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A survey on convex hypersurfaces of Riemannian manifolds

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. We survey the main extensions of the classical Hadamard, Liebmann and Cohn-Vossen rigidity theorems on convex surfaces of 3-Euclidean space to the context of convex hypersurfaces of Riemannian manifolds. The results we present include the one by Professor Renato Tribuzy (in collaboration with H. Rosenberg) on rigidity of convex surfaces of homogeneous 3-manifolds.

Keywords: convex hypersurface, Riemannian manifold.

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1 Introduction

Convexity is an ancient and fundamental geometric concept attributed to subsets of spaces. It was first considered by Archimedes in his celebrated book: *On the sphere and cylinder*, published in 225 B.C., which curiously has two of the most classical convex surfaces in its very title.

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In Euclidean space $\mathbb{R}^3$, a nonempty subset Ω is called *convex* if it contains the line segment joining any two of its points. A distinguished property of convex sets is that they can be characterized by the geometry of their boundaries. More precisely, a closed proper subset Ω in $\mathbb{R}^3$ is convex if and only if there exists at least one *supporting plane* Π at any point p on its boundary $\partial\Omega$, meaning that Ω lies in one of the half-spaces determined by $\Pi \ni p$ (see [34], pp. 137–140).

Due to this characterization of convex sets, a smooth embedded surface Σ of $\mathbb{R}^3$ is said to be *convex* if it is the boundary of an open convex set Ω in $\mathbb{R}^3$, which is then called a *convex body*. Planes, spheres, and right circular cylinders are canonical examples of convex surfaces. The *saddle*, graph of the function $z = x^2 - y^2$, is a typical example of an embedded non convex surface of $\mathbb{R}^3$.

Clearly, if $\Sigma = \partial\Omega$ is a convex surface, the supporting planes of Ω are precisely the tangent planes of Σ. In particular, for any $p \in \Sigma$, a local graph $\Sigma' \subset \Sigma$ through p — defined over an open set of $T_p\Sigma$ — is necessarily contained in one of the closed half-spaces determined by $T_p\Sigma$. This property, being local, can also be attributed to immersed surfaces (possibly having self-intersections) and is called *local convexity*.

The formula for the Gaussian curvature of a graph immediately gives that a locally convex surface Σ has nonnegative Gaussian curvature. (The converse does not hold, as we shall show in the next section.) In addition, the Gaussian curvature of Σ at a point p is positive if and only if the local graph Σ', as described above, intersects $T_p\Sigma$ only at p. If so, we say that p is an *elliptic point*, and also that Σ is *strictly convex* at p. Thusly, spheres are strictly convex everywhere, whereas right circular cylinders are nowhere strictly convex.

In [22], J. Hadamard established the striking fact that compact locally strictly convex surfaces in $\mathbb{R}^3$ are necessarily embedded, convex, and diffeomorphic to the unit sphere $\mathbb{S}^2$. The precise statement is as follows.

Hadamard Theorem. *Let Σ be a compact connected smooth surface immersed in Euclidean space $\mathbb{R}^3$ with positive Gaussian curvature. Then, Σ is* *orientable, embedded, and convex. In addition, its Gauss map $N : \Sigma \to \mathbb{S}^2$ is a diffeomorphism.*

J. Stoker [44] extended Hadamard Theorem by proving that, if Σ is a complete noncompact immersed surface in $\mathbb{R}^3$ with positive Gaussian curvature, then it is a graph over a convex open set in $\mathbb{R}^2$. Usually, these two results are put together and then called the Hadamard–Stoker Theorem.

A compact connected surface of $\mathbb{R}^3$ with positive Gaussian curvature is called an *ovaloid*. So, Hadamard Theorem tells us that ovaloids are embedded, convex and diffeomorphic to $\mathbb{S}^2$. Considering this fact, it is natural to ask on which conditions an ovaloid is necessarily a round (i.e., totally umbilical) sphere of $\mathbb{R}^3$.

H. Liebmann [32] addressed this question and provided an answer by proving that an ovaloid with either constant mean curvature or constant Gaussian curvature is a totally umbilical sphere. In the mean curvature case, a less prestigious – although stronger – result was obtained earlier by J. Jellett [28], who assumed the surface to be star shaped, instead of an ovaloid. For this reason, this Liebmann Theorem has been frequently referred to as the Liebmann–Jellett Theorem. In the Gaussian curvature case, a proof given by D. Hilbert (cf. [26, Appendix 5]) has imposed itself along the time, so that the result bears his name together with Liebmann's.

The deepest theorem regarding ovaloids is probably the one due to S. Cohn-Vossen [7], which asserts that such a surface is *rigid*. This means that, if Σ_1 and Σ_2 are isometric ovaloids, then they are congruent, that is, they coincide up to a rigid motion of $\mathbb{R}^3$. In fact, Cohn-Vossen proved his theorem assuming Σ_1 and Σ_2 analytic. Afterwards, G. Herglotz [25] provided a succinct alternate proof under the weaker assumption that Σ_1 and Σ_2 are of class C^2.

Since their appearance to date, these classical rigidity theorems on convex surfaces by Hadamard, Liebmann, and Cohn-Vossen have been extended, in many ways, to the more general context of hypersurfaces in Riemannian manifolds. In what follows, we survey the main results obtained on this matter, which have a significant contribution of Brazilian mathematicians — most notably Manfredo do Carmo — as we shall see.

We will also take this opportunity to present, in our last section, a result by Professor Renato Tribuzy in collaboration with H. Rosenberg on rigidity of convex surfaces of homogeneous 3-manifolds.

2 Preliminaries

Throughout the paper, unless otherwise stated, the Riemannian manifolds we consider are all orientable, smooth (of class C^∞), and of dimension at least 2. Given a Riemannian manifold $\overline{M}^{n+1}$, we will consider its hypersurfaces mostly as isometric immersions

$$f : M^n \to \overline{M}^{n+1},$$

where M^n is some n-dimensional Riemannian manifold.

Occasionally, we shall consider isometric immersions $f : M^n \to \overline{M}^{n+p}$ of codimension $p \geq 1$. In this setting, we will write TM and $TM^\perp$ for the tangent bundle and normal bundle of f, respectively, and

$$\alpha_f : TM \times TM \to TM^\perp$$

for its second fundamental form, that is,

$$\alpha_f(X, Y) = \overline{\nabla}_X Y - \nabla_X Y,$$

where $\overline{\nabla}$ and ∇ denote the Riemannian connections of $\overline{M}$ and M, respectively. Also, given $\xi \in TM^\perp$, we define $A_\xi : TM \to TM$ by

$$A_\xi X = -(\text{tangential component of } \overline{\nabla}_X \xi)$$

and call it the *shape operator* of f in the normal direction ξ.

As is well known, A_ξ is self-adjoint, so that, for each $x \in M$, there exists an orthonormal basis $\{X_1, \dots, X_n\} \subset T_x M$ of eigenvectors of A_ξ. Each vector X_i is called a *principal direction* of f at x (with respect to the normal ξ), and the corresponding eigenvalue λ_i is called the *principal curvature* of f at x (with respect to the normal ξ). So, we have

$$A_\xi X_i = \lambda_i X_i, \quad i = 1, \dots, n.$$

The following equality relating the second fundamental form α_f and a shape operator A_ξ holds:

$$\langle \alpha_f(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle \;\; \forall\, X, Y \in TM, \;\xi \in TM^\perp,$$

where $\langle\; ,\; \rangle$ stands for the Riemannian metric of both M and $\overline{M}$.

The second fundamental form α_f of an isometric immersion $f : M^n \to \overline{M}^{n+p}$ is said to be *semi-definite* if, for all $\xi \in TM^\perp$, the 2-form

$$(X,Y) \in TM \times TM \mapsto \langle \alpha_f(X,Y), \xi \rangle$$

is semi-definite (i.e., the nonzero eigenvalues of the shape operator A_ξ have all the same sign) on M. Also, we say that α_f is *positive* (respect. *negative*) *semi-definite* in the normal direction $\xi \in TM^\perp$ if the nonzero eigenvalues of A_ξ are all positive (respect. negative) on M.

We remark that, in the above setting, the semi-definiteness of the second fundamental form α_f does not imply that it is either positive or negative semi-definite in a given normal direction $\xi \in TM^\perp$. To see that in the case of hypersurfaces, consider a smooth plane curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$, and define the *cylinder over* γ as the immersion

$$\begin{array}{rccc} f\colon & I \times \mathbb{R}^{n-1} & \to & \mathbb{R}^2 \times \mathbb{R}^{n-1} \\ & (t,x) & \mapsto & (\gamma(t), x), \end{array}$$

which is easily seen to be an orientable hypersurface in $\mathbb{R}^{n+1}$. Also, for any of the two choices of a unit $\xi \in T(I \times \mathbb{R})^\perp$, one has

$$A_\xi = \pm \begin{bmatrix} k & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix},$$

where k is the curvature of γ. In particular, α_f has semi-definite second fundamental form. In addition, assuming that there exist $t_1, t_2 \in I$ satisfying $k(t_1) < 0 < k(t_2)$, we have that, in any normal direction $\xi \in T(I \times \mathbb{R})^{\perp}$, α_f is positive semi-definite at (t_1, x) if and only if it is negative semi-definite at (t_2, x). Therefore, α_f is neither positive semi-definite nor negative semi-definite in any normal direction (Fig. 2.1).

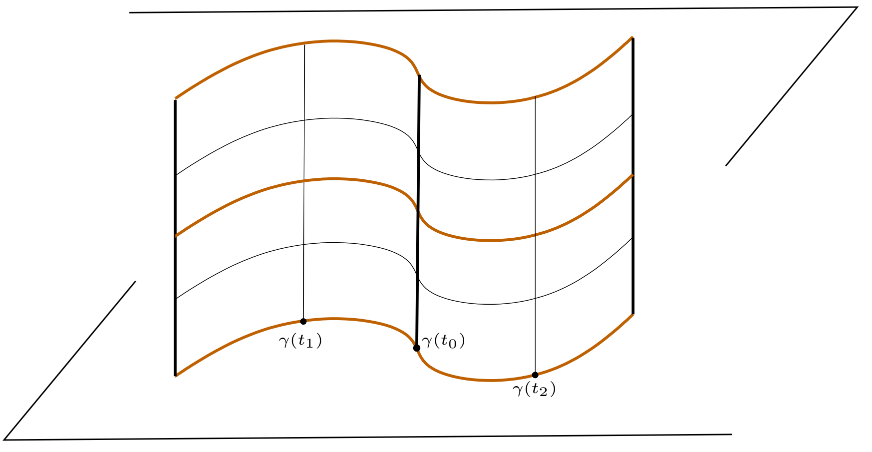

Figure 2.1: Non locally convex cylinder over a plane curve γ.

It should also be noticed that, for $n = 2$, $\Sigma := f(I \times \mathbb{R})$ is not locally convex (as we discussed in the introduction) at the points $f(t_0, x) \in \Sigma$ such that $k(t_0) = 0$, and that the Gaussian curvature Σ vanishes identically (since the same is true for one of its principal curvatures). Thus, in $\mathbb{R}^3$, nonnegativity of Gaussian curvature does not imply local convexity.

It is time for us to settle the concept of local convexity for a general orientable hypersurface $f\colon M^n \to \overline{M}^{n+1}$, and establish its relations with the second fundamental form α_f. With this purpose, let us first observe that, for any point $x \in M$, there is an open neighborhood U of x in M such that $\Sigma := f(U)$ is a graph of a function ϕ (under the normal exponential map of $\overline{M}^{n+1}$) over an open set $V \ni 0$ of the tangent space $f_*(T_xM)$. In this context, f is said to be *locally convex* (resp. *locally strictly convex*) at

x if, for a suitable normal direction $\xi \in TM^\perp$, the function ϕ is nonnegative on V (resp. positive on $V - \{0\}$).

A hypersurface $f\colon M^n \to \overline{M}^{n+1}$ is then called *locally convex* (resp. *locally strictly convex*) if it is locally convex (resp. locally strictly convex) at any point of M.

Following a suggestion by M. do Carmo, R. Bishop showed in [5] that local convexity is equivalent to semi-positiveness of the second fundamental form, as stated below.

Theorem 2.1 (Bishop [5])**.** *An orientable hypersurface $f\colon M^n \to \overline{M}^{n+1}$ is locally convex (resp. locally strictly convex) if and only if, in a suitable normal direction $\xi \in TM^\perp$, the second fundamental form α_f is positive semi-definite (resp. positive definite).*

We point out that the concepts of convex set and convex body extend naturally to subsets of any totally convex Riemannian manifold $\overline{M}^{n+1}$. (Recall that a Riemannian manifold is called *totally convex* if there exists a unique geodesic joining any two of its points.)

Remark 2.2. By adapting an argument by J. Heijenoort [24], it is possible to show that, in any Hadamard manifold (i.e., complete simply connected Riemannian manifold with nonnegative sectional curvature) $\overline{M}^{n+1}$, a complete embedded hypersurface $f : M^n \to \overline{M}^{n+1}$ is locally convex if and only if $f(M)$ is the boundary of a convex body. On the other hand, as pointed out in [3, Remark 3], there exist simply connected Riemannian manifolds whose geodesic spheres are embedded and locally convex, yet they do not bound convex bodies.

We will denote by $\mathbb{Q}_\epsilon^{n+1}$ the $(n+1)$-dimensional simply connected space form of constant sectional curvature $\epsilon \in \{0, 1, -1\}$, i.e., the Euclidean space $\mathbb{R}^{n+1}$ ($\epsilon = 0$), the unit sphere $\mathbb{S}^{n+1}$ ($\epsilon = 1$), and the hyperbolic space $\mathbb{H}^{n+1}$ ($\epsilon = -1$).

Given an oriented hypersurface $f : M \to \mathbb{Q}_\epsilon^{n+1}$, let us see how the sectional curvature of M affects the semi-definiteness of the second fundamental form α_f. To this end, choose an orthonormal frame $\{X_1, \ldots, X_n\} \subset$

TM of principal directions of f with corresponding principal curvatures $\lambda_1, \dots, \lambda_n$. Denoting by K_M the sectional curvature of M, the well known Gauss equation for hypersurfaces of space forms yields (see, e. g., [9])

$$K_M(X_i, X_j) = \epsilon + \lambda_i \lambda_j \ \ \forall i \neq j \in \{1, \dots, n\}.$$

Therefore, *the second fundamental form of* $f : M \to \mathbb{Q}_\epsilon^{n+1}$ *is semi-definite if and only if* $K_M \geq \epsilon$.

We conclude this preliminary section by introducing the fundamental concept of rigidity of isometric immersions.

Definition 2.3. We say that an isometric immersion $f : M^n \to \overline{M}^{n+p}$ is *rigid* if, for any other isometric immersion $g : M^n \to \overline{M}^{n+p}$, there exists an ambient isometry $\Phi : \overline{M}^{n+p} \to \overline{M}^{n+p}$ such that $g = \Phi \circ f$.

3 Convex hypersurfaces of $\mathbb{Q}_\epsilon^{n+1}$

Regarding extensions of Hadamard, Liebmann and Cohn-Vossen Theorems to more general ambient manifolds, a first natural step is to verify their validity for hypersurfaces of the space forms $\mathbb{Q}_\epsilon^{n+1}$.

In the Euclidean case, based on a Hadamard–Stoker type theorem due to J. Heijenoort [24], and on a local rigidity theorem due to R. Beez [4] and W. Killing [30], R. Sacksteder [41, 42] succeeded to extend both Hadamard–Stoker and Cohn-Vossen Theorems to complete hypersurfaces of $\mathbb{R}^{n+1}$ with semi-definite second fundamental form, as stated below. We stress the fact that semi-definiteness of the second fundamental form is a weaker condition than local convexity, and that in both Hadamard–Stoker and Cohn-Vossen Theorems the surfaces are assumed to be strictly convex.

Theorem 3.1 (Sacksteder [41, 42])**.** *Let* $f\colon M^n \to \mathbb{R}^{n+1}$ *be an orientable, complete, and connected hypersurface with semi-definite second fundamental form, which is strictly convex at one point. Then, the following hold:*

i) f *is an embedding and* M *is homeomorphic to either* $\mathbb{S}^n$ *or* $\mathbb{R}^n$.

ii) *$f(M)$ is the boundary of a convex body in* $\mathbb{R}^{n+1}$.

iii) *f is rigid.*

The original proofs of these Sacksteder's results are rather involved. In [15], M. do Carmo and E. Lima provided a simpler proof of (i)–(ii) by means of Morse Theory. The main idea consists in considering the tangent space of $f(M)$ at the strictly convex point $f(x) \in f(M)$, and moving it in the direction of the inner normal $\xi(x)$ until it reaches a singularity, if any. With this approach, they showed that either no singularity occurs, in which case M is homeomorphic to $\mathbb{R}^n$ (in fact, $f(M)$ is a graph over a convex open set in a hyperplane orthogonal to $\xi(x)$ in $\mathbb{R}^{n+1}$), or the singular set is a singleton, in which case $f(M)$ is an embedded topological n-sphere of $\mathbb{R}^{n+1}$. Once established the embeddedness of f, the proof of (ii) is standard.

For the proof of the rigidity assertion (iii) of Theorem 3.1, one can argue as follows. Since f has semi-definite second fundamental form, Gauss equation gives that M has nonnegative sectional curvature. Hence, any hypersurface $g : M^n \to \mathbb{R}^{n+1}$ has semi-definite second fundamental form. Assuming $n \geq 3$, and recalling that f is strictly convex at a point, one has that the set $M' \subset M$ on which the second fundamental form α_f of f has rank at least 3 is nonempty. In this case, the Beez–Killing Theorem we mentioned asserts that $f|_{M'}$ is rigid. From this, and the semi-definiteness of α_f and α_g, one has either $\alpha_f = \alpha_g$ or $\alpha_f = -\alpha_g$ on M'. The same conclusion holds in the case $n = 2$ from a result by Herglotz [25]. Now, Sacksteder's main result in [42] ensures that either of these equalities extends to the whole of M. The rigidity of f, then, follows from the Fundamental Theorem for hypersurfaces of Euclidean spaces.

The cylinders over curves (see Section 2) show that the hypothesis on the existence of an elliptic point is necessary in Sacksteder Theorem. Let us see that the same is true regarding completeness. Indeed, the graph of the function $z = x^3(1 + y^2)$, $|y| < 1/2$, is easily seen to be non convex along the line $x = 0$. However, its second fundamental form is semi-definite everywhere, as can be verified by a direct computation. Therefore, by Sacksteder Theorem, this graph cannot be a part of a complete surface with semi-definite second fundamental form.

Inspired by Sacksteder's results, M. do Carmo and F. Warner considered the analogous problem for compact hypersurfaces of $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$, obtaining the following

Theorem 3.2 (do Carmo – Warner [16]). *Let $f : M^n \to \mathbb{S}^{n+1}$ be a non-totally geodesic hypersurface with semi-definite second fundamental form, where M^n is a compact, connected, and orientable Riemannian manifold. Then, the following hold:*

i) *f is an embedding and M is homeomorphic to $\mathbb{S}^n$.*

ii) *$f(M)$ is the boundary of a convex body contained in an open hemisphere of $\mathbb{S}^{n+1}$.*

iii) *f is rigid.*

Moreover, the assertion (i) *and the convexity property in* (ii) *still hold if one replaces the sphere $\mathbb{S}^{n+1}$ by the hyperbolic space $\mathbb{H}^{n+1}$.*

The proof of do Carmo–Warner Theorem relies on the properties of the so-called *Beltrami maps*, which are central projections of open hemispheres of $\mathbb{S}^{n+1}$ (respect. hyperbolic space $\mathbb{H}^{n+1}$) on suitable hyperplanes of Euclidean space $\mathbb{R}^{n+2}$ (respect. Lorentz space $\mathbb{L}^{n+2}$).

More precisely, in the spherical case, the Beltrami map for the hemisphere $\mathcal{H}$ of $\mathbb{S}^{n+1}$ centered at $e_{n+2} := (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$ is

$$\begin{array}{rccc} \varphi : & \mathcal{H} & \to & \mathbb{R}^{n+1} \\ & x & \mapsto & \frac{x}{x_{n+2}}\,, \end{array} \tag{3.1}$$

where x_{n+2} stands for the $(n+2)$-th coordinate of x in $\mathbb{R}^{n+2}$ (Fig. 3.1).

Analogously, in the hyperbolic case, the Beltrami map is given by

$$\begin{array}{rccc} \varphi : & \mathbb{H}^{n+1} & \to & B^{n+1} \\ & x & \mapsto & \frac{x}{x_{n+2}}\,, \end{array} \tag{3.2}$$

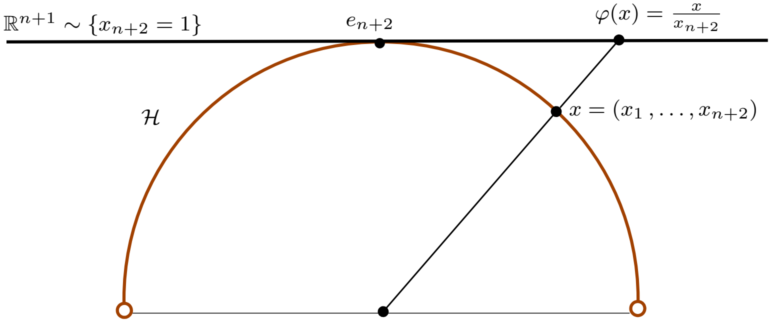

Figure 3.1: Beltrami map of the hemisphere $\mathcal{H}$ centered at e_{n+2}.

where B^{n+1} stands for the unit ball of the affine subspace $x_{n+2} = 1$ of $\mathbb{L}^{n+2}$ (Fig. 3.2).

It is easily verified that, in both cases, the Beltrami map φ is a diffeomorphism. Moreover, φ and its inverse are geodesic maps, that is, they take geodesics to geodesics and, in particular, convex sets to convex sets.

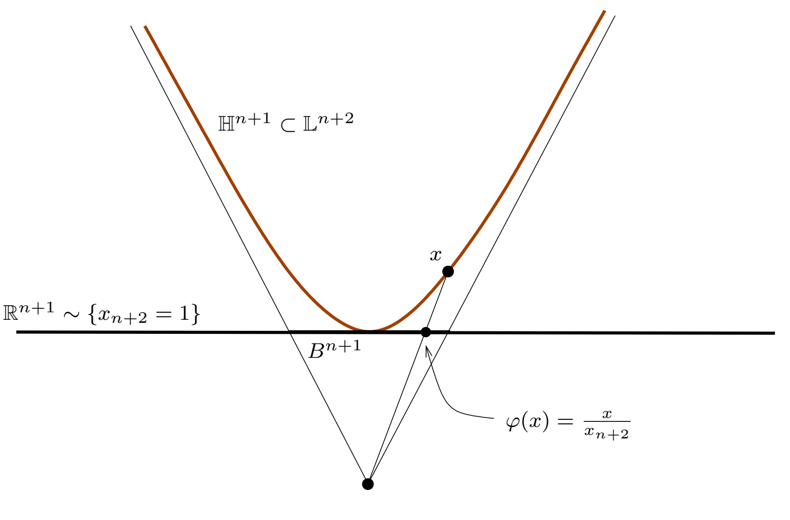

Figure 3.2: Beltrami map of hyperbolic space $\mathbb{H}^{n+1}$.

In their proof, do Carmo and Warner cleverly used Beltrami maps combined with Sacksteder Theorem 3.1 to show that, in the spherical case, there exists a point $x \in M$ at which the second fundamental form of f is definite, and also that $f(M) - \{f(x)\}$ is contained in an open hemisphere $\mathcal{H}'$ of $\mathbb{S}^{n+1}$ whose boundary meets $f(M)$ at $f(x)$. From this, since M is compact, they concluded that $f(M)$ is contained in an open hemisphere $\mathcal{H}$ of $\mathbb{S}^{n+1}$. Considering now both cases, spherical and hyperbolic, and denoting by φ the Beltrami map of either $\mathcal{H}$ or $\mathbb{H}^{n+1}$, as they show, $\varphi \circ f : M \to \mathbb{R}^{n+1}$ is complete and convex. Then, from Sacksteder Theorem, $\varphi \circ f$ is an embedding and $\varphi(f(M))$ is the body of a convex body in $\mathbb{R}^{n+1}$, which implies that the same is true for f and $f(M)$, respectively.

A similar idea, involving special mixed Beltrami-like maps, was used for the proof of the rigidity of f in the spherical case. At the very end of the paper, do Carmo and Warner conjectured that the rigidity of f holds in the hyperbolic case as well.

In [14], jointly with R. de Andrade, the author extended do Carmo and Warner theorem to immersions of arbitrary codimension, and also settled affirmatively their aforementioned conjecture. More precisely, the following result was obtained.

Theorem 3.3 (Andrade – de Lima [14]). *Let $f : M^n \to \mathbb{Q}_\epsilon^{n+p}$ be an isometric immersion of a compact connected Riemannian manifold into the $(n+p)$-dimensional space form of constant sectional curvature $\epsilon = \pm 1$. Assume that f is non-totally geodesic and has semi-definite second fundamental form. Under these conditions, the following assertions hold:*

i) *f is an embedding of M into a totally geodesic $(n+1)$-dimensional submanifold $\mathbb{Q}_\epsilon^{n+1} \subset \mathbb{Q}_\epsilon^{n+p}$.*

ii) *$f(M)$ is the boundary of a compact convex set of $\mathbb{Q}_\epsilon^{n+1}$. In particular, M is homeomorphic to $\mathbb{S}^n$.*

iii) *f is rigid.*

In the proof of Theorem 3.3, as in do Carmo–Warner Theorem, we make use of the Beltrami maps φ to recover the Euclidean situation. A fundamental result, which allowed us to apply this method, is the following relation we obtained between the second fundamental forms α_f and α_g of f and $g := \varphi \circ f \colon M \to \mathbb{R}^{n+1}$:

$$\langle \alpha_f(f_*X, f_*Y), \xi_f \rangle = \phi^2 \langle \alpha_g(g_*X, g_*Y), \xi_g \rangle, \ \ X, Y \in T\mathcal{H}^{n+p}, \tag{3.3}$$

where ξ_f and ξ_g are (bijectively related) normal fields to f and g, respectively, $\mathcal{H}^{n+p}$ denotes either an open hemisphere of $\mathbb{S}^{n+p} \subset \mathbb{R}^{n+p+1}$ or $\mathbb{H}^{n+p} \subset \mathbb{L}^{n+p+1}$ (with standard metric $\langle \, , \, \rangle$), and ϕ stands for the function

$$x = (x_1, \dots, x_{n+p+1}) \in \mathcal{H}^{n+p} \mapsto \phi(x) := 1/x_{n+p+1} \,.$$

It should be noted that no such relation was established by do Carmo and Warner for hypersurfaces, i.e., for $p = 1$.

In the spherical case, by means of a result due to Dajczer and Gromoll [10], we show that there is a point $x \in M$ at which the second fundamental form of f is positive definite. Then, considering the Beltrami map for the hemisphere of $\mathbb{S}^{n+p}$ centered at x, together with a reduction of codimension theorem for hypersurfaces of Euclidean space, due to Jonker [29], we show (i) and (ii) for $\epsilon = 1$. The case $\epsilon = -1$ is analogous.

As for the rigidity of f, we saw that the spherical case was settled by do Carmo and Warner. For the hyperbolic case, we again use Beltrami maps and identity (3.3) to show that the set of totally umbilical points of f does not disconnect M. In this case, for $n > 2$, the rigidity of f follows from another celebrated rigidity theorem by Sacksteder [42] (see [9, Theorem 6.14] for an alternate proof). The case $n = 2$ was proved by V. Fomenko and G. Gajubovin (cf. [20, Theorem 5]).

The technique of combining Beltrami maps with equality (3.3) also gives the following result, whose first part constitutes a Hadamard-type theorem. (The final statement follows from Theorem 3.3.)

Theorem 3.4 (Andrade – de Lima [14]). *Let $f : M^n \to \mathcal{H}^{n+1}$ be a compact connected hypersurface, where $\mathcal{H}^{n+1}$ is either the open hemisphere of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ centered at $e_{n+2} := (0, 0, \dots 0, 1)$ or the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$. Then, the following assertions are equivalent:*

i) *f is locally strictly convex.*

ii) *The Gauss-Kronecker curvature of f is nowhere vanishing.*

iii) *M is orientable and, for a unit normal field ξ on M, the map*

$$\begin{array}{rccc} \psi: & M^n & \to & \mathbb{S}^n \\ & x & \mapsto & \frac{\xi(x)-\langle\xi(x),e_{n+2}\rangle e_{n+2}}{\sqrt{1-\langle\xi(x),e_{n+2}\rangle^2}} \end{array}$$

is a well-defined diffeomorphism, where $\mathbb{S}^n$ stands for the n-dimensional unit sphere of the Euclidean orthogonal complement of e_{n+2} in $\mathbb{R}^{n+2}$.

Furthermore, any of the above conditions implies that f is rigid and embeds M onto the boundary of a compact convex set in $\mathcal{H}^{n+1}$.

A natural question arises from do Carmo–Warner Theorem: Could one replace compact by complete in its statement and still get to the same conclusions? In the spherical case, the question is irrelevant, since any sectional curvature of a convex hypersurface $f: M \to \mathbb{S}^{n+1}$ is at least 1, by Gauss equation. Therefore, by Bonnet–Myers Theorem, M is necessarily compact if the induced metric is complete. In the hyperbolic case, it is known that there exist complete convex hypersurfaces of $\mathbb{H}^{n+1}$ which are not embedded (see [43], pg. 124). So, in this case, the answer for the above question is negative.

In [8], J. Currier considered strictly convex hypersurfaces of $\mathbb{H}^{n+1}$ whose principal curvatures are at least 1. At points where all the principal curvatures are greater than 1, these hypersurfaces are locally supported by horospheres of $\mathbb{H}^{n+1}$. This means that, at such a point, call it x, the hypersurface is tangent to a horosphere Σ of $\mathbb{H}^{n+1}$ and, except for x itself, a neighborhood of x in the hypersurface lies in the convex connected component of $\mathbb{H}^{n+1} - \Sigma$. This geometric property allowed Currier to apply Morse Theory in the same way do Carmo and Lima did in [15], showing that such a hypersurface is necessarily embedded and rigid, being either a topological n-sphere bounding a convex body or a horosphere itself.

Again by using Beltrami maps, we were able to obtain in [14] the following version of Currier's result in arbitrary codimension, as stated below.

Theorem 3.5 (Andrade – de Lima [14]). *Let $f : M^n \to \mathbb{H}^{n+p}$ be an isometric immersion of an orientable complete connected Riemannian manifold M^n into the hyperbolic space $\mathbb{H}^{n+p}$. Assume that there is an orthonormal frame $\{\xi_1, \dots, \xi_p\}$ in $TM^\perp$ such that all the eigenvalues of the shape operators A_{ξ_i} are at least 1. Then, f admits a reduction of codimension to one, $f : M^n \to \mathbb{H}^{n+1}$. As a consequence, the following assertions hold:*

i) *f is and embedding and $f(M)$ is the boundary of a convex body in $\mathbb{H}^{n+1}$.*

ii) *M is homeomorphic to either $\mathbb{S}^n$ or $\mathbb{R}^n$.*

iii) *f is rigid and $f(M)$ is a horosphere if M is noncompact.*

Let us consider now extensions of Liebmann's theorems to $\mathbb{Q}_\epsilon^{n+1}$.

In a series of outstanding papers published between 1956 and 1962, A. Alexandrov studied constant mean curvature hypersurfaces of Euclidean space $\mathbb{R}^{n+1}$, establishing the fundamental result that such an embedded compact hypersurface is a totally umbilical n-sphere. The ingenious method employed in his proof, now called *Alexandrov reflection technique*, became one of the most powerful tools in the study of a large class of hypersurfaces of Riemannian manifolds having suitable groups of isometries. (We shall have a glimpse of this phenomenon in the next section.)

Given a constant mean curvature embedding $f : M^n \to \mathbb{R}^{n+1}$, with M compact, the Alexandrov reflection technique consists in considering an arbitrary hyperplane Π disjoint from $\Sigma := f(M)$, which moves towards it until it becomes tangent at a point $p \in \Sigma$. Then, it is shown that there exists a hyperplane parallel to $T_p\Sigma$ with respect to which Σ is symmetric. Since Π is arbitrary, Σ must be a round sphere.

Let us recall that, given an integer $k \in \{0, 1, \dots, n\}$, the k-th mean curvature of a hypersurface $f: M^n \to \overline{M}^{n+1}$ is the function

$$H_k := \binom{n}{k}^{-1} \sigma_k(\lambda_1, \dots, \lambda_n),$$

where $\lambda_1, \dots, \lambda_n$ are the principal curvatures of f, and σ_k is defined as

$$\sigma_k(\lambda_1, \dots, \lambda_n) := \begin{cases} 1 & \text{if } k = 0. \\ \displaystyle\sum_{i_1 < \cdots < i_r} \lambda_{i_1} \dots \lambda_{i_r} & \text{if } 1 \leq k \leq n. \end{cases}$$

In particular, H_1 is the mean curvature and H_n is the Gauss-Kronecker curvature. When $n = 2$, $H_2 = k_1 k_2$ is also known as the extrinsic curvature or Gaussian curvature. We will use this last terminology for the sequence of the paper in cases where $n = 2$.

Definition 3.6. When the k-th mean curvature of $f: M^n \to \overline{M}^{n+1}$ is a constant H_k, we say that f is an *H_k-hypersurface.*

In [31], N. Korevaar showed that the Alexandrov reflection technique works for H_k-hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ or in open hemispheres of $\mathbb{S}^{n+1}$. By combining his results with Sacksteder and do Carmo-Warner Theorems, one easily extends both Liebmann's theorems to $\mathbb{Q}_\epsilon^{n+1}$ as follows (see also [35]).

Theorem 3.7 (Generalized Liebmann Theorem)**.** *Let M^n be a compact orientable Riemannian manifold. If $f: M \to \mathbb{Q}_\epsilon^{n+1}$ is a locally strictly convex H_k-hypersurface for some $k \in \{1, \dots, n\}$, then $f(M)$ is a geodesic sphere of $\mathbb{Q}_\epsilon^{n+1}$.*

In Euclidean space $\mathbb{R}^3$, the Liebmann's theorems can also be proved by means of the celebrated *Minkowski formulas* (see, e.g., [36]). In fact, this method can be adapted for proving Theorem 3.7. Since we have no reference for such a proof, we shall present it here.

We will make use of the well known fact that the mean curvature functions H_k satisfy the inequality (see, e.g., [23] pg. 52)

$$H_k^{1/k} \geq H_{k+1}^{1/(k+1)} \ \ \forall k \in \{1, \dots, n-1\}, \tag{3.4}$$

and that the equality occurs if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n$.

Considering polar coordinates (r, θ) in $\mathbb{Q}_\epsilon^{n+1}$, define the *position vector* $P = s_\epsilon(r)\partial_r$, where

$$s_\epsilon(r) := \begin{cases} \sin r & \text{if} \quad \epsilon = 1. \\ r & \text{if} \quad \epsilon = 0. \\ \sinh r & \text{if} \quad \epsilon = -1. \end{cases}$$

Also, for a given compact oriented hypersurface $f : M \to \mathbb{Q}_\epsilon^{n+1}$ with inward unit normal ξ, define the *support function* $\mu = \langle P, \xi\rangle$. In this setting, the following Minkowski identity holds (cf. [1, Theorem 3.2]):

$$\int_M (c_\epsilon H_{k-1} + \mu H_k) dM = 0, \quad k = 1, \ldots, n, \tag{3.5}$$

where $c_\epsilon = c_\epsilon(r) := s'_\epsilon(r)$, and dM stands for the volume element of M. (The identity (3.5) differs from the one in [1, Theorem 3.2], where the minus sign replaces the plus sign. This is due to the fact that the definition of shape operator in [1] differs from ours by a sign. More precisely, in [1], the shape operator A_ξ is defined as $A_\xi X = \overline{\overline{\nabla}}_X \xi$, whereas we define it as $A_\xi X = -\overline{\overline{\nabla}}_X \xi$.)

Proof of Theorem 3.7. By Sacksteder and do Carmo–Warner Theorems, f is an embedding and M is homeomorphic to $\mathbb{S}^n$. We can assume f with the inward orientation, so that the support function μ is negative on M. Also, for $\epsilon = 1$, $f(M)$ is in an open hemisphere of $\mathbb{S}^{n+1}$, which implies that, for suitable polar coordinates in $\mathbb{S}^{n+1}$, one has $c_1(r) = \cos r > 0$ on $f(M)$. For $\epsilon \in \{0, -1\}$, it is clear that $c_\epsilon > 0$.

Let us consider first the case where f is an H-hypersurface. Considering the Minkowski equalities for $k = 1$ (multiplied by the constant $H > 0$) and for $k = 2$, we have that

$$\int_M (c_\epsilon H + \mu H^2) dM = 0 \quad \text{and} \quad \int_M (c_\epsilon H + \mu H_2) dM = 0.$$

Subtracting these equalities, we get

$$\int_M \mu(H^2 - H_2) dM = 0.$$

By (3.4), the integrand in this last integral is non positive. Therefore, $H^2 = H_2$ on M, which implies that f is totally umbilical, i.e., $f(M)$ is a geodesic sphere of $\mathbb{Q}_\epsilon^{n+1}$.

Now, assume that f is an H_k-hypersurface for $1 < k \leq n$. Multiplying (3.5) by $1/H_k^{(k-1)/k}$ and considering that equality also for $k = 1$, we get

$$\int_M (c_\epsilon + \mu H) dM = 0 \quad \text{and} \quad \int_M \left(c_\epsilon H_{k-1}/H_k^{(k-1)/k} + \mu H_k^{1/k} \right) dM = 0.$$

As before, subtracting these equalities, we have

$$\int_M \left(c_\epsilon (H_k^{(k-1)/k} - H_{k-1})/H_k^{(k-1)/k} + \mu(H - H_k^{1/k}) \right) dM = 0.$$

Again, by (3.4), each summand in the integrand is non positive. So,

$$H_k^{(k-1)/k} - H_{k-1} = H - H_k^{1/k} = 0,$$

giving that f is totally umbilical. This finishes the proof. □

Alternate proofs of Hilbert-Liebmann Theorem in 3-space forms can be found in [2, 21].

4 Convex hypersurfaces of Riemannian manifolds

We continue the considerations of the previous section by presenting rigidity results for locally convex hypersurfaces in Riemannian manifolds of (possibly) nonconstant sectional curvature. We start with the following Hadamard-type theorem due to S. Alexander. (Recall that a Riemannian manifold $\overline{M}^{n+1}$ is called a *Hadamard manifold* if it is complete, simply connected, and has nonnegative sectional curvature.)

Theorem 4.1 (Alexander [3]). *Let $\overline{M}^{n+1}$ be a Hadamard manifold. Suppose that $f : M^n \to \overline{M}^{n+1}$ is an oriented, compact, connected and locally convex hypersurface. Then, M is diffeomorphic to $\mathbb{S}^n$, f is an embedding, and $f(M)$ is the boundary of a convex body in $\overline{M}^{n+1}$.*

Let us outline the elegant proof Alexander provided for her theorem.

Since M is compact, there exists an open geodesic ball B containing $f(M)$ which is totally convex, for $\overline{M}^{n+1}$ is a Hadamard manifold. Then, setting $\xi \in TM^{\perp}$ for the outward unit normal to f, for each $x \in M$, the geodesic γ_x of $\overline{M}^{n+1}$ issuing from x with velocity $\xi(x)$ intersects ∂B transversely at a point $P(x)$, defining a map $P : M \to \partial B$. By an extension of the Rauch comparison theorem, due to Warner [45], M has no focal points on γ_x, so that P is, in fact, a diffeomorphism. In particular, M is diffeomorphic to $\mathbb{S}^n$.

Now, assuming that f is not an embedding, consider the one-parameter family of parallel hypersurfaces $f_t : M^n \to \overline{M}^{n+1}$ given by

$$f_t(x) = \exp_{\overline{M}}(f(x), t\xi(x)), \;\; t \in [0, +\infty).$$

The local convexity of f and the Bishop Theorem imply that the second fundamental form α_f is negative semi-definite in the outward normal direction ξ. Also, a direct computation gives that the principal curvatures of f_t are decreasing functions of t, so that, for $t > 0$, the second fundamental form of f_t is negative semi-definite as well.

Now, for a suitable $x \in M$ satisfying $f(x) = f(y)$, $x \neq y \in M$, there exists an open neighborhood $U \subset M$ of x in M such that $f_t(U)$ intersects $f(M)$ for small t, and is disjoint from $f(M)$ for a sufficiently large t. Hence, for some $t_0 > 0$, $f_{t_0}(U)$ is tangent to $f(M)$ at a point $f_{t_0}(x_0) = f(y_0)$, $x_0, y_0 \in M$ (Fig. 4.1). Moreover, the local convexity of f_t and f_{t_0} gives that their outward unit normal vectors at x_0 and y_0 coincide. In particular, $P(x_0) = P(y_0)$, which contradicts the bijectivity of P. Thus, f is an embedding and $f(M)$ is the boundary of a convex body of $\overline{M}$ (cf. Remark 2.2), which concludes the proof.

In what concerns rigidity of convex surfaces of homogeneous 3-manifolds, one of the major results obtained was the following Hadamard-Stoker type theorem by J. M. Espinar, J. A. Gálvez, and H. Rosenberg. (We keep their terminology, considering surfaces as subsets rather than isometric immersions.)

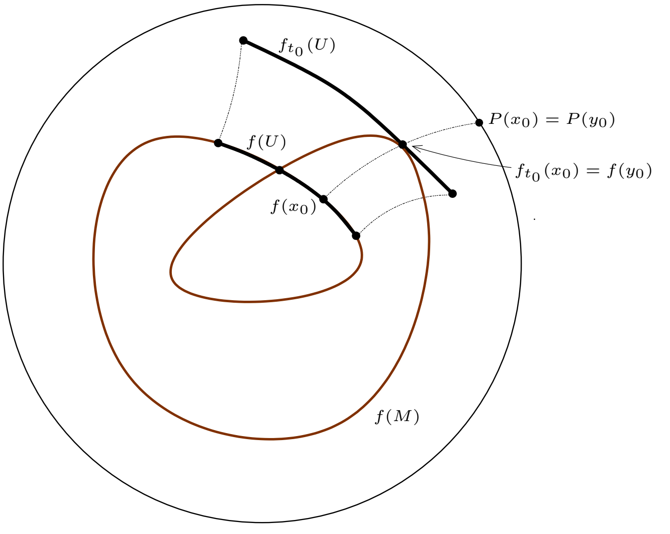

Figure 4.1: Proof of Alexander Theorem.

Theorem 4.2 (Espinar – Gálvez – Rosenberg [18]). *Let Σ be a complete connected immersed surface with positive Gaussian curvature in $\mathbb{H}^2 \times \mathbb{R}$. Then, Σ is properly embedded and bounds a convex body in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, Σ is homeomorphic to $\mathbb{S}^2$ or $\mathbb{R}^2$. In the latter case, Σ is a graph over a convex domain of $\mathbb{H}^2 \times \mathbb{R}$ or Σ has a simple end.*

By Σ having a simple end means that it has the following properties:

- The asymptotic boundary of the vertical projection of Σ over $\mathbb{H}^2$ is a singleton $\{p_\infty\}$.
- Given a complete geodesic γ in $\mathbb{H}^2$ whose "endpoints at infinity" are distinct from p_∞, the intersection of the *vertical plane* $\Gamma := \gamma \times \mathbb{R}$ with Σ is either empty or compact.

An explicit parametrization of a surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with positive Gaussian curvature and one simple end was provided in [18, Proposition 4.1]. Such a Σ is foliated by horizontal horocycles whose vertical projections on $\mathbb{H}^2$ have the same center p_∞ in the asymptotic boundary $\partial_\infty\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ (Fig. 4.2).

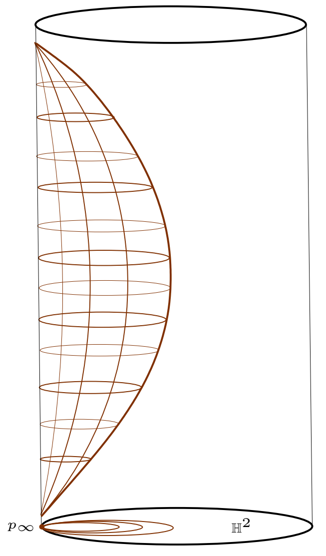

Figure 4.2: A strictly convex surface in $\mathbb{H}^2 \times \mathbb{R}$ with one simple end.

The proof of Theorem 4.2 given in [18] is divided into two parts. First, it is assumed that Σ has no vertical points, meaning that there is no $p \in \Sigma$ with $T_p\Sigma$ parallel to the vertical direction ∂_t. In this case, the intersection of a vertical plane $\Gamma = \gamma \times \mathbb{R}$ with Σ is a curve which is neither compact nor self-intersecting, since in either of these events Σ would necessarily have a vertical point. In addition, the strict convexity of Σ gives that any such embedded curve is strictly convex, that is, has positive geodesic curvature.

Now, fix a geodesic $\gamma^\perp = \gamma^\perp(s)$ of $\mathbb{H}^2$, orthogonal to γ at a point $q \in \gamma \cap \gamma^\perp$, and a family γ_s of parallel geodesics orthogonal to $\gamma^\perp$ with $\gamma_0 = \gamma$. Setting $\Gamma_s = \gamma_s \times \mathbb{R}$, one has that the intersections $\Sigma \cap \Gamma_s$ are either empty or embedded vertical graphs over γ_s. From this, one easily concludes that Σ is a graph over a convex open set of $\mathbb{H}^2$. (Notice that this graph is entire if $\Gamma_s \cap \Sigma$ is nonempty for all $s \in \mathbb{R}$.)

In the second part of the proof, it is assumed that there exists a vertical point $p \in \Sigma$. Under this assumption, writing Γ for the vertical plane tangent to Σ at p, and Γ_s for the family of parallel vertical planes with $\Gamma_0 = \Gamma$, one has that $C(s) := \Gamma_s \cap \Sigma$ is nonempty, embedded and homeomorphic to $\mathbb{S}^1$ for $s > 0$ sufficiently small. Then, a rather delicate reasoning via Morse Theory leads to the conclusion that one of the following possibilities occurs as $s \to +\infty$ (see Fig. 4.3):

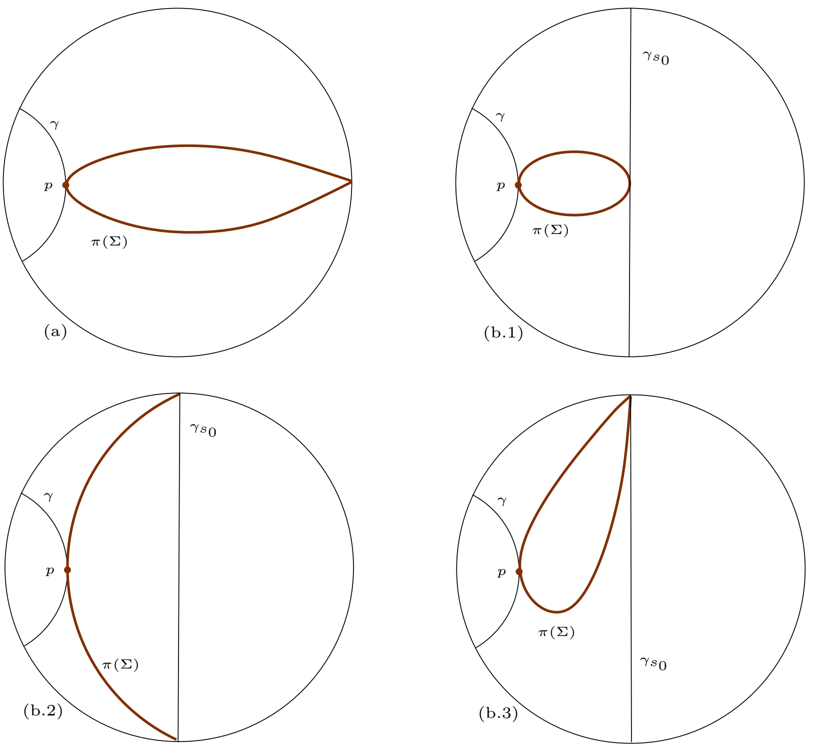

Figure 4.3: The projection $\pi(\Sigma)$ in the four cases of the proof of Theorem 4.2.

a) $C(s)$ is homeomorphic to $\mathbb{S}^1$ for all $s > 0$ — in which case Σ is homeomorphic to $\mathbb{R}^2$ and has a simple end.

b) $C(s)$ is nonempty and homeomorphic to $\mathbb{S}^1$ for all $s \in (0, s_0)$, and is empty for $s > s_0$. Then, as $s \to s_0$, one of the following occurs:

b.1) The curves $C(s)$ converge to a point $p \in \Sigma$ — in which case Σ is homeomorphic to $\mathbb{S}^2$.

b.2) The curves $C(s)$ converge to a curve in the asymptotic boundary of $\mathbb{H}^2 \times \mathbb{R}$ — in which case Σ is homeomorphic to $\mathbb{R}^2$.

b.3) The vertical projections $\pi(C(s))$ of $C(s)$ converge to a point in $\partial_\infty \mathbb{H}^2$ — in which case, as in (a), Σ is homeomorphic to $\mathbb{R}^2$ and has a simple end.

Finally, since Σ is embedded and locally strictly convex (and $\mathbb{H}^2 \times \mathbb{R}$ is a Hadamard manifold), it is necessarily the boundary of a convex body in $\mathbb{H}^2 \times \mathbb{R}$ (cf. Remark 2.2).

The main results in [18] also include horizontal and vertical height estimates for compact graphs of constant Gaussian curvature with boundary in vertical and horizontal planes, respectively. These estimates allow one to prove that a complete surface Σ in $\mathbb{H}^2 \times \mathbb{R}$ with positive constant Gaussian curvature cannot be homeomorphic to $\mathbb{R}^2$. This, together with Theorem 4.2, gives that such a Σ is embedded and homeomorphic to $\mathbb{S}^2$. Then, applying Alexandrov reflections on Σ with respect to vertical hyperplanes $\Gamma = \gamma \times \mathbb{R}$, one easily concludes that Σ must be a rotational sphere. Therefore, the following extension of the Hilbert-Liebmann Theorem for complete (rather than compact!) surfaces of $\mathbb{H}^2 \times \mathbb{R}$ holds.

Theorem 4.3 (Espinar – Gálvez – Rosenberg [18])**.** *A complete surface of $\mathbb{H}^2 \times \mathbb{R}$ with positive constant Gaussian curvature is a rotational sphere.*

Regarding the above theorem, it should be mentioned that, in [18], it was proved that there exists a unique (up to ambient isometries) rotational sphere with constant Gaussian curvature K in $\mathbb{H}^2 \times \mathbb{R}$ for any $K > 0$. Such a surface also exists in $\mathbb{S}^2 \times \mathbb{R}$, as shown by Cheng and Rosenberg [6]. Based on this fact, and by means of Alexandrov reflections, Espinar and Rosenberg [19] proved that Theorem 4.3 is valid in $\mathbb{S}^2 \times \mathbb{R}$ as well.

Nevertheless, a proof free of Alexandrov reflections (for both cases) was also provided in [18].

The method employed in the proof of Theorem 4.2 can be adapted to establish Hadamard–Stoker type theorems in other contexts. Relying on this fact, I. Oliveira and S. Schweitzer extended Theorem 4.2 to $\mathbb{H}^n \times \mathbb{R}$, $n \geq 2$. In this higher dimensional setting, Espinar and Rosenberg [19] managed to prove a Hadamard–Stoker type theorem for properly immersed, locally strictly convex hypersurfaces of $M^n \times \mathbb{R}$, where M^n is an arbitrary compact Riemannian manifold with (1/4)-pinched sectional curvature. Finally, a Hadamard–Stoker type theorem was obtained by Espinar and Oliveira [17] for certain locally strictly convex surfaces immersed in Killing submersions over Hadamard surfaces.

Inspired by these works, we obtained a Hadamard–Stoker type theorem for immersed locally strictly convex hypersurfaces of $\overline{M}^n \times \mathbb{R}$, $n \geq 3$, where $\overline{M}^n$ is either a Hadamard manifold or the unit sphere $\mathbb{S}^n$. Before stating it, let us recall that the height function of a hypersurface $f : M^n \to \overline{M}^n \times \mathbb{R}$ is the restriction to $f(M)$ of the projection $\pi_{\mathbb{R}}$ of $\overline{M}^n \times \mathbb{R}$ on its second factor $\mathbb{R}$.

Theorem 4.4 (de Lima [13]). *For $n \geq 3$, let $f : M^n \to \overline{M}^n \times \mathbb{R}$ be a complete connected oriented hypersurface with positive definite second fundamental form, where $\overline{M}^n$ is either a Hadamard manifold or the unit sphere $\mathbb{S}^n$. Then, if the height function of f has a critical point, f is a proper embedding and M is either homeomorphic to $\mathbb{S}^n$ or $\mathbb{R}^n$. In particular, $f(M)$ bounds a convex body of $\overline{M}^n \times \mathbb{R}$ in the case $\overline{M}$ is a Hadamard manifold.*

For the proof of Theorem 4.4, we apply Morse Theory together with do Carmo-Warner and Alexander Theorems to show that, under the given conditions, the height function of f has either a unique critical point, and then M is homeomorphic to $\mathbb{R}^n$, or precisely two critical points, and then M is homeomorphic to $\mathbb{S}^n$. In both cases, f is proved to be a proper embedding. (Our argument is based on the following fact: Any connected component of a transversal intersection $\Sigma_t := f(M) \cap (\overline{M}^n \times \{t\})$ is a hypersurface of the horizontal hyperplane $\overline{M}^n \times \{t\}$. In addition, such a hypersurface has positive-definite second fundamental form, since that holds for f. Hence, if Σ_t is compact, it is embedded and homeomorphic to $\mathbb{S}^{n-1}$, by do Carmo-Warner and Alexander Theorems.)

We also considered the dual case of Theorem 4.4, as stated below, in which the height function of the hypersurface has no critical points.

Theorem 4.5 (de Lima [13]). *For $n \geq 3$, let $f : M^n \to \overline{M}^n \times \mathbb{R}$ be a proper, connected, oriented, and cylindrically bounded hypersurface with positive semi-definite second fundamental form, where $\overline{M}^n$ is either a Hadamard manifold or the sphere $\mathbb{S}^n$. Then, f is an embedding, and $f(M) = \Sigma \times \mathbb{R}$, where $\Sigma \subset \overline{M}^n \times \{0\}$ is a submanifold homeomorphic to $\mathbb{S}^{n-1}$ which bounds a convex body in $\overline{M}^n \times \{0\}$.*

We recall that a hypersurface $f : M^n \to \overline{M}^n \times \mathbb{R}$ is said to be *cylindrically bounded* if there exists a geodesic ball $B \subset \overline{M}^n \times \mathbb{R}$ such that $f(M) \subset B \times \mathbb{R}$. Notice that, if $\overline{M}^n \times \mathbb{R}$ is compact, any hypersurface $f : M^n \to \overline{M}^n \times \mathbb{R}$ is cylindrically bounded.

Let us outline the proof of Theorem 4.5.

First, observe that the height function of f is unbounded, since it has no critical points and f is proper. Then, proceeding as in the proof of Theorem 4.4, we conclude from do Carmo–Warner and Alexander Theorems that $f(M)$ is foliated by embedded locally convex topological $(n-1)$-spheres, so that M is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$, and f is an embedding.

The local convexity of f implies that $f(M)$ is the boundary of a convex set of $\overline{M}^n \times \mathbb{R}$ in the case $\overline{M}^n$ is a Hadamard manifold. This, together with the cylindrical boundedness of f, easily implies that $f(M) = \Sigma \times \mathbb{R}$, as stated.

In the spherical case, we have to consider the Gauss formula for hypersurfaces $f : M \to \mathbb{S}^n \times \mathbb{R}$:

$$K_M(X, Y) = \det A|_{\mathrm{span}\{X,Y\}} + (1 - \|T(X, Y)\|^2), \ \ X, Y \in TM,$$

where A is the shape operator of f and $T(X,Y)$ is the orthogonal projection of the gradient T of its height function on $\text{span}\{X,Y\}$ (see [12]). We add that T itself is the orthogonal projection of the vertical unit field ∂_t on the tangent bundle TM.

From the above Gauss equation, the sectional curvature K_M of M is nonnegative, for $\det A|_{\text{span}\{X,Y\}} \geq 0$ (by the positive semi-definiteness of the second fundamental form of f), and $\|T(X,Y)\| \leq 1$. Since M is noncompact, for all $x \in M$, there exist orthonormal vectors $X, Y \in T_xM$ satisfying $K(X,Y) = 0$. Otherwise, the celebrated Soul Theorem, by T. Perelman [39], would give that M is homeomorphic to $\mathbb{R}^n$. By applying Gauss equation to (X,Y), one gets

$$\|T(X,Y)\|^2 = 1 + \det A|_{\text{span}\{X,Y\}} \geq 1.$$

Therefore, $\|T(X,Y)\| = 1$, which gives $T = \partial_t$, and then that $f(M)$ is a vertical cylinder over a convex hypersurface Σ of $\mathbb{S}^n$.

Once we have Theorem 4.4, we can proceed as in the proof of Theorem 4.3 to establish a Jellett–Liebmann type theorem in the products $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, $\epsilon \neq 0$. Namely, we perform Alexandrov reflections on a given compact hypersurface $f : M \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ with respect to vertical hyperplanes $\Gamma = \Gamma_0 \times \mathbb{R}$, where $\Gamma_0 \subset \mathbb{Q}_\epsilon^n$ is a totally geodesic hypersurface of Q_ϵ^n (for $\epsilon = 1$, Γ_0 is assumed to be in an open hemisphere of $\mathbb{S}^n$). The statement is as follows.

Theorem 4.6 (de Lima [13]). *For $n \geq 3$ and $\epsilon \in \{-1, 1\}$, any compact, connected, and locally strictly convex hypersurface $f : M^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$ with constant mean curvature is congruent to an embedded rotational sphere.*

Concerning Theorem 4.6, we add that Hsiang–Hsiang [27] (resp. R. Pedrosa [38]) constructed embedded and strictly convex rotational spheres in $\mathbb{H}^n \times \mathbb{R}$ (resp. $\mathbb{S}^n \times \mathbb{R}$) of constant mean curvature.

Theorem 4.4 can also be used to establish a Hilbert–Liebmann type theorem in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ for hypersurfaces of constant sectional curvature. For $n \geq 3$, such hypersurfaces were constructed and classified by F. Manfio and

R. Tojeiro in [33]. The following result, for which we provided a distinct proof, is part of their classification results.

Theorem 4.7 (de Lima [13], Manfio–Tojeiro [33]). *Let M_c^n be a complete, connected and orientable $n(\geq 3)$-dimensional Riemannian manifold with constant sectional curvature c. Given an isometric immersion $f : M_c^n \to \mathbb{Q}_\epsilon^n \times \mathbb{R}$, $\epsilon \in \{-1, 1\}$, assume that $c > (1+\epsilon)/2$. Then, f is congruent to an embedded rotational sphere.*

In our proof of Theorem 4.7, we start by noticing that the condition on c, together with Myers Theorem and Gauss equation, gives that M is compact and has positive definite second fundamental form. Hence, from Theorem 4.4, M is a topological sphere of constant curvature c and f is an embedding. Then, by using the fact that the gradient of the height function of the hypersurface f (when nonzero) is one of its principal directions (as proved in [33]), we managed to prove that $f(M)$ is the connected sum of two rotational embedded hemispheres with a common axis, showing that $f(M)$ is indeed a rotational embedded sphere.

We conclude our considerations by presenting a result due to H. Rosenberg and R. Tribuzy [40] on rigidity of convex surfaces of the homogeneous 3-manifolds known as $\mathbb{E}(k,\tau)$ *spaces.*

Given $k, \tau \in \mathbb{R}$ with $k - 4\tau^2 \neq 0$, one denotes by $\mathbb{E}(k,\tau)$ the total space of a Riemannian submersion over the simply connected two-dimensional space form of curvature k with bundle curvature τ. The unit tangent field to the fiber, which is a Killing field, is denoted by ξ. The $\mathbb{E}(k,\tau)$ spaces include the products $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, the Heisenberg space Nil_3, the Berger spheres, and the universal cover of the special linear group $\mathrm{SL}_2(\mathbb{R})$.

As defined in [40], an oriented surface $f\colon M \to \mathbb{E}(k,\tau)$ is said to be *strictly convex* if any of its principal curvatures is at least τ. With this terminology, the Rosenberg–Tribuzy Theorem reads as follows.

Theorem 4.8 (Rosenberg – Tribuzy [40]). *Let $f(t)\colon M \to \mathbb{E}(k,\tau)$ be a smooth one-parameter family of isometric immersions with $f(0) = f$. Suppose f is strictly convex, $K(f_t(x)) = K(f(x))$ for $x \in M$ and all t,* *and* $H(f_t(x)) = H(f(x))$ *at a non-horizontal point* x *of* M*. Then, there are isometries* $h(t)\colon \mathbb{E}(k,\tau) \to \mathbb{E}(k,\tau)$ *such that* $h(t)f(t) = f$*.*

In the above statement, $K(f(x))$ and $H(f(x))$ denote the Gaussian and mean curvatures, respectively, of the immersion f at a point $x \in M$. Also, a point $x \in M$ is called *horizontal* (resp. *vertical*) if ξ is orthogonal (resp. parallel) to T_xM.

The idea of the proof is to consider first the unit field

$$e_1 := \frac{P(\xi)}{\|P(\xi)\|}$$

on the open set $U \subset M$ of non horizontal points of f, where P denotes the orthogonal projection on the tangent bundle TU. Then, on U, there is a well defined differentiable angle function θ such that

$$\xi = (\cos\theta)e_1 + (\sin\theta)N,$$

where N is the unit normal to f. Moreover, it is shown that, due to the convexity of f, for any vertical point $x \in M$, θ is a submersion in a neighborhood of x. This allows one to choose a unit field $v \in TU$ orthogonal to the gradient of θ, and then define a second angle function ϕ on U as

$$v = (\cos\phi)e_1 + (\sin\phi)e_2,$$

where $e_2 = Je_1$ and J is the positive $\pi/2$-rotation.

In this setting, a computation (in which the convexity of f plays a fundamental role) shows that ϕ is the solution of an ODE which involves the function θ. From uniqueness of solutions of ODE's satisfying initial conditions, and the hypotheses on the curvature functions H e K, one concludes that the functions ϕ and θ are the same for all immersions $f(t)$.

It is also proved from the convexity of f that the horizontal points of any $f(t)$ are all isolated and have index one. From this, and the above considerations, one has that all immersions $f(t)$ satisfy the hypotheses of the fundamental theorem for immersions in homogeneous 3-manifolds, by B. Daniel [11], proving that they are all congruent to each other, as asserted.

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