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Mapping properties of geometric elliptic operators in conformally conical spaces: an introduction with examples

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Dedicated to Professor Renato Tribuzy,
on the occasion of his 75th birthday

Abstract. In this largely expository note, we discuss the mapping properties of the Laplacian (and other geometric elliptic operators) in spaces with an isolated conical singularity following the approach developed by B.-W. Schulze and collaborators. Our presentation aims at illustrating the versatility of these results by describing how certain representative (and seemingly disparate) applications in Geometric Analysis follow from a common setup.

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1 Introduction

The theory of elliptic differential operators acting on sections of a vector bundle over a compact manifold X is a well established discipline [13, 27, 49, 24]. If X is boundaryless then we may resort to the fact that any such manifold can be infinitesimally identified to euclidean space around each of its points in order to transplant the symbolic calculus of pseudo-differential operators in flat space to this "curved" arena. As a consequence, the main technical result in the theory is proved, namely, the existence of a (pseudo-differential) parametrix for the given elliptic differential operator D, from which the standard mapping properties (regularity of solutions, Fredholmness, etc) in the usual scale of Sobolev spaces may be readily derived. From this perspective, we may assert that the resulting theory is a natural outgrowth of Fourier Analysis as applied to the classical procedure of "freezing the coefficients".

In case the underlying manifold X carries a boundary ∂X, a fundamentally distinct approach is needed as the local identification to euclidean space obviously fails to hold in a neighborhood of a point in the boundary (from this standpoint, we are forced to view ∂X as the "singular locus" of X). We may, however, pass to the double of X, say $2X$, and assume that a suitable *elliptic* extension of the original operator D, say $2D$, is available. Since $\partial(2X) = \varnothing$, we have at our disposal a parametrix for $2D$ which may be employed to construct a pseudo-differential projection C acting on sections restricted to ∂X (the Calderón-Seeley projector). If the differential operator B defining the given boundary conditions is such that the principal symbol of $A = BC$ is sufficiently non-degenerate (for instance, if the pair (D, B) satisfies the so-called Lopatinsky-Shapiro condition) then a parametrix for A is available (recall that $\partial(\partial X) = \varnothing$) and from this we may deduce the expected mapping properties of the associated boundary value map

$$\mathcal{D}u = (Du, Bu|_{\partial X}) \tag{1.1}$$

acting on suitable Sobolev spaces. Thus, the theory of elliptic boundary value problems ultimately hinges on the fact that the corresponding singular locus ∂X not only is intrinsically smooth but also may be easily "resolved" after passage to $2X$[1].

We may now envisage a situation where the underlying space X displays a singular locus Y which fails to admit such a simple resolution (as a boundary does). For instance, we may agree that the singular locus $Y \subset X$ has the structure of a smooth closed manifold and that a neighborhood $U \subset X$ of Y is the total space of a fiber bundle

$$\begin{array}{ccc} \mathcal{C}^F & \hookrightarrow & U \\ & & \downarrow \pi \\ & & Y \end{array}$$

whose typical fyber is a cone $\mathcal{C}^F$ over a closed manifold F. For simplicity, we assume that this bundle is trivial, so that U carries natural coordinates (x, y, z), where $y \in Y$, $z \in F$ and x is the radial function obtained after identifying the cone generatrix to the inverval $[0, \delta]$, $\delta > 0$, with $x = 0$ along Y. We now infuse a bit of geometry in this discussion by requiring that the smooth locus $X' = X \backslash Y$ carries a Riemannian metric $\overline{g}$ so that

$$\overline{g}|_{U'} = dx^2 + x^2 g_F(z) + g_Y(y), \quad U' = U \backslash Y, \tag{1.2}$$

where g_F and g_Y are fixed Riemannian metrics on F and Y, respectively. By abuse of language, we say that x is a "defining function" for Y (with respect to $\overline{g}$).

The simplest of such "edge-type" manifolds occurs when Y collapses into a point, so we obtain a conical manifold (see Definition 2.1 below). In any case, we are led to consider *geometric* differential operators (i.e. naturally associated to $\overline{g}$ such as the Laplacian acting on functions, the Dirac operator acting on spinors, etc.) and pose the general problem of studying their mapping properties in suitable functional spaces. The main

[1] This trick of passing from the bordered manifold X to the boundaryless manifolds ∂X and $2X$ is a key ingredient in index theory [7].

purpose of this note is to illustrate through examples how useful this elliptic analysis on singular spaces turns out to be.

The problem remains of transplanting the highly successful "smooth" elliptic theory outlined above to this setting. Clearly, the edge-type structure around Y poses an obvious obstruction to a straightforward extension of the pseudo-differential calculus. Indeed, this leads us to suspect that, besides the standard ellipticity assumption on X', a complementary notion of ellipticity around Y is required in order to construct a global parametrix. In this regard, there is no canonical choice and the final formulation depends on which technique one is most familiar with. In the rather informal (and simplified) exposition below, which actually emphasizes the conical case (so that $Y = \{q\}$), we roughly follow the approach developed by B.-W. Schulze and collaborators [47, 18], as we believe it displays an adequate balance between technical subtlety and conceptual transparency. In this setting, a key ingredient is the classical Mellin transform, which allows us to pass from the restriction to U' of the given geometric operator D to its *conormal symbol* ξ_D. The appropriate complementary notion of ellipticity is then formulated by fixing $\beta \in \mathbb{R}$ and then requiring that ξ_D, viewed as a polynomial function whose coefficients are differential operators acting on the fiber F, is invertible when restricted to the vertical line $\Gamma_\beta = \{z \in \mathbb{C}; \mathrm{Re}\, z = \beta\}$[2]. Armed with this notion of ellipticity, an appropriate pseudo-differential calculus may be conceived which leads to the construction of the sought-after full parametrix, that is, an inverse for D up to a compact operator; this has as a formal consequence the Fredholmness of $D = D_\beta$ when acting on the so-called Sobolev-Mellin scale $\mathcal{H}^{\sigma,p}_\beta(X)$, *independently* of $(\sigma, p) \in \mathbb{R} \times \mathbb{Z}_+$. Moreover, the index of D_β jumps precisely at those values of β for which ellipticity fails by an integer quantity depending on the kernel of $\xi_D|_{\Gamma_\beta}$. For instance, if $D = \Delta$, the Laplacian

[2]The moral here is that, when trying to freeze the coefficients of D around the tip of the cone, we are inevitably led to contemplate the Mellin transform as the proper analogue of the Fourier transform which, as already noted, does a perfectly good job in the smooth locus.

of the underlying conical metric, which is our main concern here, then $\xi_\Delta(z) = z^2 + bz + \Delta_{g_F}$ for some $b \in \mathbb{R}$, so a jump occurs at each β satisfying the *indicial equation*

$$\beta^2 + b\beta - \mu = 0, \tag{1.3}$$

for some $\mu \in \mathrm{Spec}(\Delta_{g_F})$, and equals the multiplicity of μ as an eigenvalue. Thus, if a fairly precise knowledge of $\mathrm{Spec}(\Delta_{g_F})$ is available, the Fredholm index of Δ in the whole scale $\mathcal{H}^{s,p}_\beta(X)$ can be determined upon computation at a single value of β. This final piece of calculation may be carried out by using the fact that Δ gives rise to a densely defined, unbounded operator, say Δ_β, acting on the Hilbert sector of the scale, namely, $\mathcal{H}^{\bullet,2}_\beta(X)$. A separate argument, which boils down to identifying the minimal and maximal domains of this operator, then assures the existence of at least a β_0 such that Δ_{β_0} has a *unique* closed extension (which is necessarily Fredholm). Usually, β_0 lies in the interval determined by the indicial roots (the solutions of (1.3)) corresponding to $\mu = 0$, so that in case $b \neq 0$, Δ_β turns out to be Fredholm with the *same* index as long as β varies in the interval with endpoints 0 and $-b$; compare with Theorem 2.11.

A detailed presentation of the program outlined above for a general elliptic operator is far beyond the scope of this introductory note. Instead, we merely sketch the argument for the Laplacian in conical manifolds (Sections 2 and 3) and indicate how the method can be extended to other geometric operators by considering the case of the Dirac operator (Section 4). In fact, here we focus instead on illustrating the versatility of this theory by including a few representative applications of these mapping properties in Geometric Analysis (Sections 5 and 6). We insist, however, that the material discussed here is standard, drawn from a number of sources, so no claim is made regarding originality (except perhaps for the naive computations leading to Theorem 4.4). Indeed, this note has been written in the perspective that, after reading our somewhat informal account of a noticeably difficult subject, the diligent reader will be able to fill the formidable gaps upon consultation of the original sources. In this regard, we note that, alternatively to the path just outlined, the mapping results described below may be obtained as a consequence of the powerful "boundary fibration calculus" [40, 39, 36, 30, 23, 41, 21, 29] (a comparison of Melrose's b-calculus and Schulze's cone algebra appears in [31]). Also, direct approaches, which in a sense avoid the consideration of the corresponding pseudo-differential formalism, are available in each specific application we consider here [3, 6, 9, 12, 35, 32, 5, 42]. We believe, however, that a presentation of their mapping properties as a repertory of results stemming from a common source contributes to highlight the unifying features of geometric differential operators in singular spaces.

2 Fredholmness of the Laplacian in conformally conical manifolds

In this section, we define the class of conformally conical manifolds (this entails a slight modification of (1.2) which incorporates a conformal factor involving a suitable power of the defining function x) and discuss a few representative examples in this category. We then introduce the relevant functional spaces (the Sobolev-Mellin scale $\mathcal{H}^{\sigma,p}_{\beta}$) and then formulate a result (Theorem 2.11) which precisely locates the set of values of β for which the corresponding Laplacian is Fredholm with an explictly computable index.

2.1 Conformally conical manifolds

Given a closed Riemannian manifold (F, g_F) of dimension[3] $n-1 \geq 2$, we consider the *infinite cone* $(\mathcal{C}^{(F,g_F)}, g_{\mathcal{C},F})$ over (F, g_F):

$$\mathcal{C}^{(F,g_F)} = \mathbb{R}_{>0} \times F$$

endowed with the cone metric

$$g_{\mathcal{C},F} = dr^2 + r^2 g_F, \quad r \in \mathbb{R}_{>0}. \tag{2.1}$$

[3]In fact, the general theory also works fine for $n = 2$ and the assumption $n \geq 3$ is only needed for Theorem 2.11 and its consequences.

We then define the *truncated cones* by

$$\mathcal{C}_0^{(F,g_F)} = \{(r,z) \in \mathcal{C}^{(F,g_F)}; z \in F, 0 < r < 1\}$$

and

$$\mathcal{C}_\infty^{(F,g_F)} = \{(r,z) \in \mathcal{C}^{(F,g_F)}; z \in F, 1 < r < +\infty\},$$

both endowed with the induced metric. We also consider the *infinite cylinder* $(\mathsf{C}^{(F,g_F)}, g_{\mathsf{C},F})$ over (F, g_F):

$$\mathsf{C}^{(F,g_F)} = \mathbb{R} \times F$$

endowed with the product metric

$$g_{\mathsf{C},F} = dr^2 + g_F.$$

We now consider a compact topological space X which is smooth everywhere except possibly at a point, say q. We endow the smooth locus $X' := X \backslash \{q\}$, $\dim X' = n \geq 3$, with a Riemannian metric $\overline{g}$ and assume that there exists a neighborhood U of q (the conical region) such that $U' := U \backslash \{q\}$ is diffeomorphic to $\mathcal{C}_0^{(F,g_F)}$ and

$$\overline{g}|_{U'} = g_{\mathcal{C},F} = dx^2 + x^2 g_F, \tag{2.2}$$

where for convenience we have set $x = r$ in the description of the cone metric to emphasize that x is viewed as a defining function for $\{q\}$; compare with (2.1).

Definition 2.1. A *conformally conical manifold* is a pair (X, g_s), where X is as above, and g_s is a Riemannian metric in X' which, restricted to U', satisfies

$$g_s := x^{2s-2}(\overline{g} + o(1)), \quad s \in \mathbb{R}, \tag{2.3}$$

as $x \to 0$. We then say that (F, g_F) is the *link* of (X, g_s).

Remark 2.2. Our terminology is justified by the presence of the conformal factor next to $\overline{g} + o(1)$, which allows us to arrange the examples below in a single geometric structure.

Remark 2.3. In applications, it is often needed to append decay relations to (2.3) for the corresponding derivatives up to second order at least; see Remark 2.9 below.

Remark 2.4. (X', g_s) is complete if and only if $s \le 0$.

We will be interested in doing analysis in the open manifold (X', g_s). More precisely, we will study the mapping properties of the Laplacian Δ_{g_s} in an appropriate scale of Sobolev spaces. Before proceeding, however, we discuss a few examples, which highlight the distinguished roles played by the "rigid" spaces $\mathcal{C}_0^{(F,g_F)}$, $\mathcal{C}_\infty^{(F,g_F)}$ and $\mathsf{C}^{(F,g_F)}$ as asymptotic models.

Example 2.5. (AC_0 manifolds) Let (V, h) be an open manifold for which there exists a compact $K \subset V$ and a diffeomorphism $\psi : \mathcal{C}_0^{(F,g_F)} \to V \backslash K$ such that, as $r \to 0$,

$$|\nabla_b^k(\psi^* h - g_{\mathcal{C},F})|_b = O(r^{\nu_0 - k}), \quad 0 \le k \le m.$$

Here, $m \ge 0$ is the order and $\nu_0 > 0$ is the rate of decay. Also, the subscript b refers to invariants attached to the "rigid" conical metric in the model space (the same notation is used in the examples below). We then say that (V, h) is an *asymptotically conical manifold at the origin* (AC_0). Clearly, if we take $x = r$, this corresponds to a conformally conical manifold with $s = 1$ in (2.3).

Example 2.6. (AC_∞ manifolds) Let (V, h) be an open manifold for which there exists a compact $K \subset V$ and a diffeomorphism $\psi : \mathcal{C}_\infty^{(F,g_F)} \to V \backslash K$ such that, as $r \to +\infty$,

$$|\nabla_b^k(\psi^* h - g_{\mathcal{C},F})|_b = O(r^{-\nu_\infty - k}), \quad 0 \le k \le m.$$

Here, $m \ge 0$ is the order and $\nu_\infty > 0$ is the rate of decay. We then say that (V, h) is an *asymptotically conical manifold at infinity* (AC_∞). Clearly, if we take $x = r^{-1}$, this corresponds to a conformally conical manifold with $s = -1$ in (2.3).

Example 2.7. (ACyl manifolds) Let (V,h) be an open manifold for which there exists a compact $K \subset V$ and a diffeomorphism $\psi : \mathsf{C}_\infty^{(F,g_F)} \to V\backslash K$ such that, as $r \to +\infty$,

$$|\nabla_b^k(\psi^* h - g_{\mathsf{C},F})|_b = O(e^{-(\nu_c+k)r}), \quad 0 \le k \le m.$$

Here, $m \ge 0$ is the order and $\nu_c > 0$ is the rate of decay. We then say that (V,h) is an *asymptotically cylindrical manifold* (ACyl). Clearly, if we take $x = e^{-r}$, this corresponds to a conformally conical manifold with $s = 0$ in (2.3). These manifolds play a central role in the formulation and proof of the Atiyah-Patodi-Singer index theorem [8, 40].

Example 2.8. ($\mathrm{AC}_0/\mathrm{AC}_\infty$ manifolds) Assume more generally that $V\backslash K$ decomposes as a *finite* union of ends which are either AC_0 or AC_∞. These manifolds, which are called *conifolds* in [43], appear prominently in the study of moduli spaces of special Lagrangian submanifolds; see also [28].

Remark 2.9. In all examples above, we take $m \ge 2$.

2.2 Sobolev-Mellin spaces and Fredholmness

Given $\beta \in \mathbb{R}$ and integers $k \ge 0$ and $1 < p < +\infty$, we define $\mathcal{H}_\beta^{k,p}(X)$ to be the space of all distributions $u \in L^p_{\mathrm{loc}}(X', d\mathrm{vol}_{\overline{g}})$ such that:

- for any cutoff function φ with $\varphi \equiv 1$ near q and $\varphi \equiv 0$ outside U, we have that $(1-\varphi)u$ lies in the standard Sobolev space $H^{k,p}(X', d\mathrm{vol}_{\overline{g}})$;
- there holds

$$x^\beta \mathsf{D}^j \partial_z^\alpha(\varphi u)(x,z) \in L^p(X', d_+x d\mathrm{vol}_{g_F}), \quad j + |\alpha| \le k. \tag{2.4}$$

  Here, $\mathsf{D} = x\partial_x$ is the Fuchs operator and $d_+x = x^{-1}dx$.

Using duality and interpolation, we may define $\mathcal{H}_\beta^{\sigma,p}(X)$ for any $\sigma \in \mathbb{R}$. As usual, $\mathcal{H}_\beta^{\sigma,p}(X)$ is naturally a Banach space which is Hilbert for $p = 2$. For instance, when $k = 0$ the corresponding norm to the p^{th} power reduces to the integral

$$\int |x^\beta u(x,z)|^p d_+x d\mathrm{vol}_{g_F}(z) \tag{2.5}$$

near q. These are the weighted Sobolev-Mellin spaces considered in [46], except that there they are labeled by

$$\gamma = \frac{n}{2} - \beta. \tag{2.6}$$

In order to confirm the scale character of these spaces, we recall the relevant embedding theorem; see [16, Remark 2.2] and [45, Corollary 2.5].

Proposition 2.10. *One has a continuous embedding $\mathcal{H}^{\sigma',p}_{\beta'}(X) \hookrightarrow \mathcal{H}^{\sigma,p}_{\beta}(X)$ if $\beta' \leq \beta$ and $\sigma' \geq \sigma$, which is compact if the strict inequalities hold. Also, if $\sigma > n/p$ then any $u \in \mathcal{H}^{\sigma,p}_{\beta}(X,g)$ is continuous in X' and satisfies $u(x) = O(x^{-\beta})$ as $x \to 0$.*

It is clear that the Laplacian $\Delta_{g_s} = \mathrm{div}_{g_s} \nabla_{g_s}$ defines a bounded map

$$\Delta_{g_s,\beta} : \mathcal{H}^{\sigma,p}_{\beta}(X) \to \mathcal{H}^{\sigma-2,p}_{\beta+2s}(X), \tag{2.7}$$

and our primary concern here is to study its mapping properties. As already discussed in the Introduction, we should be aware that a key point in the analysis of an elliptic operator in a conformally conical manifold is that, differently from what happens in the smooth case, invertibility of its principal symbol does not suffice to make sure that a parametrix exists. In particular, it is not clear whether (2.7) is Fredholm for some value of the weight β. It turns out that this Fredholmness property is insensitive to the pair (σ, p) but depends crucially on β [46]. Indeed, it turns out that this map is Fredholm for all but a discrete set of values of β, with the index possibly jumping only when β reaches these exceptional values. We now state a useful result that confirms this expectation for the map (2.7). For this, we introduce the quantity

$$a = (n-2)s. \tag{2.8}$$

If $s \neq 0$ then $a \neq 0$ as well if we further assume that $n \geq 3$. We then denote by I_a the open interval with endpoints a and 0.

Theorem 2.11. *If $n \geq 4$ and $a \neq 0$ then the Laplacian map $\Delta_{g_s,\beta}$ in (2.7) is Fredholm of index 0 whenever $\beta \in I_a$.*

As already remarked, from this we can read off the Fredholm index of $\Delta_{g_s,\beta}$ as β varies if a complete knowledge of the spectrum of Δ_{g_F} is available. As another useful application of Theorem 2.11, we mention the following existence result, which is just a restatement of Fredholm alternative.

Corollary 2.12. *If $\beta \in I_a$, $a \neq 0$, then the map (2.7) is surjective if and only if it is injective.*

Remark 2.13. The case $a = 0$ may also be treated by the method leading to Theorem 2.11. It turns out that $\Delta_{g_0,\beta}$ is Fredholm for any β such that $\beta^2 \notin \mathrm{Spec}(\Delta_{g_F})$; see Remark 3.7.

3 The proof of Theorem 2.11 (a sketch)

Our aim here is to sketch the proof of Theorem 2.11. This may be confirmed in a variety of ways on inspection of standard sources; see for instance [40, 36, 47, 34, 18, 41], among others. However, since in these references the arguments leading to Theorem 2.11 appear embedded in rather elaborate theories, we include a sketch of the proof here in the setting of the Sobolev-Mellin spaces introduced above. In fact, this section may be regarded as an essay on these fundamental contributions as applied to a rather simple situation.

Since Δ_{g_s} is elliptic on X', a local parametrix may be found in this region by standard methods. Thus, analyzing the mapping properties of Δ_{g_s} involves the consideration of a suitable notion of ellipticity in the conical region U'. Starting from (2.2) and (2.3), we easily compute that the Laplacian Δ_{g_s} satisfies

$$P := x^{2s}\Delta_{g_s}|_{U'} = \mathsf{D}^2 + a\mathsf{D} + \Delta_{g_F} + o(1), \tag{3.1}$$

where $\mathsf{D} = x\partial_x$. As already noted, the needed ingredients to establish the mapping properties for Δ_{g_s} include not only its ellipticity when restricted to the smooth locus, but also the invertibility of the so-called *conormal* *symbol*, which is obtained by freezing the coefficients of P at $x = 0$, that is, passing to

$$P_0 = \mathsf{D}^2 + a\mathsf{D} + \Delta_{g_F}, \tag{3.2}$$

and then applying the Mellin transform M; see [46, 47, 18] and also (3.10) below, where this construction is actually applied to an appropriate conjugation of P_0. Recall that M is the linear map that to each well-behaved function $f : \mathbb{R}_+ \to \mathbb{C}$ associates another function $\mathsf{M}(f) : U_f \subset \mathbb{C} \to \mathbb{C}$ by means of

$$\mathsf{M}(f)(\zeta) = \int_0^{+\infty} f(x)x^\zeta d_+x, \quad d_+x = x^{-1}dx.$$

For our purposes, it suffices to know that this transform meets the following properties:

- For each $\theta \in \mathbb{R}$, the map

  $$x^\theta L^2(\mathbb{R}_+, d_+x) \xrightarrow{\mathsf{M}} L^2(\Gamma_{-\theta}),$$

  is an isometry. Here, $\Gamma_\alpha = \{\zeta \in \mathbb{C}; \operatorname{Re}\zeta = \alpha\}$, $\alpha \in \mathbb{R}$, and $x^\theta L^2(\mathbb{R}_+, d_+x)$ is endowed with the inner product

  $$\langle f, g\rangle_{x^\theta L^2(\mathbb{R}_+,d_+x)} = \langle x^{-\theta} f, x^{-\theta} g\rangle_{L^2(\mathbb{R}_+,d_+x)}. \tag{3.3}$$

  Moreover, at least if f has bounded support, each element $u = \mathsf{M}(f)$ in the image extends holomorphically to the half-space $\{\zeta \in \mathbb{C}; \operatorname{Re}\zeta > -\theta\}$ (Notation: $u \in \mathscr{H}(\{\operatorname{Re}\zeta > -\theta\})$).

- $\mathsf{M}(\mathsf{D}f)(\zeta) = -\zeta\mathsf{M}(f)(\zeta)$.

In particular, the conormal symbol

$$\xi_{\Delta_{g_s}}(\zeta) = \zeta^2 - a\zeta + \Delta_{g_F} \tag{3.4}$$

is obtained by Mellin tranforming (3.2). Note that this is a polynomial function with coefficients in the space of differential operators on the link (F, g_F).

Definition 3.1. The Laplacian Δ_{g_s} is *elliptic* (with respect to some $\beta \in \mathbb{R}$) if

$$\xi_{\Delta_{g_s}}(\zeta) : H^{\sigma,p}(F, d\mathrm{vol}_{g_F}) \to H^{\sigma-2,p}(F, d\mathrm{vol}_{g_F})$$

is invertible for any $\zeta \in \Gamma_\beta$. Here, $H^{\sigma,p}$ denotes the standard Sobolev scale.

Remark 3.2. Inherent in the discussion above is the fact that the Laplacian can be written as a polynomial in D in the conical region. More generally, we may consider any elliptic operator D satisfying, as $x \to 0$,

$$x^\nu D|_{U'} = \sum_{i=0}^{m} A_i(x)\mathsf{D}^i + o(1), \quad \nu > 0,$$

where each $A_i(x)$ is a differential operator of order at most $m-i$ acting on (sections of a vector bundle over) F [47, 34]. Definition 3.1 then applies to the corresponding conormal symbol, which is

$$\xi_D(\zeta) = \sum_{i=0}^{m}(-1)^i A_i(0)\zeta^i.$$

Besides the Laplacian, in next section we consider another most honorable example, namely, the Dirac operator acting on spinors.

Armed with this notion of ellipticity, we may setup an appropriate pseudo-differential calculus that enables the construction of a parametrix for Δ_{g_s} in the Sobolev-Mellin scale $\mathcal{H}^{\sigma,p}_\beta(X)$; the quite delicate argument can be found in [47, 18]. As in the smooth case, this turns out to be formally equivalent to the assertion that the map (2.7) is Fredholm.

Remark 3.3. The converses in the chain of implications above also hold true, so that (2.7) fails to be Fredholm precisely at those β for which the invertibility condition fails. More precisely, if we set

$$\Xi_\beta := \left\{\zeta \in \mathbb{C}; \zeta^2 - a\zeta - \mu = 0, \mu \in \mathrm{Spec}(\Delta_{g_F})\right\} \cap \Gamma_\beta. \tag{3.5}$$

then the Laplacian map in (2.7) fails to be Fredholm if and only if $\Xi_\beta \neq \varnothing$. This takes place along the discrete set formed by those $\beta = \beta_\mu$ satisfying the *indicial equation*

$$\beta_\mu^2 - a\beta_\mu - \mu = 0, \quad \mu \in \mathrm{Spec}(\Delta_{g_F}),$$

and a further argument shows that the corresponding jump in the Fredholm index equals

$$\pm \dim \ker \xi_{\Delta_{g_s}}(\beta_\mu) = \pm \dim \ker(\Delta_{g_F} + \mu). \tag{3.6}$$

From the previous remark, a first step toward computing the Fredholm index of $\Delta_{g_s,\beta}$ as β varies involves first determining it at a single value of β. A possible approach to this goal is to consider the *core* Laplacian

$$(\Delta_{g_s}, C_c^\infty(X')) : C_c^\infty(X') \subset \mathcal{H}^{0,2}_\beta(X) \to \mathcal{H}^{0,2}_\beta(X), \tag{3.7}$$

a densely defined operator whose closure is the operator $(\Delta_{g_s}, D_{\min}(\Delta_{g_s}))$, with domain $D_{\min}(\Delta_{g_s})$ formed by those $u \in \mathcal{H}^{0,2}_\beta(X)$ such that there exists $\{u_n\} \subset C_c^\infty(X')$ with $u_n \to u$ and $\{\Delta_{g_s} u_n\}$ is Cauchy in $\mathcal{H}^{0,2}_\beta(X)$. Also, we may consider $(\Delta_{g_s}, D_{\max}(\Delta_{g_s}))$, where

$$D_{\max}(\Delta_{g_s}) = \left\{ u \in \mathcal{H}^{0,2}_\beta(X); \Delta_{g_s} u \in \mathcal{H}^{0,2}_\beta(X) \right\}.$$

Regarding these notions, the following facts are well-known.

- $D_{\min}(\Delta_{g_s}) \subset D_{\max}(\Delta_{g_s})$;
- If $(\hat{\Delta}_{g_s}, \mathrm{Dom}(\hat{\Delta}_{g_s}))$ is a closed extension of $(\Delta_{g_s}, C_c^\infty(X'))$ then

$$D_{\min}(\Delta_{g_s}) \subset \mathrm{Dom}(\hat{\Delta}_{g_s}) \subset D_{\max}(\Delta_{g_s}).$$

Hence, in order to understand the set of closed extensions, we need to look at the subspaces of the *asymptotics space*

$$\mathcal{Q}(\Delta_{g_s}) := \frac{D_{\max}(\Delta_{g_s})}{D_{\min}(\Delta_{g_s})}. \tag{3.8}$$

Thus, $\mathcal{Q}(\Delta_{g_s}) = \{0\}$ implies that the Laplacian has a unique closed extension and hence the associated map (2.7) is Fredholm. In particular, it is essentially self-adjoint (hence with a vanishing index) whenever it is symmetric. From this, the remaining values of the index as β varies may be determined by means of the jump factors in (3.6).

The properties of the Mellin transform mentioned above suggest to work with the "Mellin" volume element

$$d\mathrm{vol}_{\mathsf{M}} = x^{-1}dx\,d\mathrm{vol}_{g_F}$$

instead of the volume element $x^{n-1}dx\,d\mathrm{vol}_{g_F}$ associated to $\overline{g}$. This is implemented by working "downstairs" in the diagram below, where $\tau = x^{\frac{n}{2}}$ is unitary and $\Delta_{g_s}^{\tau} = \tau\Delta_{g_s}\tau^{-1}$:

$$\begin{array}{ccc} D_{\max}(\Delta_{g_s}) \subset \mathcal{H}^{0,2}_{\beta}(X) & \xrightarrow{\Delta_{g_s}} & \mathcal{H}^{0,2}_{\beta}(X) \\ \tau\big\downarrow & & \big\downarrow\tau \\ D_{\max}(\Delta^{\tau}_{g_s}) \subset x^{\frac{n}{2}-\beta}L^2(X', d\mathrm{vol}_{\mathsf{M}}) & \xrightarrow{\Delta^{\tau}_{g_s}} & x^{\frac{n}{2}-\beta}L^2(X', d\mathrm{vol}_{\mathsf{M}}) \end{array} \tag{3.9}$$

Remark 3.4. It is immediate to check that, near the singularity,

$$\langle \Delta_{g_s}u, v\rangle_{\mathcal{H}^{0,2}_{\beta}(X)} = \int x^{2\beta-ns} v\Delta_{g_s}u\, d\mathrm{vol}_{g_s},$$

so that the horizontal maps in (3.9) define symmetric operators if and only if $\beta = ns/2$. Notice that the same conclusion holds true for any operator which is formally self-adjoint with respect to $d\mathrm{vol}_{g_s}$.

Let $u \in D_{\max}(\Delta_{g_s})$. Thus, $v := \tau u \in D_{\max}(\Delta^{\tau}_{g_s})$ satisfies $x^{\beta-n/2}v \in L^2(X', d\mathrm{vol}_{\mathsf{M}})$, so that $\mathsf{M}(v) \in \mathscr{H}(\{\mathrm{Re}\,\zeta > \beta - n/2\})$. On the other hand, if

$$P_0^{\tau} := \tau P_0 \tau^{-1} = \mathsf{D}^2 + (a-n)\mathsf{D} + \frac{n(n-2a)}{4} + \Delta_{g_F},$$

then $w := P_0^{\tau} v$ satisfies $x^{-2s+\beta-n/2}w = x^{\beta-n/2}\tau\Delta_{g_s}u \in L^2(X', d\mathrm{vol}_{\mathsf{M}})$, so that $\mathsf{M}(w) \in \mathscr{H}(\{\mathrm{Re}\,\zeta > -2s + \beta - n/2\})$. By taking Mellin transform,

$$\mathsf{M}(w)(\zeta, z, y) = \xi_{\Delta^{\tau}_{g_s}}(\zeta)\mathsf{M}(v)(\zeta, z, y),$$

where

$$\xi_{\Delta^{\tau}_{g_s}}(\zeta) = \zeta^2 + (n-a)\zeta + \frac{n(n-2a)}{4} + \Delta_{g_F} \tag{3.10}$$

is the conormal symbol of $\Delta^{\tau}_{g_s}$. The conclusion is that, at least formally,

$$\mathsf{M}(v)(\zeta, z, y) = \xi^{-1}_{\Delta^{\tau}_{g_s}}(\zeta)\mathsf{M}(w)(\zeta, z, y), \tag{3.11}$$

but we should properly handle the zeros of $\xi_{\Delta^{\tau}_{g_s}}$ located within the critical strip $\Gamma_{-2s+\beta-n/2,\beta-n/2}$, which we may gather together in the *asymptotics set*[4]

$$\Lambda^{\tau}_{\beta} := \{\zeta \in \mathbb{C}; Q_{\mu}(\zeta) = 0, \mu \in \mathrm{Spec}(\Delta_{g_F})\} \cap \Gamma_{-2s+\beta-n/2,\beta-n/2}.$$

Here, $\Gamma_{c,c'} = \{\zeta \in \mathbb{C}; c < \mathrm{Re}\,\zeta < c'\}$ for $c < c'$ and

$$Q_{\mu}(\zeta) = \zeta^2 + (n-a)\zeta + \frac{n(n-2a)}{4} - \mu.$$

Since the roots of Q_μ are explicitly given by

$$\frac{a-n}{2} \pm \delta^{\pm}_{\mu}, \quad \delta^{\pm}_{\mu} = \pm\frac{1}{2}\sqrt{a^2+4\mu}, \tag{3.12}$$

we may alternatively consider

$$\tilde{\Lambda}^{\tau,\pm}_{\beta} = \left\{\mu \in \mathrm{Spec}(\Delta_{g_F}); \delta^{\pm}_{\mu} \in \Gamma_{-2s+\beta-a/2,\beta-a/2}\right\}.$$

After applying Mellin inversion to (3.11) and using the appropriate pseudo-differential calculus [34, 46, 47], we obtain

$$v - w = \sum_{\mu \in \tilde{\Lambda}^{\tau,\pm}_{\beta}} A_{\mu}(x,z,y), \tag{3.13}$$

where the right-hand side represents a generic element in the asymptotics space $\mathcal{Q}(\Delta_{g_s})$. Thus, the elements in $\tilde{\Lambda}^{\tau,\pm}_{\beta}$ constitute the obstruction to having $v = w$ (and hence, $\mathcal{Q}(\Delta_{g_s}) = \{0\}$). From this we easily derive the next results.

Theorem 3.5. *The core Laplacian has a unique closed extension whenever* $\tilde{\Lambda}^{\tau,\pm}_{\beta} = \varnothing$.

Corollary 3.6. *Assume that $n \geq 4$. Then the core Laplacian has a unique closed, Fredholm extension if either i) $s \leq 0$ or ii) $s > 0$ and $\beta = ns/2$. In both cases, it is essentially self-adjoint for $\beta = ns/2$.*

[4] Note that $\Lambda^{\tau}_{\beta} \subset \mathbb{R}$ by (3.12) and the fact that $\mathrm{Spec}(\Delta_{g_F}) \subset [0,+\infty)$.

Proof. The case $s \leq 0$ follows from the fact that $\Gamma_{-2s+\beta-n/2,\beta-n/2} = \varnothing$, which clearly implies that $\tilde{\Lambda}_{\beta}^{\tau,\pm} = \varnothing$ as well. If $s > 0$ then

$$|\delta_{\mu}^{\pm}| \geq \frac{a}{2} = \frac{(n-2)s}{2} \geq s,$$

so that

$$\tilde{\Lambda}_{ns/2}^{\tau,\pm} = \left\{\mu \in \mathrm{Spec}(\Delta_{g_F}); \delta_{\mu}^{\pm} \in \Gamma_{-s,s}\right\} = \varnothing$$

indeed. The last assertion follows from Remark 3.4. □

In each case of Corollary 3.6, the corresponding map (2.7) is Fredholm and this turns out to be a crucial step in the proof of Theorem 2.11. Indeed, we already know that Fredholmness and the associated index do not depend on the pair (σ, p) but only on β. The key point now is that, as already explained in the slightly different (but equivalent) setting of the discussion surrounding Remark 3.3, the strategy to preserve Fredholmness as β varies involves precluding the crossing of zeros of $\xi_{\Delta_{g_s}^{\tau}}$ through the critical line $\Gamma_{\beta-n/2}$ (this is what ellipticity is all about). Precisely, we consider

$$\Xi_{\beta}^{\tau} := \{\zeta \in \mathbb{C}; Q_{\mu}(\zeta) = 0, \mu \in \mathrm{Spec}(\Delta_{g_F})\} \cap \Gamma_{\beta-n/2},$$

and the relevant result is that Δ_{g_s} remains Fredholm with the *same* index as long as $\Xi_{\beta}^{\tau} = \varnothing$; see [46, Section 3] or [47, Subsection 2.4.3]. Certainly, this is the case for all $\beta \in I_a$, $a \neq 0$. Since (the closure of) this interval always contains $ns/2$, the proof of Theorem 2.11 follows from Corollary 3.6 and the remarks above.

Remark 3.7. The case $a = 0$ follows by a similar argument observing that the roots of $Q_{\mu}(\zeta) = 0$ are $-n/2 \pm \sqrt{\mu}$, so that Fredholmness fails whenever $\beta = \pm\sqrt{\mu}$; compare with Remark 2.13 and [35, Theorem 6.2].

Remark 3.8. We emphasize that the authors in [46] and [47] work "upstairs" in respect to the diagram (3.9), that is, before applying the conjugation $\tau = x^{n/2}$, so instead of Ξ_{β}^{τ} they consider Ξ_{β} as in (3.5). Notice that the polynomial equation here is the Mellin transform of P_0 whereas the critical line is shifted to the right by $n/2$. It is immediate to check that both approaches produce the same numerical results for the Fredholmness of $\Delta_{g_s,\beta}$.

4 The Dirac operator

We now illustrate how flexible the theory described in the previous section is by explaining how it may be adapted to establish the mapping properties of the Dirac operator

$$\partial\!\!\!/_{g_s} : \mathcal{H}^{s,p}_{\beta}(S_X) \to \mathcal{H}^{s-1,p}_{\beta+1}(S_X) \tag{4.1}$$

in the appropriate scale of Sobolev-Mellin spaces. Here, X is assumed to be spin and S_X is the corresponding spinor bundle (associated to g_s). As usual, we first consider the core Dirac operator

$$(\partial\!\!\!/_{g_s}, C^\infty_0(S_X)) : C^\infty_0(S_X) \subset \mathcal{H}^{0,2}_{\beta}(S_X) \to H^{0,2}_{\beta}(S_X), \tag{4.2}$$

and our aim is to give conditions on β to make sure that the associated asymptotics space is trivial.

It follows from [1, Lemma 2.2] that, in the conical region,

$$\partial\!\!\!/_{\overline{g}} = \mathfrak{c}(\partial_x)\left(\partial_x + \frac{n-1}{2x} + \frac{1}{x}\partial\!\!\!/_F\right) + O(1),$$

where $\mathfrak{c}$ is Clifford product and $\partial\!\!\!/_F$ is the Dirac operator of the spin manifold (F, g_F). From [10, Proposition 2.31], we thus obtain

$$\partial\!\!\!/_{g_s} = x^{1-s}\mathfrak{c}(\partial_x)\left(\partial_x + \frac{\hat{a}}{x} + \frac{1}{x}\partial\!\!\!/_F\right) + O(1), \quad \hat{a} = \frac{(n-1)s}{2},$$

so that

$$\mathscr{P} := x^s\partial\!\!\!/_{g_s} = \mathfrak{c}(\partial_x)\mathscr{P}_0 + O(x),$$

where

$$\mathscr{P}_0 = \mathsf{D} + \hat{a} + \partial\!\!\!/_F$$

is the conormal symbol. By working "downstairs", we get

$$\mathscr{P}_0^\tau := \tau\mathscr{P}_0\tau^{-1} = \mathsf{D} + \hat{a} - \frac{n}{2} + \partial\!\!\!/_F,$$

and after Mellin transforming this we see that the corresponding asymptotics set is

$$\Theta_\beta^\tau := \left\{\zeta \in \mathbb{C}; \zeta + \frac{n}{2} - \hat{a} - \vartheta = 0, \vartheta \in \mathrm{Spec}(\eth_F)\right\} \cap \Gamma_{-s+\beta-n/2,\beta-n/2}.$$

By arguing exactly as above, we easily obtain the following result.

Theorem 4.1. *The core Dirac (4.2) has a unique closed extension whenever $\Theta_\beta^\tau = \varnothing$. In particular, this happens if either i) $s \leq 0$ or ii) $s > 0$, $\beta = n/2$ and the "geometric Witt assumption"*

$$\mathrm{Spec}(\eth_F) \cap \left(\frac{n}{2} - \hat{a} - s, \frac{n}{2} - \hat{a}\right) = \varnothing \tag{4.3}$$

is satisfied. In this latter case, the Dirac map (4.1) is Fredholm of index 0 if $n/2 - s < \beta < n/2$, with the core Dirac being essentially self-adjoint for $s = 1$.

Remark 4.2. This should be compared with [1, Theorem 1.1], which proves essential self-adjointness for $s = 1$ in the general edge setting. An alternate approach to this latter result, which works more generally for stratified spaces, has been recently put forward in [25].

Remark 4.3. If $D = d + d^*$, the Hodge-de Rham operator acting on differential forms, then the analogue of the Witt condition above translates into a purely topological obstruction. Precisely, if the cohomology group $H^{\frac{n-1}{2}}(F, \mathbb{R})$ is trivial (in particular, if n is even) then, after possibly rescaling the link metric g_F, D_{g_1} is essentially self-adjoint [14]. Extensions of this foundational result to general stratified spaces appear in [2].

This Fredholmness property of $\eth_{g_s}$ may be substantially improved if we assume that $\kappa_{\overline{g}}$, the scalar curvature of $\overline{g}$, is non-negative when restricted to the conical region U'. Since

$$\kappa_{\overline{g}}|_{U'} = (\kappa_{g_F} - (n-1)(n-2))\, x^{-2} + O(x^{-1}), \quad x \to 0,$$

we infer that $\kappa_{g_F} \geq (n-1)(n-2) > 0$ and a well-known estimate [19, Section 5.1] gives

$$\vartheta \in \mathrm{Spec}(\eth_F) \Longrightarrow |\vartheta| \geq \frac{n-1}{2},$$

which allows us to replace (4.3) by

$$\mathrm{Spec}(\partial\!\!\!/_F) \cap \left(\frac{1-n}{2}, \frac{n-1}{2} \right) = \emptyset, \tag{4.4}$$

a gap estimate that, remarkably, does not involve the parameter s. In this way we obtain the following specialization of Theorem 4.1.

Theorem 4.4. *If $|s| \le 1$ and $\kappa_{g_s}|_{U'} \ge 0$ then the Dirac map (4.1) is Fredholm of index 0 whenever*

$$\frac{1}{2}(n-1)(s-1) < \beta < \frac{1}{2}(n-1)(s+1). \tag{4.5}$$

Proof. If $\alpha = (s-1)(n-2)/2$, a computation shows that

$$\kappa_{g_s}|_{U'} = x^{-\frac{\alpha(n+2)}{n-2}} \left((n-1)(n-2)(1-s^2)x^{\alpha-2} + \kappa_{\overline{g}}|_{U'} x^{\alpha} \right),$$

so that $\kappa_{g_s}|_{U'} \ge 0$ implies $\kappa_{\overline{g}}|_{U'} \ge 0$ and we may appeal to (4.4) to obtain (4.5) as the interval where the index remains constant. Since $\beta = ns/2$ lies in this interval if and only if $1-n < s < 1+n$, the result follows by Remark 3.4. □

5 Applications

We now discuss a few (selected) applications of Theorems 2.11, 4.1 and 4.4 and Remark 3.7 in Geometric Analysis.

5.1 The Laplacian in AC_0 manifolds

This class of manifolds appears in Example 2.5 above, so that $s = 1$ in (2.3). Thus, Theorem 2.11 applies with $h = g_1$ and $a = n-2$. It is convenient here to pass from β to γ as in (2.6), so the Sobolev-Mellin norm in (2.5) becomes

$$\int |x^{\frac{n}{2}-\gamma} u(x,z)|^p x^{-1} dx d\mathrm{vol}_{g_F}(z), \tag{5.1}$$

which gives rise to the Sobolev-Mellin spaces $\mathcal{H}_p^{\sigma,\gamma}(V)$ considered in [46]. The following result is an immediate consequence of Theorem 2.11.

Theorem 5.1. *If $n \geq 4$ then the Laplacian map*

$$\Delta_{h,\gamma} : \mathcal{H}_p^{\sigma,\gamma}(V) \to \mathcal{H}_p^{\sigma-2,\gamma-2}(V)$$

is Fredholm of index 0 if $(4-n)/2 < \gamma < n/2$.

This result is used in [17, Section 2] as a key step in the argument toward proving that a function which is negative somewhere is the scalar curvature of some conical metric in V.

5.2 The Laplacian in AC_∞ manifolds

This class of manifolds appears in Example 2.6 above, so that $s = -1$ in (2.3). Thus, Theorem 2.11 applies with $h = g_{-1}$ and $a = 2-n$. If $r = x^{-1}$ then the Sobolev-Mellin norm in (2.5) becomes

$$\int |r^{-\beta} u(r,z)|^p r^{-n} d\mathrm{vol}_h(r,z), \tag{5.2}$$

which gives rise to the weighted Sobolev spaces $L^p_{\sigma,\beta}(V)$ considered in [33, Section 9]. The following result is an immediate consequence of Theorem 2.11; compare with [33, Theorem 9.2 (b)].

Theorem 5.2. *If $n \geq 4$ then the Laplacian map*

$$\Delta_{h,\beta} : L^p_{\sigma,\beta}(V) \to L^p_{\sigma-2,\beta-2}(V)$$

is Fredholm of index 0 if $2-n < \beta < 0$.

Remark 5.3. Consider the case in which the link is the round sphere $(\mathbb{S}^{n-1}, \delta)$ and $\nu_\infty > (n-2)/2$. Thus, we are in the *asymptotically flat* case so dear to practitioners of Mathematical Relativity [33, 9]. Here, an asymptotic invariant for (V,h), the ADM mass $\mathfrak{m}_{(V,h)}$, is defined by

$$\mathfrak{m}_{(V,h)} = \lim_{r \to +\infty} \int_{S_r^{n-1}} (h_{ij,j} - h_{jj,i})\, \eta^i dS_r^{n-1},$$

where h_{ij} are the coefficients of h in the given coordinate system, the comma denotes partial differentiation, S_r^{n-1} is the coordinate sphere of radius r in the asymptotic region and η is its outward unit normal (with respect to the flat metric). The problem remains of checking that the expression above does *not* depend on the particular coordinate system chosen near infinity. The first step in confirming this assertion involves the construction of harmonic coordinates; this is explained in [33, Theorem 9.3], which is a rather straightforward consequence of Theorem 5.2.

5.3 The Laplacian in ACyl manifolds

This class of manifolds appears in Example 2.7 above, so that $s = 0$ in (2.3). Thus, Theorem 2.11 applies with $h = g_0$ and $a = 0$. If $x = e^{-r}$ and $\delta = -\beta$ then the Sobolev-Mellin norm in (2.5) becomes

$$\int |e^{\delta r} u(r,z)|^p d\mathrm{vol}_h(r,z),$$

which gives rise to the weighted Sobolev spaces $W^p_{\sigma,\delta}(V)$ considered in [35], but notice that these authors use $\log r$ instead of r. The following result is an immediate consequence of Remark 3.7.

Theorem 5.4. *If $n \geq 4$ then the Laplacian map*

$$\Delta_{h,\delta} : W^p_{\sigma,\delta}(V) \to W^p_{\sigma-2,\delta}(V)$$

is Fredholm if $0 < \delta < \sqrt{\mu_{g_F}}$, where μ_{g_F} is the first (positive) eigenvalue of Δ_{g_F}.

The Hölder counterpart of this result is used in [26] to study asymptotically cylindrical Calabi-Yau manifolds.

5.4 The Laplacian in $\mathrm{AC}_0/\mathrm{AC}_\infty$ manifolds

This class of manifolds appears in Example 2.8 above, so that $s = \pm 1$ in (2.3) depending on the nature of the end. The corresponding mapping properties for the Laplacian are formulated in weighted Sobolev spaces incorporating the norms induced by (5.1) and (5.2) above. These properties, including the extra information coming from the jumps in the Fredholm index, are used in [43] to study the moduli space of special Lagrangian conifolds in $\mathbb{C}^m$; see also [28].

5.5 The Dirac operator in AC_0 spin manifolds

For this class of manifolds, Theorems 4.1 and 4.4 apply with $s = 1$ (and $h = g_1$) so if we further assume that $\kappa_h|_{U'} \geq 0$ then the Dirac operator $\not{\partial}_h$ is Fredholm of index 0 for $0 < \beta < n - 1$ with the core Dirac being essentially self-adjoint for $\beta = n/2$. Now recall that if n is even then the spinor bundle decomposes as $S_V = S_V^+ \oplus S_V^-$, with a corresponding decomposition for $\not{\partial}_h$:

$$\not{\partial}_h = \begin{pmatrix} 0 & \not{\partial}_h^- \\ \not{\partial}_h^+ & 0 \end{pmatrix}$$

where $\not{\partial}_h^\pm : \Gamma(S_V^\pm) \to \Gamma(S_V^\mp)$, the chiral Dirac operators, are adjoint to each other. Thus, it makes sense to consider the *index* of $\not{\partial}_h^+$:

$$\operatorname{ind} \not{\partial}_h^+ = \dim\ker \not{\partial}_h^+ - \dim\ker \not{\partial}_h^-.$$

This fundamental integer invariant can be explicitly computed in terms of topological/geometric data of the underlying AC_0 manifold by means of heat asymptotics [1, 15, 34]. The resulting formula has been used in [17] to exhibit obstructions for the existence of conical metrics with positive scalar curvature.

5.6 The Dirac operator in asymptotically flat spin manifolds

If an asymptotically flat manifold V as in Remark 5.3 (with $h = g_{-1}$) is spin and satisfies $\kappa_h \geq 0$ in the asymptotic region then Theorem 4.4 applies and $\not{\partial}_h : L^p_{\sigma,\beta}(S_V) \to L^p_{\sigma-1,\beta-1}(S_V)$ is Fredholm of index 0 for $1 - n < \beta < 0$. If we further assume that $\kappa_h \geq 0$ *everywhere* then integration by parts starting with the Weitzenböck formula for the Dirac Laplacian $\not{\partial}_h^2$ shows that $\not{\partial}_h$ is injective and hence surjective by Fredholm alternative. We now take a *parallel* spinor ϕ_∞ in $\mathbb{R}^n$, $|\phi_\infty| = 1$, and transplant it to the asymptotic region by means of the diffeomorphism ψ in Example 2.6. If we still denote by ϕ_∞ a smooth extension of this spinor to the whole of V, then a computation shows that, as $r \to \infty$,

$$\not{\partial}_h \phi_\infty = O(|\partial h|) = O(r^{-\nu_\infty - 1}) \in L^p_{\sigma-1,\beta-1}(S_V), \quad \beta \in \left[1 - \frac{n}{2}, 0\right),$$

so there exists $\phi_0 \in L^p_{\sigma,\beta}(S_V)$ with $\partial\!\!\!/_h\phi_0 = -\partial\!\!\!/_h\phi_\infty$. It follows that $\phi = \phi_0 + \phi_\infty$ is harmonic ($\partial\!\!\!/_h\phi = 0$) and $|\phi - \phi_\infty| = O(r^\beta)$. With this spinor ϕ at hand, another (more involved!) integration by parts yields Witten's remarkable formula for the ADM mass of (V, h):

$$\mathfrak{m}_{(V,h)} = c_n \int_V \left(|\nabla\phi|^2 + \frac{\kappa_h}{4}|\phi|^2 \right) d\text{vol}_h, \quad c_n > 0.$$

From this we easily deduce the following fundamental positive mass inequality.

Theorem 5.5. *[48] If (V, h) is asymptotically flat and spin as above and $\kappa_h \geq 0$ everywhere then $\mathfrak{m}_{(V,h)} \geq 0$. Moreover, the equality holds only if $(V, h) = (\mathbb{R}^n, \delta)$ isometrically.*

The details of the argument above may be found in [33, Appendix].

6 Further applications

The techniques described above may be adapted to handle more general situations. We only briefly discuss here three interesting cases.

6.1 Conformally conical manifolds with boundary

Here we consider conformally conical manifolds carrying a non-empty boundary ∂X which is allowed to reach the tip of the cone. The formal definition is as in Example 2.5, except that the link F itself carries a non-empty boundary ∂F. The key observation now is that both ∂X and the double $2X$ along the boundary ∂X are conformally conical manifolds as in Definition 2.1 (the links of these "boundaryless" manifolds are ∂F and $2F$, respectively). Thus, we are led to ask whether the Calderón-Seeley technique mentioned in the Introduction (for smooth manifolds) may be adapted to this context. This program has been carried out in [16], where it is shown, among other things, that the realizations of the Laplacian under standard boundary conditions (Dirichlet/Neumann) may be treated as well, at least in the "straight" case where the link metric is not allowed to vary with x [16, Section 5]; see also [20] for an approach in the setting of fibred cusp operators. If we invert the conical singularity as in Example 2.6, we obtain an asymptotically flat manifold with a *non-compact* boundary and this theory provides a (rather sophisticated) approach to the results obtained "by hand" in [3, Appendix A]. Finally, we mention that the setup in [16] also applies to the realization of the Dirac operator acting on spinors under MIT bag boundary condition, so after inversion we recover the analytical machinery underpinning the positive mass theorems for asymptotically flat initial data sets in [3, 5].

6.2 Asymptotically hyperbolic spaces

We may consider an edge space (X, g_s) with $g_s = x^{2s-2}(\overline{g} + o(1))$ and $\overline{g}$ as in (1.2). Here,

$$x^{2s}\Delta_{g_s}|_{U'} = \mathsf{D}^2 + \tilde{a}\mathsf{D} + \Delta_{g_F} + x^2\Delta_{g_Y} + o(1), \quad \tilde{a} = a - d, \quad d = \dim Y.$$

If we specialize to the "pure" edge case in which the cone fiber $\mathcal{C}^F$ degenerates into a line (F becomes a point) then $d = n-1$ and $\tilde{a} = a + 1 - n$, $n = \dim X \geq 3$, and

$$g_s|_{U'} = x^{2s-2}\left(\overline{g} + o(1)\right), \quad \overline{g} = dx^2 + g_Y.$$

If we further take $s = 0$ then (X, g_0) is *conformally compact* with $(Y, [g_Y])$ as its *conformal boundary* and x is the corresponding *defining function*. In particular, since $|dx|_{\overline{g}} = 1$ along Y, a computation shows that g_0 is *asymptotically hyperbolic* in the sense that its sectional curvature approaches -1 as $x \to 0$. Since $a = (n-2)s = 0$, the conormal symbol gets replaced by

$$\xi_{\Delta_{g_0}}(\zeta) = \zeta^2 + (n-1)\zeta, \tag{6.1}$$

whose roots define a *unique* interval $(1-n, 0)$ where the weight parameter β is allowed to vary. Here, it is convenient to set

$$\beta = -\delta + \frac{1-n}{p}, \quad p > 1,$$

so the Sobolev-Mellin norm in (2.5) becomes

$$\int |x^{-\delta}u(x,y)|^p x^{-n} dx d\mathrm{vol}_{g_Y} = \int |x^{-\delta}u(x,y)|^p d\mathrm{vol}_{g_0},$$

which defines the weighted Sobolev spaces $H^{\sigma,p}_\delta(X)$ considered in [6, 32]. A variation of the procedure above then yields the following result, which should be compared to [32, Proposition F] and [6, Corollary 3.13]; this latter reference only treats the case $p = 2$.

Theorem 6.1. *The Laplacian map*

$$\Delta_{g_0,\delta} : H^{\sigma,p}_\delta(X) \to H^{\sigma-2,p}_\delta(X) \tag{6.2}$$

is Fredholm of index 0 *if*

$$\frac{1-n}{p} < \delta < \frac{(n-1)(p-1)}{p}. \tag{6.3}$$

Remark 6.2. Since in this asymptotically hyperbolic case the Laplacian of the total space of the restricted fiber bundle $F\times Y = \{\mathrm{pt}\}\times Y \to Y$ endowed with the metric $g_F \oplus g_Y = g_Y$ does not show up in (6.1), we are led to suspect that (6.2) fails to be Fredholm if δ does not satisfy (6.3). This is the case indeed and a proof of this claim may be found in [32]. Notice also that for $p = 2$, (6.3) becomes $|\delta|^2 < (n-1)^2/4$, a bound that also appears in MacKean's estimate [38], which in particular provides a sharp lower bound for the bottom of the spectrum of the Laplacian in the model space (this is of course hyperbolic n-space $\mathbb{H}^n$, which is obtained in the formalism above by taking (Y, g_Y) to be a round sphere). In fact, asymptotic versions of this estimate are used in [6] as a key ingredient in directly establishing the mapping properties of geometric operators in asymptotically hyperbolic spaces.

Remark 6.3. In an asymptotically hyperbolic manifold as above, the operator $\mathcal{L}^{(t)}_{g_0} := \Delta_{g_0} + t(n-1-t)$, $t \in \mathbb{C}$, whose conormal symbol is

$$\xi_{\mathcal{L}^{(t)}_{g_0}}(\zeta) = \zeta^2 + (n-1)\zeta + t(n-1-t),$$

also plays a distinguished role [37, 22, 11]. Let us assume that $2t \in \mathbb{R}\backslash\{n-1\}$ so that, by symmetry, we may take $2t > n-1$. Proceeding as above, we see that

$$\mathcal{L}^{(t)}_{g_0,\delta} : H^{\sigma,p}_{\delta}(X) \to H^{\sigma-2,p}_{\delta}(X)$$

is Fredholm of index 0 if

$$\frac{(n-1-t)p+1-n}{p} < \delta < \frac{tp+1-n}{p}.$$

If g_0 is Einstein ($\mathrm{Ric}_{g_0} = -(n-1)g_0$), the special choice $t = n$ is particularly important: the so-called *static potentials* (that is, solutions of $\nabla^2_{g_0} V = V g_0$) all lie in the kernel of $\mathcal{L}^{(n)}_{g_0}$. In this case, the Hölder counterpart of this result (which is obtained by sending $p \to +\infty$ in the assertion above) has been used in [44] to establish the existence of "approximate" static potentials. As a consequence, an asymptotically hyperbolic Einstein manifold with the round sphere as its conformal infinity was shown to be isometric to hyperbolic n-space, at least if $4 \le n \le 7$.

6.3 Asymptotically hyperbolic spaces with boundary

A more general kind of asymptotically hyperbolic space is obtained by assuming that the underlying conformally compact space X carries a boundary decomposing as $\partial X = Y \cup Y_{\mathrm{f}}$, with the intersection $\Sigma = Y \cap Y_{\mathrm{f}}$ being a (intrinsically smooth) co-dimension two corner. We assume further that there exists a tubular neighborhood U of Y on which a defining function x for Y exists so that

$$g|_U = x^{-2}(\overline{g} + o(1)), \quad \overline{g} = dx^2 + g_Y(y),$$

where g_Y is a metric in Y. Thus, the conformal boundary $(Y, [g_Y])$ itself carries a boundary, namely, $(\Sigma, [g_Y|_{\Sigma}])$, whereas the other piece of the boundary, Y_{f}, remains at a finite distance. Finally, we impose that $\nabla_{\overline{g}} x$ is tangent to Y_{f} along $U \cap Y_{\mathrm{f}}$, so that Y and Y_{f} meet orthogonally along Σ. It is immediate to check that: i) the "finite" boundary $\partial X_{\mathrm{f}} := Y_{\mathrm{f}}$ is a pure edge space as above with conformal infinity $(\Sigma, [g_Y|_{\Sigma}])$; ii) the double $2X_{\mathrm{f}} :=$

$X \sqcup_{Y_{\rm f}} -X$ is naturally a pure edge space as above with conformal boundary having the closed manifold $2Y$ as carrier and conformal structure induced by g_Y. Thus, at least in principle, the appropriate version of the Calderón-Seeley approach mentioned in the Introduction should apply here. Very likely, this follows from the corresponding adaptation of the general setup in [16, 20], so that both the realizations of the Laplacian and the Killing Dirac operator (under Dirichlet/Neumann boundary and chiral boundary conditions, respectively, imposed along $\partial X_{\rm f}$) may be shown to be Fredholm in suitable Sobolev scales. This should provide an alternate approach to the analysis underlying the positive mass theorems for asymptotically hyperbolic initial data sets in [4, 5].

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