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Recent developments on fully nonlinear PDEs with unbalanced degeneracy

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Abstract. In this survey we present some recent advances regarding existence and regularity results (more specifically L^∞, $C^{0,\gamma}$ and $C^{1,\gamma}$ estimates and non-degeneracy results) for a class of fully nonlinear elliptic PDEs with unbalanced variable degeneracy. In a precise way, the degeneracy law of the model switches between two different kinds of degenerate elliptic operators of variable order, according to the null set of a modulating function $\mathfrak{a}(\cdot) \geq 0$. The model case in question is given by

$$\left[|Du|^{p(x)} + \mathfrak{a}(x)|Du|^{q(x)}\right] \Delta u = f(x) \quad \text{in} \quad \Omega$$

for a bounded, regular and open set $\Omega \subset \mathbb{R}^n$ (to be specified), and appropriate continuous data $p(\cdot), q(\cdot)$ and $f(\cdot)$. Such sharp regularity estimates generalize and improve, to some extent, earlier ones via geometric treatments.

Our results are consequences of geometric tangential methods and make use of compactness, localized oscillating and scaling techniques. We further show how our findings apply to the study of a wide class of nonlinear models and free boundary problems.

Keywords: Sharp regularity estimates, unbalanced degenerate operators, Variable-exponent fully nonlinear models, Geometric free boundary problems.

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1 Introduction

Diffusion processes are ubiquitous in science and engineering; roughly speaking, they are characterized by a sort of averaging phenomenon which has its most simple and fundamental mathematical expression in the so called mean value property satisfied by harmonic functions, i.e., functions whose Laplacian is equal to 0:

$$\Delta u(x) = 0,\ x \in \Omega \quad \Leftrightarrow \quad u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy,\ B_r(x) \subset \Omega. \tag{1.1}$$

There is a broad range of problems in which diffusion processes make their appearance, from physics and biology to material sciences, Information

Theory and even Data Science. Moreover, its mathematical importance is clear by its deep connections with many areas such as Probability, Differential Geometry and Harmonic Analysis among many others. In this direction, equation (1.1) captures the fundamental feature shared by a broad class of differential equations which are the interest of the present survey: uniformly elliptic second order differential equations. One of the most appealing signatures of this type of equations is that they enjoy some version the aforementioned averaging property which improves the smoothness properties of solutions, as they are required to somehow equate their values at a point with those of their neighbors. More specifically, we will be interested in a subclass of elliptic equations in which this specific characteristic (uniform ellipticity) is lost along some parts of the domain of definition of solutions, precisely where the gradient of the solutions is vanishing, namely *degenerate elliptic* equations as follows:

$$\left\{\begin{array}{rcl} \mathcal{L}\,u & := & -\mathrm{div}(\mathrm{A}(x,\nabla u)) \\ & \text{or} & \\ \mathcal{G}(x,Du,D^2u) & := & \mathbb{A}(x,|Du|)\mathrm{Tr}(\mathrm{M}(x)D^2u), \end{array}\right.$$

where

$$|z|^{p-2}\mathrm{Id}_n \lesssim \partial_z \mathrm{A}(x,z) \lesssim |z|^{p-2}\mathrm{Id}_n \quad \forall\ x\in\Omega, \quad \text{for} \quad p>2,$$

and

$$0 \le \mathbb{A} \in C^0(\Omega\times\mathbb{R}^+) \quad \text{with} \quad \mathbb{A}(x,0)=0 \quad \forall\ x\in\Omega \quad \text{and} \quad \mathrm{M}\in \mathrm{Sym}(n),$$

whose degeneracies laws of such models collapse in the region of singular points of existing solutions (i.e., where the gradient vanishes), thereby obstructing the regularizing effect for such solutions (i.e., the impossibility of classical solutions). Such a phenomena has been studied since the early 60's by several authors, e.g. Ladyzhenskaya-Ural'tseva, Marceline, Serrin, Stampacchia, Trudinger, Ural'tseva-Urdaletova and Zhikov, in the scenario of non-uniformly elliptic models, see [46], [47], [50], [59], [61], [66], [67], [69] [72] and [75] just to cite a few classical contributions.

We also recommend to reader the Teixeira's works [64] and [65] for modern enlightening surveys concerning geometric regularity insights in elliptic PDE models and applications.

1.1 Models with degenerate signature

Firstly, we will revisit some well-known elliptic PDEs models with divergence structure (for a symmetric, uniformly elliptic and continuous matrix $\mathbb{A}$):

Scenario	**Divergence form**
Uniformly Elliptic	$\mathrm{div}(\mathbb{A}(x)\nabla u)$
Single Degeneracy Law	$\mathrm{div}(\|\nabla u\|^{p-2}\nabla u) \quad (p>2)$
Double Degeneracy Law	$\mathrm{div}((\|\nabla u\|^{p-2}+\mathfrak{a}(x)\|\nabla u\|^{q-2})\nabla u) \quad (2<p\le q)$

Regarding the non-variational counterpart of the previous models we have, respectively (for a symmetric, uniformly elliptic and continuous matrix $\mathbb{A}$):

Scenario	**Non-divergence form**
Uniformly Elliptic	$\mathrm{Tr}(\mathbb{A}(x)D^2u)$
Single Degeneracy Law	$\|Du\|^p\mathrm{Tr}(\mathbb{A}(x)D^2u) \quad (p>0)$
Double Degeneracy Law	**What should be the model case?**

At this point, it will be natural to consider the following non-homogeneous model:

$$\mathcal{G}[u]=[|Du|^p+\mathfrak{a}(x)|Du|^q]\,\mathrm{Tr}(\mathbb{A}(x)D^2u) \quad \text{for} \quad 0<p<q<\infty \quad \text{and} \quad 0\le\mathfrak{a}\in C^0(\Omega),$$

i.e., the counterpart of certain variational problems from Calculus of Variations with double phase structure (cf. [9], [20], [21], [52], [69], [72] and [75]).

Concerning such class of models, in the last four decades there has appeared a wide amount of literature on **double phase problems** as follows

$$(w,f)\mapsto \min\int_\Omega\left(\frac{1}{p}|\nabla w|^p+\frac{\mathfrak{a}(x)}{q}|\nabla w|^q-fw\right)dx,$$

whose Euler-Lagrange PDE is driven by

$$\mathcal{L}_0\,[w] = -\mathrm{div}\left(\left(|\nabla w|^{p-2} + \mathfrak{a}(x)|\nabla w|^{q-2}\right)\nabla w\right) = f(x).$$

Such mathematical models appear in many applications in the contexts of applied sciences: material sciences, related to the behavior of certain strongly anisotropic materials [43]; elasticity theory [72]; transonic flows [8]; quantum physics [10] and many others.

We must highlight that studies on existence, boundedness and regularity issues involving non-uniformly elliptic operators given by

$$\mathcal{L}w := -\mathrm{div}(\partial_w F(x, \nabla w)) = f(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^n,$$

whose "bulk energy density" does not fulfill a standard "polynomial growth" condition, i.e., a sort of double-sided bound of the type

$$|\xi|^p \lesssim F(x,\xi) \lesssim |\xi|^p + 1, \qquad \text{for } \; p > 1, \; \text{ and } \; |\xi| \; \text{ large}$$

are currently very much a classical topic, and they have attracted attention of many researchers in the last decades (cf. [46] for a Ladyzhenskaya-Ural'tseva's long-established treatise). Particularly, several results were developed to functionals with non-standard growth conditions of type

$$|\xi|^p \lesssim F(x,\xi) \lesssim |\xi|^q + 1, \qquad \text{for } \; 1 < p < q, \; \text{ and } \; |\xi| \; \text{ large},$$

which are nowadays known as **autonomous functional** with $(p, q)-$growth.

An archetypical model of such functionals is given by multi-phase variational integrals

$$w \mapsto \mathcal{F}_0(w,\Omega) := \int_\Omega \left(|\nabla w|^p + \sum_{i=1}^{n} |\nabla w|^{q_i} - fw\right) dx, \quad 1 < p < q_1 \leq \cdots \leq q_n, \tag{1.2}$$

that were firstly examined in the Ural'tseva-Urdaletova's groundbreaking work [69], whose special case of bounded minimizers was studied. In addition, for a regularity treatment, we must cite Marcellini's pioneering work

[50], which established that minimizers of (1.2) (with $f \equiv 0$ and under $(p,q)-$growth) belong to $W^{1,\infty}_{\text{loc}}(\Omega)$, where the condition

$$\frac{p}{q} < 1 + \mathrm{o}(n)$$

proves to be necessary and sufficient for obtaining such estimates.

Recently, many authors have introduced a new class of **Functional of Double Phase** type, whose simplest model case is given by

$$W^{1,p}_{\text{loc}}(\Omega) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega \mathrm{H}_0(x,\nabla w)dx, \quad \text{for} \quad \mathrm{H}_0(x,\xi) := |\xi|^p + \mathfrak{a}(x)|\xi|^q, \tag{1.3}$$

where $0 \le \mathfrak{a} \in C^{0,\alpha}(\Omega)$ (for $\alpha \in (0,1]$) and p and q are positive constants.

In this direction, by considering a wide class of double phase functionals

$$w \mapsto \mathcal{G}(w,\Omega) := \int_\Omega \mathrm{H}(x,w,\nabla w)dx, \tag{1.4}$$

where $\mathrm{H} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying a growth condition

$$L_1\mathrm{H}_0(x,\xi) \le \mathrm{H}(x,w,\xi) \le L_2\mathrm{H}_0(x,\xi) \quad \text{for constants} \quad 0 < L_1 \le L_2,$$

in a series of seminal papers, Mingione *et al.* (see also Koch's work [44]) proved a number of boundedness and local sharp regularity results:

Theorem 1.1 ([9], [20] and [21]). *Let $w \in W^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimizer of the functional*

$$w \mapsto \int_\Omega \left(|\nabla w|^p + \mathfrak{a}(x)|\nabla w|^q\right) dx$$

where $0 \le \mathfrak{a} \in C^{0,\alpha}(\Omega)$ and p and q are positive constants such that

$$1 < p \le q \quad \textit{and} \quad \frac{q}{p} < 1 + \frac{\alpha}{n} \quad \textit{with} \quad \alpha \in (0,1].$$

Then, $|\nabla w| \in C^{0,\gamma}_{loc}(\Omega)$ for a universal exponent.

We must also to cite De Filippis-Mingione's recent results for minima of non-differentiable functionals:

Theorem 1.2 (**[35]**). *Let $w \in W^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimizer of the functional*

$$w \mapsto \int_\Omega (\mathrm{F}(Dw) + h(x,w))\, dx$$

where

$$|z|^{p-2} Id_n \lesssim \partial_{zz}\mathrm{F}(z) \lesssim |z|^{q-2} Id_n,$$

$$|h(x,w_1) - h(x,w_2)| \lesssim |w_1 - w_2|^\alpha \quad \textit{for} \quad \alpha \in (0,1]$$

and

$$\frac{q}{p} \le 1 + \frac{1}{5}\left(1 - \frac{\alpha}{p}\right)\frac{\alpha}{n}.$$

Then, Du is locally Hölder continuous.

We also refer to De Filippis-Mingione's recent works [33] and [34] for optimal Lipschitz bounds in the context of nonautonomous integrals. Finally, existence/multiplicity issues of solutions to double phase problems like (1.3) were addressed by Liu-Dai's work [48].

Studies on double phase problems as (1.3) date back to Zhikov's works [73] and [74], which describe the behavior of certain strongly anisotropic materials, whose hardening estates, connected to the exponents of the gradient's growth, change in a point-wise fashion. In this scenario, a mixture of two heterogeneous materials, with hardening (p,q)-exponents, can be performed according to the intrinsic geometry of the null set of the modeling function:

$$\mathcal{L}w(x_0) = \begin{cases} -\Delta_p w(x_0) & \text{if} \quad \mathfrak{a}(x_0) = 0 \\ -\Delta_p w(x_0) - \mathrm{div}(\mathfrak{a}(x_0)|\nabla w(x_0)|^{q-2}\nabla w(x_0)) & \text{if} \quad \mathfrak{a}(x_0) \neq 0. \end{cases}$$

Additionally, Zhikov's works also give new instances for the occurrence of the so-called *Lavrentiev's phenomenon*: for a suitable boundary datum u_0

$$\inf_{w \in u_0 + W^{1,p}(\Omega)} \int_\Omega F(x,\nabla w)dx < \inf_{w \in u_0 + W^{1,q}(\Omega)} \int_\Omega F(x,\nabla w)dx.$$

In another words, it is not possible to attain the minimum of the functional (1.3) in a more regular fashion, i.e., a natural obstruction to regularity (cf. [9], [20], [21] and [33] for related works). In particular, Zhikov showed in [75] that functionals with $p(x)-$growth manifest the Lavrentiev's phenomenon if and only if the critical continuity condition is violated (cf. [37, Section 4.1] and [53, Part 1]):

$$\limsup_{s\to 0} \hat{\omega}(s)\ln(s^{-1}) \le \mathrm{C}_1,$$

for a universal modulus of continuity $\hat{\omega}:[0,\infty)\to[0,\infty)$ such that

$$|p(x)-p(y)| \le \mathrm{C}_2\cdot\hat{\omega}(|x-y|).$$

On the other hand, these functionals (1.3) also appear in a variety of physical models; see [76] for applications in the elasticity theory, [8] for transonic flows, [10] for model of static solutions for elementary particle.

Along recent directions, different generalized functionals have been considered in the literature. For instance, Ragusa and Tachikawa in [54] analysed minimizers to a class of integral functionals of double phase type with variable exponents:

$$w \mapsto \int_\Omega \left(|\nabla w|^{p(x)} + \mathfrak{a}(x)|\nabla w|^{q(x)}\right)dx, \ \ \text{for} \ \ q(x)\ge p(x)\ge p_0>1, \ \ \mathfrak{a}(x)\ge 0,$$

where $p(\cdot)$, $q(\cdot)$ and $\mathfrak{a}(\cdot)$ are assumed to be Hölder continuous functions. In such a context, the authors prove (under appropriate assumptions on data) that local minimizers are $C^{1,\gamma}_{\text{loc}}(\Omega)$ for some $\gamma\in(0,1)$. These results somehow extend the theory put forward by Zhang and Zhou in [71].

We recommend reading the Mingione-Rădulescu's survey [52] on recent developments for classes of problems with non-standard growth and nonuniform ellipticity regime.

1.2 Advances in fully nonlinear degenerate models

In turn, in the light of recent non-divergence researches (i.e., a nonvariational counterpart of certain variational integrals of the calculus of variations with non-standard growth (1.4)), De Filippis' manuscript [31] was the pioneering work in considering fully nonlinear problems with non-homogeneous degeneracy. Precisely, De Filippis proved $C^{1,\gamma}_{\text{loc}}$ regularity estimates for viscosity solutions of

$$[|Du|^p + \mathfrak{a}(x)|Du|^q]\, F(D^2u) = f \in C^0(\overline{\Omega}), \quad \text{for} \quad q \geq p > 0, \;\; \mathfrak{a}(x) \geq 0,$$

for some $\gamma \in (0,1)$ depending on universal parameters. Aftermath, Da Silva-Ricarte in [26] improved De Filippis' result and addressed a variety of applications in nonlinear elliptic models and related free boundary problems.

Concerning fully nonlinear models with (single) degeneracy law (i.e., $\mathfrak{a} \equiv 0$)

$$\mathcal{G}_p[u] := |Du|^p F(D^2u) = f(x) \quad \text{in} \quad \Omega$$

the list of contributions is fairly diverse, including aspects such as existence/uniqueness issues, Harnack inequality, ABP estimates [25] and [11], Liouville type results [23], local Hölder and Lipschitz estimates, local gradient estimates [3], [12], [13], [14], and [40], as well as their connections with a variety of free boundary problems of Bernoulli type [25], obstacle type models [29] and [30], singular perturbation models [4], [11] and [56], and dead-core type [23], just to cite a few.

We must also place emphasis on Bronzi et al.'s result [15, Theorem 2.1], where the authors show that viscosity solutions for the class of variable-exponent fully nonlinear models

$$|Du|^{\theta(x)} F(D^2u) = f(x) \quad \text{in} \quad B_1$$

are of class $C^{1,\gamma}(B_{1/2})$, where $f \in C^0(B_1)$, and $\theta \in C^0(B_1)$ enjoys minimal assumptions such that $\inf\limits_{B_1} \theta(x) > -1$ (structural law of singular/degenerate type). Moreover, the following estimate holds

$$\|u\|_{C^{1,\gamma}\left(B_{\frac{1}{2}}\right)} \leq \mathrm{C}(n, \lambda, \Lambda, \gamma) \cdot \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}^{\frac{1}{\|\theta^+\|_{L^\infty(B_1)} + \|\theta^-\|_{L^\infty(B_1)} + 1}} + 1 \right).$$

We must also cite the recent work [38] due to Fang-Rădulescu-Zhang, which by combining the approaches from [15] and [31] established local $C^{1,\gamma}$ regularity estimates to doubly degenerate fully nonlinear elliptic PDEs with variable exponent

$$\left[|Du|^{p(x)} + \mathfrak{a}(x)|Du|^{q(x)}\right] F(D^2u) = f \in C^0(\overline{\Omega}),$$

for $q(x) \geq p(x) \geq p_0 > 0$ and $\mathfrak{a}(x) \geq 0$.

In the sequel, Da Silva et al. extend the Fang-Rădulescu-Zhang's results for the up to the boundary setting in [24], presenting a number of consequences in nonlinear models and free boundary problems.

Finally, completing this mathematical state-of-the-art, we recall a very recent work by De Filippis [32], in which she studied free transmission type problems of the form

$$\left(|Du|^{p^+ \mathbb{1}_{\{u>0\}} + p_- \mathbb{1}_{\{u<0\}}} + a(x)\mathbb{1}_{\{u>0\}}|Du|^q + b(x)\mathbb{1}_{\{u<0\}}|Du|^s\right) F(D^2u) = f \text{ in } \Omega$$

(for $0 \leq p_+ \leq q < \infty$ and $0 \leq p_- \leq s < \infty$) obtaining existence and interior $C^{1,\gamma}$ regularity of viscosity solutions. See also Jesus' work [42], for degenerate fully nonlinear free transmission problems, which proved optimal point-wise regularity estimates, where the degeneracy law varies in the domain.

1.3 Mathematical purpose

The aim of this survey is to present some recent results regarding existence and geometric regularity estimates for solutions of a class of nonlinear elliptic equations having a non-homogeneous degeneracy and variable exponent, whose general mathematical model is given by

$$\left\{\begin{array}{rcll} \mathcal{H}(x, Du)F(x, D^2u) & = & f(x) & \text{in} \quad \Omega \\ u(x) & = & g(x) & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1.5}$$

for a bounded and regular open set $\Omega \subset \mathbb{R}^n$, suitable data f and g. Throughout this work, $\mathcal{H}$ enjoys an appropriated degeneracy law and F is assumed to be a second order fully nonlinear (uniformly elliptic) operator, i.e., it is nonlinear in its highest order derivatives (to be specified soon).

The primary model case to (1.5) we have in mind is

$$\left[|Du|^{p(x)} + \mathfrak{a}(x)|Du|^{q(x)}\right] \operatorname{Tr}(\mathbb{A}(x)D^2u) = f(x) \quad \text{in} \quad B_1, \tag{1.6}$$

for a symmetric matrix $\mathbb{A} \in \operatorname{Sym}(n)$ with $\lambda|\xi|^2 \leq \xi^t\mathbb{A}(x)\cdot\xi \leq \Lambda|\xi|^2$ and constants $0 < \lambda \leq \Lambda < \infty$. In a mathematical perspective, (1.6) enjoys distinct types of degeneracy laws under a variable exponent regime, depending on the values of the modulating function $\mathfrak{a}(\cdot) \geq 0$, as well as of magnitude of vectorial component

$$\text{If} \quad \mathfrak{a}(x) \neq 0 \quad \Rightarrow \quad |\xi|^{p(x)} \leq \mathcal{H}(x,\xi) \leq \begin{cases} \left(1+\|\mathfrak{a}\|_{L^\infty(\Omega)}\right)|\xi|^{p(x)} & \text{if} \quad |\xi| \leq 1 \\ \left(1+\|\mathfrak{a}\|_{L^\infty(\Omega)}\right)|\xi|^{q(x)} & \text{if} \quad |\xi| > 1. \end{cases}$$

$$\text{If} \quad \mathfrak{a}(x) = 0 \quad \Rightarrow \quad \mathcal{H}(x,\xi) = |\xi|^{p(x)}.$$

As a consequence, diffusion properties of the model exhibit a sort of non-uniformly elliptic and doubly (non-homogeneous) degenerate signature, which combines two different type operators of variable order (cf. [3], [12], [13], [14], [15] and [40] for single degeneracy scenarios).

In conclusion, in this research framework, we account for the following contributions:

1. Existence/uniqueness of solutions (Theorem 3.1);
2. Uniform bounds, i.e., ABP estimates (Theorem 3.5)
3. Hölder estimates (Theorem 3.7);
4. Gradient estimates (improved and sharp estimates) (Theorem 3.9);
5. Geometric non-degeneracy properties of solutions (Theorem 3.16);
6. Finer regularity estimates (Theorem 4.1);
7. Applications in nonlinear elliptic models and free boundary problems (Section 4).

The structure of the manuscript is as follows: in Section 2 we present some preliminary definitions and detail the main assumptions we are going imposed to get the results; in Section 3 we account for the main results. Section 4 we present several applications and further results.

2 Assumptions and preliminary definitions

In this section we will present some definitions and the main structural conditions we are going to imposed in the problem; precisely, it is assumed:

(A0′) (**Regularity of the domain**) In this work, we will assume that $\Omega \subset \mathbb{R}^n$ is a bounded, open domain, which satisfies a uniform exterior sphere condition. That is, for each $z \in \partial\Omega$, we can choose $x_z \in \mathbb{R}^n \setminus \Omega$ such that

$$B_r(x_z) \cap \overline{\Omega} = \{z\} \quad \text{with} \quad r = |z - x_z|.$$

Notice that C^2 domains (i.e., with C^2 boundary) fulfill a uniform exterior sphere condition.

(A0) (**Continuity and normalization condition**)

$$\text{Fixed } \Omega \ni x \mapsto F(x, \cdot) \in C^0(\Omega) \quad \text{and} \quad F(\cdot, 0_{n\times n}) = 0, \ \text{(null matrix)}.$$

(A1) (**Uniform ellipticity**) For any pair of matrices $\mathrm{X}, \mathrm{Y} \in Sym(n)$

$$\mathcal{M}^-_{\lambda,\Lambda}(\mathrm{X} - \mathrm{Y}) \le F(x, \mathrm{X}) - F(x, \mathrm{Y}) \le \mathcal{M}^+_{\lambda,\Lambda}(\mathrm{X} - \mathrm{Y})$$

where $\mathcal{M}^\pm_{\lambda,\Lambda}$ are the *Pucci's extremal operators* given by

$$\mathcal{M}^-_{\lambda,\Lambda}(\mathrm{X}) := \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(X) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

for *ellipticity constants* $0 < \lambda \le \Lambda < \infty$, where $\{e_i(\mathrm{X})\}_{i=1}^n$ are the eigenvalues of X.

Additionally, for our Theorem 3.9, we must require some sort of continuity assumption on the coefficients:

(A2) ($\omega-$**continuity of the coefficients**) There exist a uniform modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ and a constant $\mathrm{C_F} > 0$ such that

$$\Omega \ni x, x_0 \mapsto \Theta_{\mathrm{F}}(x, x_0) := \sup_{\substack{\mathrm{X} \in Sym(n) \\ \mathrm{X} \neq 0}} \frac{|F(x, \mathrm{X}) - F(x_0, \mathrm{X})|}{\|\mathrm{X}\|} \leq \mathrm{C_F}\,\omega(|x - x_0|),$$

which measures the oscillation of coefficients of F around x_0. For simplicity purposes, we shall often write $\Theta_{\mathrm{F}}(x, 0) = \Theta_{\mathrm{F}}(x)$.
Finally, for notation purposes we define

$$\|F\|_{C^{\omega}(\Omega)} := \inf\left\{\mathrm{C_F} > 0 : \frac{\Theta_{\mathrm{F}}(x, x_0)}{\omega(|x - x_0|)} \leq \mathrm{C_F}, \ \ \forall\, x, x_0 \in \Omega,\ x \neq x_0\right\}.$$

(A3) (**Assumptions on data**) Throughout this manuscript we will assume the following assumptions on data: $f \in L^{\infty}(\overline{\Omega})$ and $g \in C^0(\partial\Omega)$. Moreover, when necessary, we invoke continuity assumption on f. (see Theorem 3.1).

(A4) (**Non-homogeneous degeneracy**) In our studies, we enforce that the diffusion properties of the model (1.5) degenerate along an *a priori* unknown set of singular points of existing solutions:

$$\mathcal{S}_0(u, \Omega) := \{x \in \Omega : |Du(x)| = 0\}.$$

Consequently, we will impose that $\mathcal{H} : \Omega \times \mathbb{R}^n \to [0, \infty)$ – non-homogeneous degeneracy law – one behaves as

$$\mathrm{L}_1 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x, |\xi|) \leq \mathcal{H}(x, \xi) \leq \mathrm{L}_2 \cdot \mathcal{K}_{p,q,\mathfrak{a}}(x, |\xi|) \tag{2.1}$$

for constants $0 < \mathrm{L}_1 \leq \mathrm{L}_2 < \infty$, where

$$\mathcal{K}_{p,q,\mathfrak{a}}(x, |\xi|) := |\xi|^{p(x)} + \mathfrak{a}(x)|\xi|^{q(x)}, \ \text{ for } \ (x, \xi) \in \Omega \times \mathbb{R}^n. \tag{2.2}$$

(A5) (**Assumptions on variable exponents**) In turn, for the degeneracy law in (2.2), we suppose that the functions $p, q \in C^0(\Omega)$ and the modulating function $\mathfrak{a}(\cdot)$ fulfill

$$0 < p_{\min} \leq p(x) \leq p_{\max} \leq q(x) \leq q_{\max} < \infty \quad \text{and} \quad 0 \leq \mathfrak{a} \in C^0(\overline{\Omega}). \tag{2.3}$$

Now, let us introduce the notion of viscosity solution for our operators.

Definition 2.1 (Viscosity solutions). A function $u \in C^0(\Omega)$ is a viscosity super-solution (resp. sub-solution) to (1.5) if whenever $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $u - \varphi$ has a local minimum (resp. a local maximum) at x_0, then

$$\mathcal{H}(x_0, D\varphi(x_0))F(x_0, D^2\varphi(x_0)) \leq f(x_0) \qquad \text{resp. } (\cdots \geq f(x_0))$$

Finally, u is said to be a viscosity solution if it is simultaneously a viscosity super-solution and a viscosity sub-solution.

In order to measure the smoothness of solutions in suitable spaces, we are going to use the following norms and semi-norms (see, [45, Section 1]):

Definition 2.2 ($C^{1,\gamma}$ norm and semi-norm). For $\gamma \in (0,1]$, $C^{1,\gamma}(\Omega)$ denotes the space of all functions u whose spacial gradient $Du(x)$ there exists in the classical sense for every $x \in \Omega$ such that the norm

$$\|u\|_{C^{1,\gamma}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \|Du\|_{L^\infty(\Omega)} + [u]_{C^{1,\gamma}(\Omega)}$$

is finite. Moreover, we have the semi-norm (see [45])

$$[u]_{C^{1,\gamma}(\Omega)} := \sup_{\substack{x_0 \in \Omega \\ 0<r\leq \operatorname{diam}(\Omega)}} \inf_{\mathfrak{l} \in \mathcal{P}_1} \frac{\|u - \mathfrak{l}\|_{L^\infty(B_r(x_0)\cap\Omega)}}{r^{1+\gamma}},$$

where $\mathcal{P}_1$ denotes the spaces of polynomial functions of degree at most 1. As a result, $u \in C^{1,\gamma}(\Omega)$ implies every component of Du is $C^{0,\gamma}(\Omega)$ (see, [45, Main Theorem]).

3 Theorems

As our first result, we present the existence of viscosity solutions.

Theorem 3.1 (Existence of solutions - [24, Theorem 1.1]). *Suppose assumptions (A0)-(A5) are in force for continuous data f and g. Then, there exists a unique viscosity solution $u \in C^0(\overline{\Omega})$ to (1.5).*

Remark 3.2. Regarding the existence of viscosity solutions we must present a Comparison Principle result. For this reason, additionally to assumptions (A0)-(A2) and (A4)-(A5), aiming to obtain such a comparison result, we will assume the following condition: $p, q, \mathfrak{a} \in C^{0,1}(\Omega)$ and there exist a constant

$$\mathrm{C}_{p,q,\mathfrak{a}} = \mathrm{C}_{p,q,\mathfrak{a}}\left(\|Dp\|_{L^\infty(\Omega)}, \|Dq\|_{L^\infty(\Omega)}, \|D\mathfrak{a}\|_{L^\infty(\Omega)}, \|\mathfrak{a}\|_{L^\infty(\Omega)}, L_1, L_2\right) > 0$$

and a modulus of continuity $\omega_{p,q,\mathfrak{a}} : [0, \infty) \to [0, \infty)$ such that

$$|\mathcal{H}(x, \xi) - \mathcal{H}(y, \xi)| \leq \mathrm{C}_{p,q,\mathfrak{a}}(|\xi|)^{p_{\min}}\left(|\ln(|\xi|)| + 1\right)\omega_{p,q,\mathfrak{a}}(|x - y|). \tag{3.1}$$

for all $(x, y) \in \Omega$ and $\xi \in B_1 \setminus \{0\}$.

Example 3.3. By way of illustration, when

$$\mathcal{H}(x, \xi) = |\xi|^{p(x)} + \mathfrak{a}(x)|\xi|^{q(x)}$$

then (3.1) becomes

$$|\mathcal{H}(x, \xi) - \mathcal{H}(y, \xi)| \leq \mathfrak{A}_1(x, y, \xi) + \mathfrak{A}_2(x, y, \xi),$$

where

$$\mathfrak{A}_1(x, y, \xi) := \|Dp\|_{L^\infty(\Omega)}|\xi|^{p_{\min}}|\ln(|\xi|)||x - y|$$

and

$$\mathfrak{A}_2(x, y, \xi) := \left(\|Dq\|_{L^\infty(\Omega)}\|\mathfrak{a}\|_{L^\infty(\Omega)}|\ln(|\xi|)| + \|D\mathfrak{a}\|_{L^\infty(\Omega)}\right)|\xi|^{q_{\min}}|x - y|.$$

Theorem 3.4 (Comparison Principle - [24, Theorem 2.3]). *Assume that assumptions (A0)-(A2), (A4)-(A5) are in force. Let $f \in C^0(\overline{\Omega})$. Suppose u and v are respectively a viscosity supersolution and subsolution of (1.5). If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Therefore, by making use of Comparison Principle (Theorem 3.4) and Perron methods (cf. [22]) we are able to present the existence of viscosity solutions to (1.5).

Now, we present an ABP estimate adapted to our context of fully nonlinear models with unbalanced variable degeneracy. Such an estimate is pivotal in order to obtain universal bounds for viscosity solutions in terms of data of the problem.

Theorem 3.5 (Alexandroff-Bakelman-Pucci estimate - [24, Theorem 2.1]). *Suppose that assumptions (A0)-(A2) there hold. Then, there exists* $\mathrm{C} = \mathrm{C}(n, \Lambda, p_{min}, q_{max}, \mathrm{diam}(\Omega)) > 0$ *such that for any* $u \in C^0(\overline{\Omega})$ *viscosity sub-solution (resp. super-solution) of* (1.5) *in* $\{x \in \Omega : u(x) > 0\}$ *(resp.* $\{x \in \Omega : u(x) < 0\}$*), satisfies*

$$\sup_{\Omega} u(x) \leq \sup_{\partial\Omega} g^+(x) + \mathrm{C} \cdot \mathrm{diam}(\Omega) \max\left\{ \left\| \frac{f^-}{1+\mathfrak{a}} \right\|_{L^n(\Gamma(u^+)=u)}^{\frac{1}{p_{min}+1}}, \left\| \frac{f^-}{1+\mathfrak{a}} \right\|_{L^n(\Gamma(u^+)=u)}^{\frac{1}{q_{max}+1}} \right\},$$

respectively,

$$\left(\sup_{\Omega} u^-(x) \leq \sup_{\partial\Omega} g^-(x) + \mathrm{C} \cdot \mathrm{diam}(\Omega) \max\left\{ \left\| \frac{f^+}{1+\mathfrak{a}} \right\|_{L^n(\Gamma(u^-)=u)}^{\frac{1}{p_{min}+1}}, \left\| \frac{f^+}{1+\mathfrak{a}} \right\|_{L^n(\Gamma(u^-)=u)}^{\frac{1}{q_{max}+1}} \right\} \right)$$

where $\Gamma(u^{\pm})$ *is the convex hull of* $-u^{\pm}$ *extended by* 0 *on* B_{2d}*, and* B_d *is a ball of radius* $d = \mathrm{diam}(\Omega)$ *such that* $\Omega \subset B_d$ *and we have extended* $u^{\pm} \equiv 0$ *outside* Ω*.*

Particularly, we conclude that

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)} + \mathrm{C} \cdot \mathrm{diam}(\Omega) \max\left\{ \left\| \frac{f}{1+\mathfrak{a}} \right\|_{L^n(\Omega)}^{\frac{1}{p_{min}+1}}, \left\| \frac{f}{1+\mathfrak{a}} \right\|_{L^n(\Omega)}^{\frac{1}{q_{max}+1}} \right\}.$$

Remark 3.6. The proof of the Alexandroff-Bakelman-Pucci estimate follows the same ideas as in [11, Theorem 8.6], which strongly rely on the recent work [41].

We also present the following interior Hölder regularity result. The proof follows the same lines as [11, Theorem 8.5] and [24, Theorem 2.2].

Theorem 3.7 (Local Hölder estimates). *Let u be a viscosity solution to*

$$\mathcal{H}(x, Du)F(x, D^2u) = f(x) \quad in \quad B_1.$$

where f is a continuous and bounded function. Then, $u \in C^{0,\alpha'}_{loc}(B_1)$ for some universal $\alpha' \in (0,1)$. Moreover,

$$\|u\|_{C^{0,\alpha'}(\Omega')} \leq \mathrm{C}(\mathtt{universal}, \mathrm{dist}(\Omega', \partial\Omega)) \left(\|u\|_{L^\infty(\Omega)} + \max\left\{\left\|\frac{f}{1+\mathfrak{a}}\right\|^{\frac{1}{p_{min}+1}}_{L^n(\Omega)}, \left\|\frac{f}{1+\mathfrak{a}}\right\|^{\frac{1}{q_{max}+1}}_{L^n(\Omega)}\right\}\right).$$

Remark 3.8. Now, making use of the Harnack's inequality and following the arguments presented in [18, Proposition 4.10], we can obtain the interior Hölder regularity result; we refer the reader to [11, Theorem 8.5]. See also [31, Theorem 2] for similar estimates.

Now we are in a position to state an optimal geometric estimate. In effect, any bounded viscosity solution of (1.5) belongs to $C^{1,\alpha}$ at interior points, where

$$\alpha \in (0, \alpha_{\mathrm{F}}) \cap \left(0, \frac{1}{p_{\max}+1}\right], \tag{3.2}$$

where $\alpha_{\mathrm{F}} = \alpha_{\mathrm{F}}(n, \lambda, \Lambda) \in (0, 1]$ is the optimal exponent to (local) Hölder continuity of gradient of solutions to homogeneous problem with "frozen coefficients" $F(D^2\mathfrak{h}) = 0$ (see, [17], [18, Ch.5 §3] and [68]).

Theorem 3.9 (Interior improved regularity - [24, Theorem 1.3]). *Assume that assumptions (A0)-(A5) there hold. Let u be a bounded viscosity solution to (1.5) with $f \in L^\infty(\Omega)$. Then, u is $C^{1,\alpha}$, at interior points, for α as in (3.2). More precisely, for any point $x \in B_r(x_0) \Subset \Omega$ there holds*

$$[u]_{C^{1,\alpha}(B_r(x_0))} \leq \mathrm{C} \cdot \left(\|u\|_{L^\infty(\Omega)} + \|f\|^{\frac{1}{p_{min}+1}}_{L^\infty(\Omega)} + 1\right),$$

for $r \in \left(0, \frac{1}{2}\right)$ where $\mathrm{C} > 0$ is a universal constant[1].

[1]A constant is said to be universal if it depends only on dimension, degeneracy and ellipticity constants, α_{F}, β, L_1, L_2 and $\|F\|_{C^\omega(\Omega)}$.

Remark 3.10. Now, concerning the geometric estimates, different from $C^{1,\alpha}$ regularity estimates from linear setting, we can no longer proceed with an iterative scheme, i.e.,

$$\sup_{B_{\rho^k}(x_0)} \frac{|u(x)-\mathfrak{l}_k(x)|}{\rho^{k(1+\alpha)}} \leq 1 \quad \overset{\text{Dini-Campanato}}{\underset{\Longrightarrow}{\text{embedding}}} \quad u \text{ is } C^{1,\alpha} \quad \text{at} \quad x_0,$$

because *a priori* we do not know the equation which is satisfied by

$$B_1(0) \ni x \mapsto \frac{(u-\mathfrak{l}_k)(\rho^k x)}{\rho^{k(1+\alpha)}},$$

for $\{\mathfrak{l}_k\}_{k\in\mathbb{N}}$ affine functions, since

$$\mathcal{H}(x, Dv)F(x, D^2 v) \text{ is not invariant by affine maps.}$$

For this reason, an alternative approach must be undertaken: quantitative information on the oscillation of u

$$\sup_{B_\rho(x_0)} \frac{\rho^{-1}|u(x)-u(x_0)|}{\rho^\alpha + |Du(x_0)|} \leq 1 \quad \overset{\text{Iteration}}{\Longrightarrow} \quad \sup_{B_{\rho^k}(x_0)} \frac{\rho^{-k}|u(x)-u(x_0)|}{\rho^{k\alpha} + \frac{|Du(x_0)|(1-\rho^{(k-1)\alpha})}{1-\rho^\alpha}} \leq 1,$$

which proves to be the proper estimate for continuing with an iterative process, provided we get a sort of suitable control under the magnitude of the gradient (point-wisely).

Remark 3.11. We must stress that Theorem 3.9 is an improved version of the one addressed in [38, Theorem 1.1] (see also [15, Theorem 2.1] and [31, Theorem 1]).

Remark 3.12 (**[24, Corollary 1.1]**)**.** Suppose F to be a concave or convex operator. Then, u is $C^{1,\frac{1}{p_{\max}+1}}_{\text{loc}}(\Omega)$ (cf. [3, Corollary 3.2]). Moreover, for $\Omega' \Subset \Omega$ there holds

$$\|u\|_{C^{1,\frac{1}{1+p_{\max}}}(\Omega')} \leq \mathrm{C}\cdot\left(\|u\|_{L^\infty(\Omega)} + \|f\|^{\frac{1}{p_{\min}+1}}_{L^\infty(\Omega)} + 1\right).$$

Remark 3.13. One of the key points of our studies consists of removing the restriction of analyzing $C^{1,\alpha}_{\text{loc}}$ regularity estimates just along the *a priori* unknown set of singular points of solutions $\mathcal{S}_0(u,\Omega)$, where the "ellipticity of the model" collapses (cf. [3] and [4] for such an introductory approach). See e.g. [27], [28] and [62], where improved regularity estimates were addressed along certain sets of degenerate points of existing solutions.

Remark 3.14. It is worth to highlight that our geometric approach is a byproduct of an oscillation-type estimate (see [5], [26] and [30]) combined with a localized analysis, whose proof is conducted by studying two cases:

- ✓ If the gradient is small with a controlled magnitude (a "singular zone"), then a perturbation of the $\mathfrak{F}-$harmonic profile leads to the inhomogeneous problem at the limit via a stability argument in a C^1-fashion. Thus, geometric iteration yields the desired estimate at interior points of singular zone.
- ✓ On the other hand, if the gradient has a uniform lower bound, i.e., $|Du| \geq L_0 > 0$, then classical estimates (see [17], [18] and [68]) can be enforced since the operator becomes uniformly elliptic:

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) \leq \mathrm{C}_0(L_0^{-1}, \|f\|_{L^\infty(\Omega)}) \text{ and } \mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) \geq -\mathrm{C}_0(L_0^{-1}, \|f\|_{L^\infty(\Omega)}).$$

We must cite the works [15], [31], [38] and [40], which explore a deviation by planes strategy to prove Hölder gradient estimates.

Remark 3.15. Finally, an important fact concerning the Theorem 3.9 is that our estimates do not depend on Hölder continuity of the modulating function $\mathfrak{a}$ neither of the compatibility conditions imposed on p and q as the ones in the Theorems 1.1 and 1.2, [44], [50] and [54, Theorem 1.2] for the double phase variational scenarios.

From now on, we will label the singular zone of existing solutions as

$$\mathcal{S}_{r,\theta}(u,\Omega') := \left\{x_0 \in \Omega' \Subset \Omega : |Du(x_0)| \leq r^{\theta},\ \text{ for } \ 0 \leq r \ll 1\right\}.$$

A geometric inspection to Theorem 3.9 ensures that if u satisfies (1.5) and $x_0 \in \mathcal{S}_{r,\alpha}(u,\Omega')$, then near x_0 we get

$$\sup_{B_r(x_0)} |u(x)| \leq |u(x_0)| + \mathrm{C}\cdot r^{1+\alpha}.$$

Nonetheless, from a geometric perspective, it plays an essential qualitative role to obtain the (counterpart) sharp lower bound estimate for such PDEs with non-homogeneous degeneracy law. This feature is designated *non-degeneracy property* of solutions.

Therefore, under a natural, non-degeneracy hypothesis on f, we present the precise behavior of solutions at certain (interior) "singular zones".

Theorem 3.16 (**Non-degeneracy property - [24, Theorem 1.4]).** *Suppose that the assumptions of Theorem 3.9 are in force. Let u be a bounded viscosity solution to (1.5) with $f(x) \geq \mathfrak{m} > 0$ in Ω. Given $x_0 \in \mathcal{S}_{r,\alpha}(u, B_r(x_0))$ for $r \in \left(0, \frac{1}{2}\right)$, there exists a constant $\mathrm{c}_0 > 0$, such that*

$$\sup_{\partial B_r(x_0)} \frac{u(x) - u(x_0)}{r^{1+\frac{1}{p_{max}+1}}} \geq \mathrm{c}_0(\mathfrak{m}, \|\mathfrak{a}\|_{L^\infty(\Omega)}, \mathrm{L}_1, n, \lambda, \Lambda, p_{min}, q_{min}, \Omega).$$

Such a quantitative information plays an essential role in the development of several analytic and geometric problems, as in the study of blow-up procedure and related weak geometric and free boundary analysis (cf. [27] and [28] for related topics).

Remark 3.17. Finally, when developing the non-degeneracy property, for $x_0 \in \Omega' \Subset \Omega$ and $\varepsilon > 0$ fixed, we introduce the scaled function

$$u^{\varepsilon}_{r,x_0}(x) := \frac{u(x_0 + rx) - u(x_0) + \varepsilon}{r^{\frac{p_{\min}+2}{p_{\min}+1}}} \quad \text{for} \quad x \in B_1(0),$$

which satisfies in the viscosity sense the equation

$$\mathcal{H}_{r,x_0}(x, Du_{r,x_0})F_{r,x_0}(x, D^2u_{r,x_0}) = f_{r,x_0}(x) \quad \text{in} \quad B_1(0),$$

where

$$\begin{cases} F_{r,x_0}(x, \mathrm{X}) & := & r^{\frac{p_{\min}}{p_{\min}+1}} F\left(x_0 + rx, r^{-\frac{p_{\min}}{p_{\min}+1}}\mathrm{X}\right) \\ f_{r,x_0}(x) & := & f(x_0 + rx) \\ \mathcal{H}_{r,x_0}(x, \xi) & := & r^{-\frac{p_{\min}}{p_{\min}+1}}\mathcal{H}\left(x_0 + rx, r^{\frac{1}{p_{\min}+1}}\xi\right) \\ \mathfrak{a}_{r,x_0}(x) & := & r^{\frac{q_{\max}-p_{\min}}{p_{\max}+1}}\mathfrak{a}(x_0 + rx). \end{cases}$$

Also, we define the barrier function

$$\Theta(x) := \mathfrak{c} \cdot |x|^{\frac{p_{\min}+2}{p_{\min}+1}},$$

where we choose the constant $\mathfrak{c} > 0$ such that

$$\mathcal{H}_{x_0,r}(x, D\Theta)F_{x_0,r}(x, D^2\Theta) < f_{x_0,r}(x) \quad \text{in} \quad B_R(0) \Subset \Omega.$$

In conclusion, under suitable adjustments in the constants, the next step is making use of the Comparison Principle (Theorem 3.4) for the profiles u^{ε}_{r,x_0} and Θ to reach a contradiction as $\varepsilon \to 0$; see [24, Theorem 1.4] for more details.

4 Further results and applications

Suppose that $p(\cdot)$ and $q(\cdot)$ enjoy a universal modulus of continuity, i.e., there is a non-decreasing function $\hat{\omega} : [0, +\infty) \to [0, +\infty)$ such that

$$|p(x) - p(y)| + |q(x) - q(y)| \le \mathfrak{L}_0 \hat{\omega}(|x - y|) \quad \text{with} \quad \hat{\omega}(0) = 0,$$

which satisfies the balancing condition:

$$\limsup_{s \to 0^+} \ \hat{\omega}(s) \ln(s^{-1}) \le \mathfrak{L}_1 < \infty. \tag{4.1}$$

Thus, we can present a finer regularity result (cf. [15, Theorem 7.2]):

Theorem 4.1 (Improved point-wise estimates - [24, Theorem 5.1]). *Assume that assumptions of Theorem 3.9 are in force. Suppose F to be a concave/convex operator and assumption (4.1) holds. Then, for any $x_0 \in B_{\frac{1}{2}}$ we have that u is $C^{1, \frac{1}{p(x_0)+1}}_{loc}$ at x_0. Moreover, there holds*

$$\sup_{B_r(x_0)} \frac{|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)|}{r^{1+\frac{1}{1+p(x_0)}}} \le \mathrm{C}(n, \lambda, \Lambda, \mathfrak{L}_0, \mathfrak{L}_1) \cdot \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p_{min}+1}} + 1 \right).$$

Remark 4.2. We will summarize the proof. First, we may suppose that $x_0 = 0$. Now, following [15, Theorem 7.2] we can estimate

$$\rho^{1-\frac{1+p(\rho x)}{1+p(0)}} \le \left[\left(\rho^{-1}\right)^{\mathfrak{L}_0 \omega(\rho)} \right]^{\frac{1}{1+p(0)}} \quad \text{for all} \quad x \in B_1.$$

Moreover, assumption (4.1) yields

$$\left(\rho^{-1}\right)^{\omega(\rho)} \leq \left(\rho^{-1}\right)^{\sqrt{2}\mathfrak{L}\ln(\rho^{-1})} \leq e^{\sqrt{2}\mathfrak{L}}.$$

As a consequence, for some $\rho_0 \ll 1$ we obtain

$$\rho^{1-\frac{1+p(\rho x)}{1+p(0)}} \leq \Phi(\mathfrak{L}_0, \mathfrak{L}) \quad \text{for all} \quad \rho \ll \rho_0. \tag{4.2}$$

Thus, as in the proof of Theorem 3.9 we can establish

$$\sup_{B_{\rho^k}(0)} |u(x) - u(0)| \leq \rho^{k\left(1+\frac{1}{1+p(0)}\right)} + |Du(0)| \sum_{j=0}^{k-1} \rho^{k+\frac{j}{1+p(0)}} \quad \forall\, k \in \mathbb{N}, \tag{4.3}$$

provided a smallness regime and (4.2) are in force

$$\max\left\{\Theta_{\mathrm{F}}(x), \|f\|_{L^\infty(B_1)}\right\} \leq \frac{\delta_\iota}{\Phi(\mathfrak{L}_0, \mathfrak{L}) + 1}.$$

The above estimates in (4.3) imply:

$$\sup_{B_r(0)} \frac{|u(x) - u(0)|}{r^{1+\frac{1}{1+p(0)}}} \leq \mathrm{M}_0 \cdot \left(1 + |Du(0)| r^{-\frac{1}{1+p(0)}}\right), \ \forall r \in (0, \rho),$$

which completes the desired estimate around singular points.

Remark 4.3. In contrast with Theorem 3.9, in the above result, we obtain a finer point-wise estimate, because

$$1 + \frac{1}{1 + p(x_0)} \geq 1 + \frac{1}{1 + p_{\max}} \quad \forall\, x_0 \in B_{1/2}.$$

Furthermore, different from [15, Theorem 7.2], in our approach we withdraw the restriction of analyzing $C^{1,\alpha}$ estimates along an *a priori* unknown set of singular points of solutions, i.e., $\mathcal{S}_0(u, \Omega')$, where the uniform ellipticity character of the operator is lost.

Example 4.4. Let us consider variable exponents fulfilling the Log-condition:

$$|p(x) - p(y)| + |q(x) - q(y)| \leq \frac{\omega^*(|x-y|)}{|\ln(|x-y|^{-1})|} \quad \forall\ x, y \in \Omega,,\ x \neq y,$$

for a universal modulus of continuity $\omega^* : [0, +\infty) \to [0, +\infty)$. Notice that the function $s \mapsto \frac{\omega^*(s)}{|\ln(s^{-1})|}$ is non-decreasing on $(0, s^*)$ with

$$\lim_{s \to 0} \frac{\omega^*(s)}{|\ln(s^{-1})|} = 0,$$

thereby satisfying the assumption (4.1).

Example 4.5. In particular, such a class in (4.1) embraces the so-called *Log-Hölder condition*, i.e.,

$$|p(x) - p(y)| + |q(x) - q(y)| \le \frac{\mathfrak{c}_{p,q}}{-\ln(|x-y|)} \quad \forall\, x, y \in \Omega, \text{ with } |x-y| \le \frac{1}{2},$$

which plays an important role in obtaining regularity estimates in many context of elliptic PDEs with non-standard growth (see [37, Section 4.1], [52, Section 4] and [53, Part 1]).

4.1 Regularity for Strong $p(x)-$Laplace under constraints

Remember that the strong $p(x)-$Laplace operator was introduced in the Adamowicz-Hästö's fundamental works [1] and [2], which were inspired on the strong form of the $p-$Laplace operator, i.e.,

$$\hat{\Delta}_p u := |Du|^{p-4}(|Du|^2 \Delta u + (p-2)\Delta_\infty u),$$

where

$$\Delta_\infty u := \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = Du \cdot D^2 u Du^t$$

is the $\infty-$*Laplacian operator* (see [6] for a description on this topic).

Now, replacing p by $p(x)$ in the above definition, it yields a kind of generalization of the $p-$Laplace operator, i.e., the Strong $p(x)-$Laplace:

$$-\Delta^S_{p(x)} u = -\text{div}(|Du|^{p(x)-2} Du) + |Du|^{p(x)-2} \log(|Du|) Du \cdot Dp.$$

Siltakoski in [60] has considered the notion of viscosity to the so-called normalized $p(x)-$Laplace equation given by

$$-\Delta^N_{p(x)} u := -\Delta u - \frac{p(x)-2}{|Du|^2} \Delta_\infty u = 0, \tag{4.4}$$

whose interest in such a class of equation stems from its connection between PDEs and Probability Theory (stochastic tug-of-war games with spatially varying probabilities, see [7] for related topic).

Formally, one has

$$-|Du|^{p(x)-2}\Delta^N_{p(x)}u = -\mathrm{div}(|Du|^{p(x)-2}Du) + |Du|^{p(x)-2}\log(|Du|)Du\cdot Dp.$$

As a result, a notion of weak and viscosity solutions can be based on the Strong $p(x)-$Laplacian equation

$$-\Delta^S_{p(x)}u = 0. \tag{4.5}$$

In fact, Siltakoski proved (see [60, Theorem 4.1]) that viscosity solutions of (4.4) are equivalent to viscosity solutions of (4.5), provided $p\in C^{0,1}(\Omega)$ and $\inf\limits_{\Omega} p(x) > 1$. In particular, viscosity solutions to (4.4) are $C^{1,\alpha'}_{\text{loc}}$ for some $\alpha'\in(0,1)$ (cf. [71]). Furthermore, Siltakoski addressed the result:

Lemma 4.6 ([60, Lemma 6.2])**.** *A function u is a viscosity solution to (4.4) if and only if it is a viscosity solution to*

$$-|Du|^2\Delta u - (p(x)-2)\Delta_\infty u = -\mathrm{Tr}\left((|Du|^2\mathrm{Id}_n + (p(x)-2)Du\otimes Du)D^2u\right) = 0, \tag{4.6}$$

As an immediate consequence of Lemma 4.6, [60, Theorem 4.1] and our findings (i.e., Theorem 3.9) we are able to present the sharp estimates:

Theorem 4.7 ([24, Theorem 5.2]). *Let u be a bounded weak solution to (4.5). Assume further the assumptions on [60, Theorem 4.1] and Lemma 4.6 are in force, and $\mathcal{G}[u] = (p(x)-2)\Delta_\infty u \in L^\infty(B_1)$. Then, $u\in C^{1,\frac{1}{3}}_{loc}(B_1)$. Moreover, there holds*

$$[u]_{C^{1,\frac{1}{3}}\left(B_{\frac{1}{2}}\right)} \le \mathrm{C}(\mathtt{universal})\cdot\left[\|u\|_{L^\infty(B_1)} + \sqrt[3]{\|\mathcal{G}[u]\|_{L^\infty(B_1)}}\right].$$

Remark 4.8. Theorem 4.7 might be interpreted as a sort of weaker form of the longstanding $C^{1,\frac{1}{3}}$ conjecture for inhomogeneous $\infty-$Laplacian equation, which states that if v is a bounded viscosity solution to (cf. [3])

$$\Delta_\infty v = f\in L^\infty(B_1) \qquad \Rightarrow \qquad v\in C^{1,\frac{1}{3}}_{\text{loc}}(B_1).$$

4.2 Regularity for models with $(p(x), q(x))-$growth

Our methods also allow us to access regularity results for solutions to non-linear elliptic problems with $(p(x), q(x))-$growth as follows (cf. [54]):

$$-\mathcal{L}w(x) := \operatorname{div}\left(|Dw|^{p(x)-2}Dw + \mathfrak{a}(x)|Dw|^{q(x)-2}Dw\right) = 0.$$

Remember that such solutions might be obtained as minimizers for functionals of double phase with variable exponents given by

$$w \mapsto \min_{u_0+W_0^{1,1}(\Omega)} \int_\Omega \left(\frac{1}{p(x)}|Dw|^{p(x)} + \frac{\mathfrak{a}(x)}{q(x)}|Dw|^{q(x)}\right) dx.$$

Recently, Ragusa-Tachikawa in [54, Theorem 1.2] addressed a $C^{1,\gamma}_{\text{loc}}(\Omega)$-regularity result for such minimizers provided that $p(\cdot), q(\cdot)$ and $a(\cdot)$ are real-valued functions on Ω satisfying:

1. $q(x) \geq p(x) \geq p_0 > 1$ and $\mathfrak{a}(x) \geq 0$;
2. $p(\cdot), q(\cdot) \in C^{0,\sigma}(\Omega)$, $a(\cdot) \in C^{0,\alpha'}(\Omega)$ $(\alpha', \sigma \in (0,1])$;
3. If $\beta' := \min\{\alpha', \sigma\}$ then

$$\sup_{x\in\Omega} \frac{q(x)}{p(x)} < 1 + \frac{\beta'}{n}.$$

Now, from equivalence of notion solutions in [39] and [60], and by straightforward computations such an operator can be re-written as:

$$-\mathcal{L}w(x) := \mathcal{J}_1(Du, D^2w) + \mathcal{J}_2(Du, D^2w) = 0, \tag{4.7}$$

where

$$\left\{\begin{array}{lcl} \mathcal{J}_1(Dw, D^2w) & := & \left[|Dw|^{p(x)-2} + \mathfrak{a}(x)|Dw|^{q(x)-2}\right]\Delta w(x) \\ \mathcal{J}_2(Dw, D^2w) & := & \left[(p(x)-2)\,|Dw|^{p(x)-2} + (q(x)-2)\,\mathfrak{a}(x)|Dw|^{q(x)-2}\right]\Delta_\infty^N w(x) \\ & + & \left(|Du|^{p(x)-2} + |Du|^{q(x)-2}\right)\log(|Du|)Du\cdot(Dp+Dq) \\ & + & |Du|^{q(x)-2}Du\cdot D\mathfrak{a}, \end{array}\right.$$

and

$$\Delta_\infty^N w(x) := \frac{1}{|Du|^2}\sum_{i,j} u_{x_i}u_{x_j}u_{x_ix_j}$$

is the *normalized* $\infty-$*Laplacian operator* (see, [49]).

Therefore, in contrast with [54, Theorem 1.2] and under constraints by invoking our sharp estimates in Theorem 3.9 we are able to present (as a direct consequence) the optimal regularity result (cf. [29] - see also [5] for related results to $C^{p'}-$conjecture for $p > 2$).

Theorem 4.9 (**[24, Theorem 5.3]**). *Let* $w \in C^{0,1}(B_1)$ *be a bounded viscosity solution to* (4.7)*. Assume further* $\Delta_\infty^N w, D\mathfrak{a}, Dp, Dq \in L^\infty(B_1)$*. Then,* $w \in C^{1,\frac{1}{p_{max}-1}}_{loc}(B_1)$*. Additionally, the following estimate holds*

$$[w]_{C^{1,\frac{1}{p_{max}-1}}\left(B_{\frac{1}{2}}\right)} \leq \mathrm{C}(\texttt{universal}) \cdot \left[\|w\|_{L^\infty(B_1)} + \|\mathcal{J}_2\|_{L^\infty(B_1)}^{\frac{1}{p_{min}-1}}\right].$$

4.3 Geometric free boundary problems: sharp estimates

We stress that our geometric approach is particularly refined to be carried out in other classes of problems. In particular, we may study dead-core problems for fully nonlinear models with unbalanced variable degeneracy:

$$\left[|Du|^{p(x)} + \mathfrak{a}(x)|Du|^{q(x)}\right]\Delta u = f_0(x)\cdot u^{\varsigma}\chi_{\{u>0\}} \quad \text{in} \quad \Omega, \tag{4.8}$$

where $\varsigma \in [0, p_{\max}+1)$ is the order of reaction, f_0 is bounded away from zero and infinity and assumptions (A0)-(A5) are in force. We refer the reader to [27] and [28] for a mathematical description of such dead-core problems governed by quasi-linear degenerate operators.

In contrast with Theorem 3.9 we are able to present an improved regularity estimate for non-negative solutions of (4.8) along their touching ground boundary $\partial\{u > 0\}$ (cf. [23], [27], [28] and [63] for similar results). The proof for such improved estimate follows the same lines as [23, Theorem 1.2] and [28, Theorem 1.2].

Theorem 4.10 (**Improved regularity along free boundary - [24, Theorem 5.4]**)**.** *Let* u *be a nonnegative and bounded viscosity solution to* (4.8)*. Then, given* $r_0 > 0$ *there exists a constant* $\mathrm{C}_0 > 0$ *depending only on*

$n, p_{min}, q_{max}, r_0, \inf_\Omega f(x)$ such that for any $x_0 \in \Omega$ such that $B_{r_0}(x_0) \subset \Omega$ and any $r \leq \frac{r_0}{2}$, the following estimate holds

$$\sup_{B_r(x_0)} u(x) \leq \mathrm{C}_0 \cdot \min\left\{\inf_{B_r(x_0)} u(x),\ r^{\frac{p_{max}+2}{p_{max}+1-\varsigma}}\right\}.$$

In particular, if $x_0 \in \partial\{u > 0\} \cap \Omega$ (i.e., a free boundary point), then

$$\sup_{B_r(x_0)} u(x) \leq \mathrm{C}_0 \cdot r^{\frac{p_{max}+2}{p_{max}+1-\varsigma}}.$$

Remark 4.11. Such improved regularity result come out by combining (in an iterative way) the Serrin's Harnack inequality (see [57] and [58]) with the scaling invariance of the equation and the optimal scaling for solutions around free boundary points $v_r(x) = \frac{u(x_0+rx)}{r^{\frac{p_{\max}+2}{p_{\max}+1-\varsigma}}}$ for $x_0 \in \partial\{u > 0\} \cap \Omega$.

Another context in which we can apply our findings is to a family of solutions to inhomogeneous singular perturbation problems in the non-variational framework; these appear naturally when studying high energy activation model in combustion and flame propagation theories (cf. [19] and [70] for the stationary divergence setting and [4], [55] and [56] for related non-divergence topics). The simplest instance is given by looking for non-negative profiles u^ε (for each $\varepsilon > 0$ fixed) satisfying

$$\begin{cases} \left[|Du^\varepsilon|^{p_\varepsilon(x)} + \mathfrak{a}_\varepsilon(x)|Du^\varepsilon|^{q_\varepsilon(x)}\right]\Delta u^\varepsilon & = & \zeta_\varepsilon(u^\varepsilon) + f_\varepsilon(x) & \text{in} & \Omega, \\ u^\varepsilon(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

in the viscosity sense, for suitable data $p_\varepsilon(\cdot), q_\varepsilon(\cdot)$, $\mathfrak{a}_\varepsilon(\cdot)$, g, where

$$\zeta_\varepsilon(s) := \frac{1}{\varepsilon}\zeta\left(\frac{s}{\varepsilon}\right)$$

behaves singularly of order $\mathrm{o}\left(\varepsilon^{-1}\right)$ near ε-surfaces and $0 < c_0 \leq f_\varepsilon \leq c_0^{-1}$.

In this scenario, existing solutions are globally (uniformly) Lipschitz continuous (see [11, Theorem 1.4] and [55, Theorem 1] for specific results) such that

$$\|Du^{\varepsilon}\|_{L^{\infty}(\overline{\Omega})} \leq \mathrm{C}(n, (p_{\varepsilon})_{\min}, (q_{\varepsilon})_{\max}, \|\zeta\|_{L^{\infty}(\Omega)}, \|\mathfrak{a}_{\varepsilon}\|_{L^{\infty}(\Omega)}, \|g\|_{C^{1,\kappa}(\partial\Omega)}, \|f_{\varepsilon}\|_{L^{\infty}(\Omega)}, \Omega).$$

As a result, up to a subsequence, there exists a function u_0, obtained as the uniform limit of $u^{\varepsilon_{k_j}}$, as $\varepsilon_{k_j} \to 0$. Moreover, such a limiting profile satisfies in the viscosity sense

$$\left[|Du_0|^{p_0(x)} + \mathfrak{a}_0(x)|Du_0|^{q_0(x)}\right] \Delta u_0 = f_0(x)$$

for an appropriate $f_0 \geq 0$ (see [11, Theorem 1.7]). For these applications, we refer the reader to [4], [11] and [25].

Finally, we might consider to address regularity estimates to nonlinear free boundary problems for fully non-linear elliptic problem with unbalanced variable degeneracy as follows

$$\begin{cases} \left[|Du|^{p(x)} + \mathfrak{a}(x)|Du|^{q(x)}\right] \Delta u &= f(x) \quad \text{in} \quad \Omega_+(u), \\ |Du| &= \mathrm{Q}(x) \quad \text{on} \quad \mathfrak{F}(u), \end{cases} \tag{4.9}$$

where the assumptions (A0)-(A5) are satisfied, $\mathrm{Q} \geq 0$ is a continuous function and

$$u \geq 0 \text{ in } \Omega, \quad \Omega_+(u) := \{x \in \Omega : u(x) > 0\} \quad \text{and} \quad \mathfrak{F}(u) := \partial\Omega_+(u) \cap \Omega.$$

In such a way, by following the approach developed in the authors' work [25, Theorems 1.3 and 1.4] we might address fine properties of the free boundary:

Flat/Lipschitz free boundaries are $C^{1,\gamma}_{\text{loc}}$ for some $\gamma(\texttt{universal}) \in (0,1)$.

Precisely, we adapt the technique presented by De Silva in [36] to prove that flat free boundaries are $C^{1,\beta}$:

Theorem 4.12 (Flatness implies $C^{1,\beta}$ - [25, Theorem 1.3]). *Let u be a viscosity solution to (4.9) in B_1. Suppose that $0 \in \mathfrak{F}(u)$, $\mathrm{Q}(0) = 1$ and $F(0, X)$ is uniformly elliptic. Then, there exists a constant $\tilde{\varepsilon}(\texttt{universal}) > 0$ such that, if the graph of u is $\tilde{\varepsilon}$-flat in B_1, i.e.,*

$$(x_n - \tilde{\varepsilon})^+ \leq u(x) \leq (x_n + \tilde{\varepsilon})^+ \quad \textit{for} \quad x \in B_1,$$

and

$$\max\left\{\|f\|_{L^\infty(B_1)},\quad [\mathrm{Q}]_{C^{0,\alpha}(B_1)}, \|F\|_{C^\omega(B_1)}\right\} \le \tilde{\varepsilon},$$

then $\mathfrak{F}(u)$ *is* $C^{1,\beta}$ *in* $B_{1/2}$ *for some (universal)* $\beta \in (0,1)$.

Remark 4.13. The Flatness assumption implies

$$\left(\langle x, \nu\rangle - r\frac{\varepsilon}{2}\right)^+ \le u(x) \le \left(\langle x, \nu\rangle + r\frac{\varepsilon}{2}\right)^+ \quad \text{for all } x \in B_r,$$

where $r \le r_0$(universal), $\varepsilon \le \varepsilon_0(r)$, $\nu \in \mathbb{S}^{n-1}$ and $|\nu - e_n| \le \mathrm{c}\varepsilon^2$. Then, be performing an iterative argument we obtain

$$(\langle x, \nu_k\rangle - \varepsilon_k)^+ \le u_k(x) \le (\langle x, \nu_k\rangle + \varepsilon_k)^+ \quad \text{for all } x \in B_1,$$

where $|\nu_k - \nu_{k+1}| \le \mathrm{c}\frac{\varepsilon_0}{2^k}$. Therefore, $u := \lim\limits_{k\to\infty} u_k$ and

$$\partial\{u > 0\} \cap B_{r^k} \subset \left\{|\langle x, \nu_k\rangle| \le \frac{\varepsilon_0}{2^k} r^k\right\},$$

which implies $\mathfrak{F}(u) \cap B_{\frac{3}{4}}$ is a $C^{1,\beta}-$ graph.

As stated in [36], the previous improvement of flatness strongly rely on a blow-up argument, which depend on the following key devices:

1. **Compactness via Harnack type inequality**: via a contradiction argument and smallest regime on data

$$v_k(x) := \frac{u_k(x) - x_n}{\varepsilon_k} \quad \text{satisfies} \quad -1 \le v_k \le 1.$$

   Thus, $\mathfrak{F}(u_k) \to \{x_n = 0\} \cap B_1$ in the Hausdorff distance. Hence, graph of v_k in $\Omega_{1/2}(u_k)$ converges (in the Hausdorff distance) for a graph of a Hölder continuous function u_∞ (due to a Harnack type inequality).

2. **Characterizing of limiting profiles**: u_∞ satisfies in the viscosity sense

$$\begin{cases} F_\infty(D^2 u_\infty) & = & = & \text{in} & B_{\frac{1}{2}} \cap \{x_n > 0\} \\ \frac{\partial u_\infty}{\partial x_n} & = & 1 & \text{on} & B_{\frac{1}{2}} \cap \{x_n = 0\}, \end{cases}$$

which enjoys good a priori estimates

$$\sup_{B_\rho^+} \frac{|u_\infty(x) - u_\infty(0) - Du_\infty(0)\cdot x|}{\rho^{1+\alpha}} \leq \mathrm{C}_0 \quad \text{for} \quad \rho \in \left(0, \frac{1}{2}\right).$$

for universal constants $\alpha \in (0,1)$ and $\mathrm{C}_0 > 0$ due to Milakis-Silvestre's work [51].

Through a blow-up argument from Theorem 4.12 and the approach used in [16] and [36], we obtain our last main result:

Theorem 4.14 (Lipschitz implies $C^{1,\beta}$ - [25, Theorem 1.4]). *Let u be a viscosity solution for the free boundary problem (4.9). Assume further that $0 \in \mathfrak{F}(u)$, $f \in L^\infty(B_1)$ is continuous in $B_1^+(u)$ and $\mathrm{Q}(0) > 0$. If $\mathfrak{F}(u)$ is a Lipschitz graph in a neighborhood of 0, then $\mathfrak{F}(u)$ is $C^{1,\beta}$ in a (smaller) neighborhood of 0.*

Remark 4.15. Following the blow-up argument employed in the proof of Theorem 4.12, the Lipschitz assumption on $\mathfrak{F}(u_k)$ implies that $\mathfrak{F}(u_0)$ is Lipschitz, where $u_0 := \lim\limits_{k\to\infty} u_k$. Hence,

$$(x_n - \tilde{\varepsilon})^+ \leq u_k(x) \leq (x_n + \tilde{\varepsilon})^+ \quad \forall \ k \geq 1.$$

Therefore, by Theorem 4.12 we conclude that $\mathfrak{F}(u_k)$ is $C^{1,\beta}_{\text{loc}}(B_1)$ for some exponent $\beta \in (0,1)$ universal, thereby concluding the result.

Remark 4.16. In Theorem 4.14, the size of the neighborhood where $\mathfrak{F}(u)$ is $C^{1,\beta}$ depends on the radius r of the ball B_r where $\mathfrak{F}(u)$ is Lipschitz, the Lipschitz norm of $\mathfrak{F}(u)$, n and $\|f\|_{L^\infty(B_1)}$. We also emphasize that to obtain the Theorems 4.12 and 4.14 via the *improvement of flatness property* for the graph of u, we need a version of Hopf type estimate, Harnack inequality, Lipschitz regularity, and non-degeneracy for u.

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