infimum is strictly positive. In fact, any $u \in \mathcal{S}_r$ satisfies

$$\|u\|^2 \leqslant \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} u^2 + \int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1 - |\nabla \phi_u|^2}} = \int_{\mathbb{R}^3} |u|^{p+1} \leqslant C \|u\|^{p+1},$$

and therefore

$$\inf_{u \in \mathcal{S}_r} \|u\| > 0.$$

Finally from $I(u) \geqslant c\|u\|^2$ for all $u \in \mathcal{S}_r$, we conclude.

As a final step, one easily shows that the infimum is achieved and the existence of a ground state solution for ($\mathcal{SBI}$) is proved.

5 Sketch of proof of Theorem 2.3

To treat the case $p < 5/2$ we use a different approach. The previous one based on the monotonicity trick reveals not useful, since it was based on the fundamental Lemma 4.1 used at the end of the proof of Proposition 4.3. We just resume here the main points used in our arguments.

The "approximated" equation we use now is not (4.1), but the following one

$$-\Delta u + u + \phi_u u + \lambda\|u\|_2 u = |u|^{p-1}u + \lambda|u|^{q-1}u, \qquad u \in H^1_r(\mathbb{R}^3)$$

where

- $\lambda \in (0,1]$,
- $q \in (\max\{p+1, 4\}, 6)$.

In this context we consider the approximating functional

$$J_\lambda(u) = I(u) + \frac{\lambda}{3}\Big(\int_{\mathbb{R}^3} u^2\Big)^{3/2} - \frac{\lambda}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}$$

where I is the original functional defined before. The next step is then to pass, this time, to the limit as $\lambda \to 0^+$, instead of $\lambda \to 1$. What happens is that, after straightforward computations, the following properties hold:

- there is a Mountain Pass Geometry for J_λ which is uniform in λ;
- the Mountain Pass level $c_\lambda > 0$ is controlled:

$$0 < m \le c_\lambda \le M \quad \text{and} \quad c_\lambda \to c_* \text{ as } \lambda \to 0;$$

- J_λ satisfies the PS condition.

Indeed the following result holds

Lemma 5.1. *We have*

1. *there exist* $\rho, \delta > 0$ *such that, for any* $\lambda \in (0,1]$, $J_\lambda(u) \geq \delta$ *for every* $u \in S_\rho = \{u \in E : \|u\| = \rho\}$*;*

2. *there is* $v \in H^1_r(\mathbb{R}^3)$ *with* $\|v\| > \rho$ *such that, for any* $\lambda \in (0,1]$, $J_\lambda(v) < 0$.

The above properties were possible due to the "new" perturbation we used.

Then, having also the Palais-Smale condition satisfied, there is u_λ such that $J'_\lambda(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$.

Now the family $\{u_\lambda\}_{\lambda\in(0,1]}$ furnishes a bounded PS sequence for the original functional I, and after some computations there is a critical point u^* for I at the level c_*. The fact that I possesses a ground state is addressed as in the previous section.

Remark 5.2. We point out that in [18] a more general nonlinearity has been considered. Indeed we studied the problem

$$\begin{cases} -\triangle u + u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\mathrm{div}\left(\dfrac{\nabla \phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0, \ \phi(x) \to 0, & \text{as } x \to \infty, \end{cases}$$

with the following assumptions on the nonlinearity f:

1. $f \in C(\mathbb{R},\mathbb{R})$ and $\lim_{s\to 0} f(s)/s = 0$;

2. $|f(s)| \leq C(1 + |s|^p)$ for $p \in (2,5)$;

3. for any $s > 0$, $0 < \varrho F(s) \leq f(s)s$, where $\varrho \in (3,4)$ and $F(s) = \int_0^s f(\tau)d\tau$,

which are quite natural when dealing with variational methods. Moreover we get a multiplicity result of solutions, in fact the problem admits infinitely many solutions $(u_j, \phi_j) \subset H^1_r(\mathbb{R}^3) \times \mathcal{X}_r$ such that the energy functional F tends to infinity.

Finally it is worth to note that also the critical case is treated. However a further assumption due to compactness issues is necessary:

4. there exist $D > 0$ and $2 < r < 6$ such that $F(t) \geq Dt^r$ for $t \geq 0$.

We prove then that under the set of assumption (1)-(4) the system has a ground state solution if (i) $r \in (4, 6)$, or (ii) $r \in (2, 4]$ and D is sufficiently large.

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