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Existence and approximation of solutions for a class of degenerate elliptic equations with Neumann boundary condition

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Abstract. In this work we study the equation $Lu = f$, where L is a degenerate elliptic operator, with Neumann boundary condition in a bounded open set Ω. We prove the existence and uniqueness of weak solutions in the weighted Sobolev space $\mathrm{W}^{1,2}(\Omega, \omega)$ for the Neumann problem. The main result establishes that a weak solution of degenerate elliptic equations can be approximated by a sequence of solutions for non-degenerate elliptic equations

Keywords: Neumann problem, weighted Sobolev spaces.

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1 Introduction

In this paper, we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $W^{1,2}(\Omega, \omega)$ for the Neumann problem

$$(P) \begin{cases} Lu(x) = f(x) \ \text{ in } \ \Omega, \\ \langle A(x)\nabla u, \vec{\eta}(x)\rangle = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\vec{\eta}(x) = (\eta_1(x), ..., \eta_n(x))$ is the outward unit normal to $\partial\Omega$ at x, $\langle ., . \rangle$ denotes the usual inner product in $\mathbb{R}^n$, the symbol ∇ indicates the gradient and

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L is a degenerate elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_j\big(a_{ij}\, D_i u\big) + \sum_{i=1}^{n} b_i\, D_i u + g\, u + \theta\, u\, \omega, \tag{1.1}$$

with $D_j = \dfrac{\partial}{\partial x_j}$, $(j = 1, ..., n)$, θ is a positive constant, the coefficients a_{ij}, b_i and g are measurable, real-valued functions, the coefficient matrix $A(x) = (a_{ij}(x))$ is symmetric and satisfies the *degenerate ellipticity condition*

$$\lambda|\xi|^2\omega(x) \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2\omega(x) \tag{1.2}$$

for all $\xi\in\mathbb{R}^n$ and almost every $x\in\Omega\subset\mathbb{R}^n$ a bounded open set with boundary $\partial\Omega$, ω is a weight function (that is, locally integrable and nonnegative function on $\mathbb{R}^n$), λ and Λ are positive constants

Remark 1.1. In the case where $A(x) = Id$ (where Id denotes the Identity matrix in $\mathbb{R}^n$) then the second equation of (P) is $\dfrac{\partial u}{\partial \vec{\eta}} = 0$ on $\partial\Omega$ namely the normal derivative of u vanishes on $\partial\Omega$. In the general case and with the summation convention the second equation of (P) can be written $a_{ij}(x)\dfrac{\partial u}{\partial x_j}\,\eta_i = 0$ on $\partial\Omega$. This expression is called *conormal derivative of* u.

The main purpose of this paper (see Theorem 3.3) is to establish that a weak solution $u\in W^{1,2}(\Omega,\omega)$ for the Neumann problem (P) can be approximated by a sequence of solutions of non-degenerate elliptic equations.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations (see [10]). For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2], [3], [6], [8], [11] and [16]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1] and [5]).

A class of weights, which is particularly well understood, is the class of A_p weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see

[14]). These weights have found many useful applications in harmonic analysis (see [15]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [13]). There are, in fact, many interesting examples of weights (see [11] for p-admissible weights).

2 Definitions and basic results

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

Let ω be a weight and $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define $L^p(\Omega,\omega)$ as the set of measurable function f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x) dx\right)^{1/p} < \infty.$$

The class of A_p weight was introduced by B. Muckenhoupt (see [14]), where he showed that the A_p weights are precisely those weights w for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N,\omega)$ to $L^p(\mathbb{R}^N,\omega)$ $(1 < p < \infty)$, that is

$$M : L^p(\mathbb{R}^n,\omega) \to L^p(\mathbb{R}^n,\omega)$$
$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x;r)|} \int_{B(x;r)} |f(y)|\,dy,$$

is bounded if and only if $\omega \in A_p$ $(1 < p < \infty)$ (where $B(x;r) = \{z \in \mathbb{R}^n : |z-x| < r\}$ and $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$). When $p = 1$, then $M : L^1(\mathbb{R}^n,\omega) \to L^1_{wk}(\mathbb{R}^n,\omega)$ is bounded, where $L^1_{wk}(\mathbb{R}^n,\omega)$ as the set of measurable functions f on $\mathbb{R}^n$ satisfying $\|f\|_{L^1_{wk}(\mathbb{R}^n,\omega)} = \sup\limits_{\lambda>0} \lambda\omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty$.

Definition 2.1. Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there exists a positive constant C such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\,dx\right)^{p-1} \le C, \text{ if } 1 < p < \infty,$$
$$\left(\frac{1}{|B|}\int_B \omega\,dx\right)\left(\operatorname*{ess\,sup}_{x \in B} \frac{1}{\omega(x)}\right) \le C, \text{ if } p = 1.$$

for all balls $B \subset \mathbb{R}^n$. The infimum over all such constants C is called the A_p constant of ω and this constant will be denoted by $C(\omega,p) = C_{p,\omega}$.

The union of all Muckenhoupt class A_p is denoted by A_∞, i.e., $A_\infty = \bigcup_{p\geq 1} A_p$.

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x\in\mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [15]).

Remark 2.2. (a) If $\omega \in A_p$, $1 < p < \infty$, then $\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega}\frac{\mu(E)}{\mu(B)}$ for all measurable subset E of B (see 15.5 strong doubling property in [11]). Therefore $\mu(E) = 0$ if and only if $|E| = 0$; so there is no need to specify the measure when using the ubiquitous expression *almost everywhere* and *almost every*, both abbreviated a.e..
(b) If $\omega \in A_p$, $1 < p < \infty$, then there exist $\delta > 0$ and $C > 0$ depending only on n, p and $C_{p,\omega}$ such that, every time we have a measurable set E contained in a cube K_0, the following inequality holds: $\frac{\mu(E)}{\mu(K_0)} \leq C\left(\frac{|E|}{|K_0|}\right)^\delta$ (see Theorem 2.9, Chapter IV in [9] or Lemma 15.8 in [11]).

If $\omega\in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [16]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.3. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $1 < p < \infty$, $k\in\mathbb{N}$ and $\omega\in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$, $1\leq|\alpha|\leq k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p\omega\,dx + \sum_{1\leq|\alpha|\leq k}\int_\Omega |D^\alpha u|^p\omega\,dx\right)^{1/p}. \tag{2.1}$$

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Proposition 3.5 in [4], Theorem 2.5 in [12] or Theorem 2.1.4 in [16]). The space $W^{k,p}_0(\Omega,\omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}_0(\Omega,\omega)} = \left(\sum_{1\leq|\alpha|\leq k}\int_\Omega |D^\alpha u|^p\omega\,dx\right)^{1/p}.$$

The spaces $W^{k,p}(\Omega,\omega)$ and $W^{k,p}_0(\Omega,\omega)$ are Banach spaces and for $k = 1$ and $p = 2$ the spaces $W^{1,2}(\Omega,\omega)$ and $W^{1,2}_0(\Omega,\omega)$ are Hilbert spaces.

It is evident that the weight function ω which satisfy $0 < c_1\leq\omega(x)\leq c_2$ for $x\in\Omega$, give nothing new (the space $W^{k,p}(\Omega,\omega)$ is then identical with the classical

Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

Remark 2.4. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$. Using integration by parts, with $u,\varphi\in W^{1,2}(\Omega,\omega)$, if u satisfies the boundary condition in problem (P), we have (by Remark 1.1)

$$\begin{aligned}\int_\Omega \varphi\, Lu\, dx &= \int_\Omega a_{ij}\, D_i u\, D_j\varphi\, dx + \int_\Omega b_i\,\varphi\, D_i u\, dx + \int_\Omega g\, u\,\varphi\, dx \\ &+ \theta\int_\Omega u\,\varphi\,\omega\, dx + \underbrace{\int_{\partial\Omega} a_{ij}\,\frac{\partial u}{\partial x_j}\,\eta_i\,\varphi\, dx}_{=\ 0} \\ &= B(u,\varphi) + \theta\int_\Omega u\,\varphi\,\omega\, dx.\end{aligned}$$

where $B(u,\varphi) = \int_\Omega a_{ij}\, D_i u\, D_j\varphi\, dx + \int_\Omega b_i\,\varphi\, D_i u\, dx + \int_\Omega g\, u\,\varphi\, dx$ is a bilinear form.

We introduce the following definition of weak solution of the Neumann problem (P).

Definition 2.5. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and $\dfrac{f}{\omega}\in L^2(\Omega,\omega)$. A function $u\in W^{1,2}(\Omega,\omega)$ is a weak solution of the Neumann problem (P) if

$$\begin{aligned}&\sum_{i,j=1}^n \int_\Omega a_{ij} D_i u D_j\varphi\, dx + \sum_{i=1}^n \int_\Omega \varphi b_i\, D_i u\, dx + \int_\Omega g\, u\,\varphi\, dx + \theta\int_\Omega u\,\varphi\,\omega\, dx \\ &= \int_\Omega f\varphi\, dx \end{aligned} \tag{2.2}$$

for all $\varphi\in W^{1,2}(\Omega,\omega)$.

Before we prove the main result of this section, Theorem 2.7, we need the following lemma.

Lemma 2.6. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Suppose that*
(**H1**) $\omega\in A_2$;
(**H2**) $f/\omega\in L^2(\Omega,\omega)$;
(**H3**) $b_i/\omega\in L^\infty(\Omega)$ *(i=1,...,n) and* $g/\omega\in L^\infty(\Omega)$.
Then, there exists a constant $\mathbf{C}>0$ *such that*

$$B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)} \geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}$$

for all $u \in W^{1,2}(\Omega,\omega)$.

Proof. Using (1.2) and (H3), for all $u \in W^{1,2}(\Omega,\omega)$, we have

$$
\begin{aligned}
B(u,u) &= \int_\Omega a_{ij} D_i u D_j u\, dx + \int_\Omega b_i u D_i u\, dx + \int_\Omega u^2\, g\, dx \\
&\geq \lambda \int_\Omega |\nabla u| + \int_\Omega b_i u D_i u\, dx + \int_\Omega u^2\, g\, dx \\
&\geq \lambda \int_\Omega |\nabla u|^2 \omega\, dx - C_1 \|u\|_{L^2(\Omega,\omega)} \|u\|_{\mathrm{W}^{1,2}(\Omega,\omega)} - C_2 \|u\|^2_{L^2(\Omega,\omega)} \qquad (2.3)
\end{aligned}
$$

where $C_1 = \max\left\|\dfrac{b_i}{\omega}\right\|_{L^\infty(\Omega)}$ $(i = 1, ..., n)$ and $C_2 = \left\|\dfrac{g}{\omega}\right\|_{L^\infty(\Omega)}$. Using the elementary inequality $ab \leq \varepsilon a^2 + \dfrac{1}{4\varepsilon} b^2$, for all $\varepsilon > 0$, we obtain in (2.3),

$$
\begin{aligned}
&B(u,u) \\
&\geq \lambda \int_\Omega |\nabla u|^2 \omega\, dx - C_1\left(\varepsilon \|u\|^2_{L^2(\Omega,\omega)} + \frac{1}{4\varepsilon}\|u\|^2_{W^{1,2}(\Omega,\omega)}\right) - C_2 \|u\|^2_{L^2(\Omega,\omega)} \\
&= \left(\lambda - \frac{C_1}{4\varepsilon}\right) \|u\|^2_{W^{1,2}(\Omega,\omega)} - (C_1\varepsilon + C_2 + \lambda)\|u\|^2_{L^2(\Omega,\omega)}. \qquad (2.4)
\end{aligned}
$$

If $C_1 > 0$, we can choose $\varepsilon > 0$ such that $\lambda - \dfrac{C_1}{4\varepsilon} = \dfrac{\lambda}{2}$, that is, $\varepsilon = \dfrac{C_1}{2\lambda}$. Thus, in (2.4) we obtain

$$
B(u,u) \geq \frac{\lambda}{2} \|u\|^2_{W^{1,2}(\Omega,\omega)} - \mathbf{C} \|u\|^2_{L^2(\Omega,\omega)},
$$

where $\mathbf{C} = C_1\varepsilon + C_2 + \lambda = \dfrac{C_1^2}{2\lambda} + C_2 + \lambda > 0$. Therefore,

$$
B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)} \geq \frac{\lambda}{2} \|u\|^2_{W^{1,2}(\Omega,\omega)}.
$$

If $C_1 = 0$ (that is, $b_i(x) \equiv 0$, $i = 1, ..., n$) then (2.3) reduces to

$$
\begin{aligned}
B(u,u) &\geq \lambda \int_\Omega |\nabla u|^2 \omega\, dx - C_2 \|u\|^2_{L^2(\Omega,\omega)} \\
&= \lambda\left(\int_\Omega |u|^2 \omega\, dx + \int_\Omega |\nabla u|^2 \omega\, dx\right) \\
&- (C_2 + \lambda)\|u\|^2_{L^2(\Omega,\omega)} \\
&\geq \frac{\lambda}{2} \|u\|^2_{\mathrm{W}^{1,2}(\Omega,\omega)} - \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Therefore $B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,\omega)} \geq \dfrac{\lambda}{2} \|u\|^2_{W^{1,2}(\Omega,\omega)}$. □

Theorem 2.7. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set. Suppose that (H1) - (H3) holds. Then, there exists a constant $\mathbf{C}>0$ such that for all $\theta\geq\mathbf{C}$ the Neumann problem (P) has a unique solution $u\in W^{1,2}(\Omega,\omega)$. Moreover, we have*

$$\|u\|_{W^{1,2}(\Omega,\omega)}\leq\frac{2}{\lambda}\left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)}.$$

Proof. We define bilinear form

$$\tilde{B}:W^{1,2}(\Omega,\omega)\times W^{1,2}(\Omega,\omega)\to\mathbb{R}$$
$$\tilde{B}(u,\varphi)=B(u,\varphi)+\theta\int_\Omega u\,\varphi\,\omega\,dx$$

and a linear mapping

$$T:W^{1,2}(\Omega,\omega)\to\mathbb{R}$$
$$T(\varphi)=\int_\Omega f\,\varphi\,dx.$$

Then $u\in W^{1,2}(\Omega,\omega)$ is a weak solution of the Neumann problem (P) if

$$\tilde{B}(u,\varphi)=T(\varphi),\quad\text{for all }\varphi\in W^{1,2}(\Omega,\omega).$$

Step 1. If $\theta\geq\mathbf{C}$ then $\tilde{B}$ is coercive. In fact, by Lemma 2.6 there exists a constant $\mathbf{C}>0$ such that $B(u,u)+\mathbf{C}\|u\|^2_{L^2(\Omega,\omega)}\geq\frac{\lambda}{2}\|u\|_{\mathrm{W}^{1,2}(\Omega,\omega)}$. Hence, if $\theta\geq\mathbf{C}$, we have

$$\begin{aligned}\tilde{B}(u,u) &= B(u,u)+\theta\int_\Omega u^2\omega\,dx\\ &= B(u,u)+\theta\|u\|^2_{L^2(\Omega,\omega)}\\ &\geq B(u,u)+\mathbf{C}\|u\|^2_{L^2(\Omega,\omega)}\\ &\geq \frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)}.\end{aligned}$$

Therefore, for $\theta\geq\mathbf{C}$, we have that

$$\tilde{B}(u,u)\geq\frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)},\tag{2.5}$$

for all $u\in W^{1,2}(\Omega,\omega)$.

Step 2. $\tilde{B}$ is bounded. In fact, using that the coefficient matrix $A=(a_{ij})$ is symmetric, (1.2) and (H3), we obtain

$$|\tilde{B}(u,\varphi)|$$

$$
\begin{aligned}
&\le |B(u,\varphi)| + \theta\left|\int_\Omega u\varphi\,\omega\,dx\right| \\
&\le \int_\Omega |\langle A\nabla u, \nabla\varphi\rangle|\,dx + \int_\Omega |b_i||\varphi||D_i u|\,dx + \int_\Omega |g||\varphi||u|\,dx + \theta\int_\Omega |u||\varphi|\,\omega\,dx \\
&\le \left(\Lambda + \max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)} + \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)} + \theta\right)\|u\|_{W^{1,2}(\Omega,\omega)}\|\varphi\|_{W^{1,2}(\Omega,\omega)} \\
&= \tilde{C}\,\|u\|_{W^{1,2}(\Omega,\omega)}\|\varphi\|_{W^{1,2}(\Omega,\omega)},
\end{aligned}
$$

where $\tilde{C} = \left(\Lambda + \max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)} + \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)} + \theta\right)$, for all $u,\varphi \in W^{1,2}(\Omega,\omega)$.

Step 3. The linear mapping T is bounded (that is, $T \in [W^{1,2}(\Omega,\omega)]^*$). In fact, using (H2), we have

$$
\begin{aligned}
|T(\varphi)| &\le \int_\Omega |f||\varphi|dx \\
&\le \left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)} \|\varphi\|_{W^{1,2}(\Omega,\omega)},
\end{aligned}
$$

for all $\varphi \in W^{1,2}(\Omega,\omega)$.

Therefore the bilinear form $\tilde{B}$ and the linear functional T satisfy the hypotheses of the Lax-Milgram Theorem. Thus, for every f, with $f/\omega \in L^2(\Omega,\omega)$, there is a unique solution $u \in W^{1,2}(\Omega,\omega)$ such that $\tilde{B}(u,\varphi) = T(\varphi)$ for all $\varphi \in W^{1,2}(\Omega,\omega)$, that is, u is a unique solution of the Neumann problem (P). In particular, by setting $\varphi = u$, we have $\tilde{B}(u,u) = \int_\Omega f\,u\,dx$. Using the definition of $\tilde{B}$, we obtain

$$
\begin{aligned}
\tilde{B}(u,u) &= B(u,u) + \theta\int_\Omega u^2\omega\,dx \\
&\le \|u\|_{W^{1,2}(\Omega,\omega)}\|f/\omega\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Using (2.5), we obtain

$$
\begin{aligned}
\frac{\lambda}{2}\|u\|^2_{W^{1,2}(\Omega,\omega)} &\le \tilde{B}(u,u) \\
&\le \|u\|_{W^{1,2}(\Omega,\omega)}\|f/\omega\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Therefore,

$$
\|u\|_{W^{1,2}(\Omega,\omega)} \le \frac{2}{\lambda}\left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)}. \tag{2.6}
$$

□

3 Approximation of solution

In this section we present our main result: the weak solution to the problem (P) can be approximated by a sequence of solutions for non-degenerate elliptic equations.

The following lemma provides a general approximation theorem for A_p weights $(1 \leq p < \infty)$ by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constant of ω. Lemma 3.2 is the key point for Theorem 3.3, and the crucial point consists of showing that a weak limit of a sequence of solutions of approximate problems is in fact a solution of the original problem.

The proof of Lemma 3.2 (main result) is based on the Calderon-Zygmund Decomposition performed for dyadic cubes of $\mathbb{R}^n$ (and see also Lemma 2.1 in [7]). Throughout this paper, however, we used balls instead of cubes in our estimates. This deviation from the standard proof causes only minor technical complications (see for example the Calderon-Zygmund decomposition for doubling weights in [11]).

Theorem 3.1. *(The Jones Factorization Theorem) For* $1 < p < \infty$*. Then* $\omega \in A_p$ *if and only if* $\omega = \omega_0\, \omega_1^{1-p}$*, where* ω_0 *and* ω_1 *are* A_1 *- weights.*

Proof. See Corollary 5.3, Chapter IV in [9]. □

For $k \in \mathbb{Z}$, we consider the lattice $\Gamma_k = 2^{-k}\mathbb{Z}^n$ formed by those points of $\mathbb{R}^n$ whose coordinate are integer multiples of 2^{-k}, i.e.,

$$\Gamma_k = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_j = p2^{-k},\, p \in \mathbb{Z}, j = 1, ..., n\}.$$

Let D_k be a collection of cubes determined by Γ_k that is, those cubes with side length 2^{-k} and vertices in Γ_k. The cubes belonging to $D = \bigcup_{k=-\infty}^{\infty} D_k$ are called *dyadic cubes*. Note that if $Q_1, Q_2 \in D$ and $|Q_1| \leq |Q_2|$, then either $Q_1 \subset Q_2$ or else Q_1 and Q_2 do not overlap (by which we mean that their interiors are disjoint).

Lemma 3.2. *Let* $\alpha, \beta > 1$ *be given and let* $\omega \in A_p$ $(1 \leq p < \infty)$*, with* A_p*-constant* $C(\omega, p)$ *and let* $a_{ij} = a_{ji}$ *be measurable, real-valued functions satisfying*

$$\lambda\, \omega(x) |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda\, \omega(x)\, |\xi|^2, \tag{3.1}$$

for all $\xi \in \mathbb{R}^n$ *and a.e.* $x \in \Omega$ *(where* $\Omega \subset \mathbb{R}^n$ *is a bounded open set) . Then there exist weights* $\omega_{\alpha\beta} \geq 0$ *a.e. and measurable real-valued functions* $a_{ij}^{\alpha\beta}$ *such that the following conditions are met.*

(i) $c_1(1/\beta) \leq \omega_{\alpha\beta} \leq c_2\,\alpha$ *in* Ω*, where* c_1 *and* c_2 *depend only on* ω *and* Ω*.*

(ii) *There exist weights* $\tilde{\omega}_1$ *and* $\tilde{\omega}_2$ *such that* $\tilde{\omega}_1 \leq \omega_{\alpha\beta} \leq \tilde{\omega}_2$*, where* $\tilde{\omega}_i \in A_p$ *and* $C(\tilde{\omega}_i, p)$ *depends only on* $C(\omega, p)$ *(*$i = 1, 2$*).*

(iii) $\omega_{\alpha\beta} \in A_p$*, with constant* $C(\omega_{\alpha\beta}, p)$ *depending only on* $C(\omega, p)$ *uniformly on* α *and* β*.*

(iv) *There exists a closed set* $F_{\alpha\beta}$ *such that* $\omega_{\alpha\beta} \equiv \omega$ *in* $F_{\alpha\beta}$ *and* $\omega_{\alpha\beta} \sim \tilde{\omega}_1 \sim \tilde{\omega}_2$ *in* $F_{\alpha\beta}$ *with equivalence constants depending on* α *and* β *(i.e., there are positive constants* $c_{\alpha\beta}$ *and* $C_{\alpha\beta}$ *such that* $c_{\alpha\beta}\,\tilde{\omega}_i \leq \omega_{\alpha\beta} \leq C_{\alpha\beta}\,\tilde{\omega}_i$*,* $i = 1, 2$*). Moreover,* $F_{\alpha\beta} \subset F_{\alpha'\beta'}$ *if* $\alpha \leq \alpha'$ *and* $\beta \leq \beta'$*, and the complement of* $\bigcup_{\alpha,\beta \geq 1} F_{\alpha\beta}$ *has zero measure.*

(v) $\omega_{\alpha\beta} \to \omega$ *a.e. in* $\mathbb{R}^n$ *as* $\alpha, \beta \to \infty$*.*

(vi) $\lambda\,\omega_{\alpha\beta}(x)\,|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta}(x)\,\xi_i\xi_j \leq \Lambda\omega_{\alpha\beta}(x)\,|\xi|^2$*,* $\forall\,\xi \in \mathbb{R}^n$ *and a.e.* $x \in \Omega$*, and* $a_{ij}^{\alpha\beta}(x) = a_{ji}^{\alpha\beta}(x)$*.*

(vii) $a_{ij}^{\alpha\beta}(x) = a_{ij}(x)$ *in* $F_{\alpha\beta}$*.*

Proof. **Case 1.**: Suppose first $\omega \in A_1$. Since we are interested to approximate in Ω, we may assume, without loss of generality, that $\omega \in L^1(\mathbb{R}^n)$. For each $\alpha > 1$, we define $U_\alpha^+ = \{x \in \mathbb{R}^n \;:\; M(\omega)(x) > \alpha\}$, where $M(\omega)$ is the usual Hardy-Littlewood maximal operator for ω, i.e. $(M\omega)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \omega(y)\,dy$, where the supremum is taken over all cubes Q containing x (cube will always mean a compact cube with sides parallel to the axes and nonempty interior). By Calderon-Zygmund decomposition (see Theorem 1.12, Chapter II in [9]), there exists a family of non-overlapping cubes $\left\{Q_j^\alpha\right\}$ consisting of those maximal dyadic cubes over which the average of ω is greater than α, i.e.,

$$(CZ1) \qquad U_\alpha^+ = \bigcup_{j=1}^{\infty} Q_j^\alpha,$$

$$(CZ2) \qquad \alpha < \frac{1}{|Q_j^\alpha|} \int_{Q_j^\alpha} \omega(x)dx \leq 2^n\alpha,$$

$$(CZ3) \qquad \omega(x) \leq \alpha, \; \forall x \in F_\alpha^+ = (U_\alpha^+)^c,$$

$$(CZ4) \qquad |U_\alpha^+|{\leq}\frac{c}{\alpha}\int_{\mathbb{R}^n}\omega(x)dx.$$

Then,

$$\{x{\in}\mathbb{R}^n\ :\ M(\omega)(x)>4^n\alpha\}\subset\bigcup_j 3Q_j^\alpha. \tag{3.2}$$

We explicitly note that, if $\alpha<\beta$, then $U_\beta^+\subset U_\alpha^+$.
Define the weights ω_α by

$$\omega_\alpha(x)=\sum_{k=1}^{\infty}\left(\frac{1}{|Q_k^\alpha|}\int_{Q_k^\alpha}\omega(y)dy\right)\chi_{Q_k^\alpha}(x)+\omega(x)\chi_{F_\alpha^+}(x),$$

where χ_E denotes the characteristic function of a measurable set $E{\subset}\mathbb{R}^n$.
We will show that
(I) $\omega_\alpha{\in}A_1$ and $C(\omega_\alpha,1)$ depends only on $C(\omega,1)$.
(II) $\omega_\alpha{\to}\omega$ a.e. in $\mathbb{R}^n$ as $\alpha{\to}\infty$.
(III) $\min\{1,\omega\}{\in}A_1$ and $C(\min\{1,\omega\},1)$ depends only on $C(\omega,1)$.
Moreover, we also have $\min\{1,\omega\}(x)\leq\omega_\alpha(x){\leq}C\,\omega(x)$, where C depends only on $C(\omega,1)$.
(IV) $2^n\alpha\geq\omega_\alpha(x)\geq C=\dfrac{1}{|Q_0|}\displaystyle\int_{Q_0}\min\{1,\omega\}(y)dy$, $x{\in}\Omega$ and Q_0 is a fixed cube containing Ω $(\Omega\subset Q_0)$.

First of all we note that, if $\omega\in L^1_{\rm loc}$ is a weight function such that for any dyadic cubes Q we have $\omega(3Q)\leq C_0\,\omega(Q)$ (C_0 depends only on N and $C(\omega,1)$), then we can restrict ourselves to test the A_1 condition only on dyadic cubes. In fact, denoting by $C^*(\omega,1)$ the *constant* A_1 *for dyadic cubes* , if $f\in L^1(\mathbb{R}^n,\omega)$ and $\{C_j^t\}$ is the Calderon-Zygmung decomposition for f, we have

$$\begin{aligned}
\omega(\{x\ :\ M(f)(x)>4^n t\}) &\leq \sum_j\omega(3C_j^t)\\
&\leq C_0\sum_j\omega(C_j^t)\\
&\leq C_0\,C^*(\omega,1)\sum_j|C_j^t|\operatorname*{ess\,inf}_{C_j^t}\,\omega\\
&\leq C_0C^*(\omega,1)\frac{1}{t}\sum_j\int_{C_j^t}|f(x)|\,\omega(x)\,dx\\
&\leq C_0\,C^*(\omega,1)\,\frac{1}{t}\int_\Omega|f(x)|\,\omega(x)\,dx,
\end{aligned}$$

and hence $\omega\in A_1$ by Theorem 2.1, Chapter IV in [9] and $C(\omega,1)=C_0\,C^*(\omega,1)$.

Thus, let us prove first that $\omega_\alpha(3Q) \le C_0\, \omega_\alpha(Q)$ for any dyadic cube Q and for any $\alpha > 1$, where C_0 depends only on n and $C(\omega, 1)$. Let Q be a dyadic cube. We have either $|Q \cap U_\alpha^+| < \frac{1}{2}|Q|$ or $|Q\cap U_\alpha^+| \ge \frac{1}{2}\,|Q|$. First, let us suppose that $|Q\cap U_\alpha^+| < \frac{1}{2}|Q|$. We will prove later, in (III), that $\omega_\alpha \le \max\{1, C(\omega,1)\}\omega = C_1\omega$; we have

$$\begin{aligned}
\omega_\alpha(3Q) &\le C_1\,\omega(3Q) = C_1\,\frac{|3Q|}{|3Q|}\int_{3Q}\omega(x)\,dx \\
&= 3^n C_1 |Q| \frac{1}{|3Q|}\int_{3Q}\omega(x)\,dx \\
&\le C_1\,C(\omega,1)3^n|Q| \operatorname{ess\,inf}_{3Q}\omega \\
&\le C_1\,C(\omega,1)3^n|Q| \operatorname{ess\,inf}_{Q\cap F_\alpha^+}\omega \\
&= C_1\,C(\omega,1)3^n \frac{|Q|}{|Q\cap F_\alpha^+|}|Q\cap F_\alpha^+| \operatorname{ess\,inf}_{Q\cap F_\alpha^+}\omega \\
&\le 2C_1\,C(\omega,1)3^n |Q\cap F_\alpha^+| \operatorname{ess\,inf}_{Q\cap F_\alpha^+}\omega \\
&\le 2C_1\,C(\omega,1)3^n \int_{Q\cap F_\alpha^+}\omega(x)\,dx \\
&= 2C_1\,C(\omega,1)3^n \int_{Q\cap F_\alpha^+}\omega_\alpha(x)\,dx,
\end{aligned}$$

since $\omega_\alpha \equiv \omega$ on F_α^+, and therefore

$$\begin{aligned}
\omega_\alpha(3Q) &\le 2C_1\,C(\omega,1)3^n \int_{Q\cap F_\alpha^+}\omega_\alpha(x)\,dx \\
&\le 2C_1\,C(\omega,1)3^n\omega_\alpha(Q) = C_0\;\omega_\alpha(Q),
\end{aligned}$$

(where $C_0 = 2C_1\,C(\omega,1)\,3^n$) and hence, in this case we are done.

Suppose now $|Q\cap U_\alpha^+| \ge \frac{1}{2}\,|Q|$. Then either $Q \subset Q_{j_0}^\alpha$ for some j_0 (which in turn is unique) or $Q_j^\alpha \subset Q$ for $j \in \mathcal{J}$ (a set of indices). By definition of ω_α, it follows from (CZ2) e (CZ3) that $\omega_\alpha(x) \le 2^n\alpha$ a.e.. Therefore, if $Q \subset Q_{j_0}^\alpha$ we get

$$\begin{aligned}
\omega_\alpha(3Q) &= \int_{3Q}\omega_\alpha(x)dx \\
&\le |3Q|2^n\alpha = 3^n\,2^n\,\alpha|Q| \\
&\le 3^n\,2^n|Q|\frac{1}{|Q_{j_0}^\alpha|}\int_{Q_{j_0}^\alpha}\omega(x)\,dx = 6^n\int_Q\omega_\alpha(x)dx = 6^n\omega_\alpha(Q),
\end{aligned}$$

since $\omega_\alpha \equiv \dfrac{1}{|Q_{j_0}^\alpha|}\displaystyle\int_{Q_{j_0}^\alpha} \omega(x)\,dx$ on Q. Otherwise, we obtain

$$\begin{aligned}
\omega_\alpha(3Q) &= \int_{3Q} \omega_\alpha(x)\,dx \\
&\leq 2^n\alpha|3Q| = 2^n\,3^n\alpha|Q| \\
&\leq 6^n\,2|Q\cap U_\alpha^+|\alpha = 2\,6^n\int_{Q\cap U_\alpha^+}\alpha\,dx \\
&= 2\,6^n\sum_{j\in\mathcal{J}}\int_{Q_j^\alpha}\alpha\,dx \\
&\leq 2\,6^n\sum_{j\in\mathcal{J}}\int_{Q_j^\alpha}\omega_\alpha(x)\,dx \\
&\leq 2\,6^n\int_{\cup Q_j^\alpha}\omega_\alpha(x)\,dx \\
&\leq 2\,6^n\int_{Q}\omega_\alpha(x)\,dx = 2\,6^n\omega_\alpha(Q).
\end{aligned}$$

Therefore $\omega_\alpha(3Q) \leq C_0\,\omega_\alpha(Q)$, $\forall Q$ dyadic cube.

Now we are ready to prove (I). Let Q be a fixed dyadic cube, then one of the three cases can happen:

(I1) $Q\cap Q_j^\alpha = \emptyset,\ \forall j$;

(I2) $Q\subset Q_j^\alpha$ for one and only one j;

(I3) $Q_j^\alpha\subset Q$ for some index $j\in\mathcal{J}$.

In case (I1), $Q\subset F_\alpha^+$ and hence $\omega_\alpha\equiv\omega$ in Q and we are done.

In case (I2), we have

$$\omega_\alpha(Q) = \int_Q \omega_\alpha(x)\,dx = |Q|\left(\frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha}\omega\,dx\right) = |Q|\operatorname{ess}\inf_{x\in Q}\omega_\alpha(x),$$

since $\omega_\alpha = \dfrac{1}{|Q_j^\alpha|}\displaystyle\int_{Q_j^\alpha}\omega(x)\,dx$ over Q_j^α.

Finally in case (I3), we have

$$\begin{aligned}
\omega_\alpha(Q) &= \int_Q \omega_\alpha(x)dx = \int_{Q\cap U_\alpha^+}\omega_\alpha(x)\,dx + \int_{Q\cap F_\alpha^+}\omega_\alpha(x)\,dx \\
&= \sum_{j\in\mathcal{J}}\int_{Q_j^\alpha}\omega_\alpha(x)\,dx + \omega_\alpha(Q\cap F_\alpha^+) \\
&= \sum_{j\in\mathcal{J}}\left(\frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha}\omega(x)dx\right)|Q_j^\alpha| + \omega(Q\cap F_\alpha^+)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in\mathcal{J}} \int_{Q\cap Q_j^\alpha} \omega(x)\,dx + \omega(Q\cap F_\alpha^+) \\
&\le \omega(Q) \\
&\le C(\omega,1)\,|Q|\operatorname{ess\,inf}_{y\in Q} \omega(y).
\end{aligned}
$$

On the other hand we note that if $y\in U_\alpha^+$, by definition we have $\omega_\alpha(y) > \alpha$. Thus, if $y\in Q\cap U_\alpha^+$ and Q_k^α is any cube contained in Q we have

$$
\operatorname{ess\,inf}_{Q} \omega \le \operatorname{ess\,inf}_{Q_k^\alpha} \omega \le \frac{1}{|Q_k^\alpha|}\int_{Q_k^\alpha} \omega(x)\,dx \le 2^n\,\alpha < 2^n\omega_\alpha(y).
$$

In addition, if $y\in Q\cap F_\alpha^+$, $\omega_\alpha(y) = \omega(y) \ge \operatorname{ess\,inf}_Q \omega$.
Hence, $\operatorname{ess\,inf}_{Q} \omega \le 2^n \operatorname{ess\,inf}_{Q} \omega_\alpha(y)$.
Therefore, in both cases,

$$
\omega_\alpha(Q) \le C(\omega,1)|Q|\operatorname{ess\,inf}_{y\in Q}\omega(y) \le 2^n\,C(\omega,1)|Q|\operatorname{ess\,inf}_{y\in Q}\omega_\alpha(y),
$$

that is,

$$
\frac{1}{|Q|}\int_Q \omega_\alpha(x)\,dx \le 2^n\,C(\omega,1)\operatorname{ess\,inf}_{y\in Q}\omega_\alpha(y).
$$

Consequently, in (I1), (I2) and (I3) we have

$$
\frac{1}{|Q|}\int_Q \omega_\alpha(x)\,dx \;\sim\; \operatorname{ess\,inf}_{Q}\omega_\alpha
$$

i.e., $\omega_\alpha \in A_1$ and (I) is proved.

To prove (II), by definition we have

$$
\omega_\alpha(x) = \sum_{j=1}^{\infty}\left(\frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha}\omega(x)\,dx\right)\chi_{Q_j^\alpha}(x) + \omega(x)\chi_{F_\alpha^+}(x).
$$

Hence, $\omega_\alpha \equiv \omega$ in F_α^+ and that F_α^+ increases as α tends to infinity. Moreover, $|\cap(F_\alpha^+)^c| = 0$. Then $\omega_\alpha \to \omega$ a.e. as $\alpha\to\infty$.
Finally, to show (III) we know that, for any cube Q, either $\operatorname{ess\,inf}_{Q}\omega \ge 1$ or $\operatorname{ess\,inf}_{Q}\omega < 1$. In the first case,

$$
\frac{1}{|Q|}\int_Q \min\{\omega,1\}(x)\,dx = \frac{1}{|Q|}\int_Q 1dx = 1 \le \min\{\omega(y),1\}
$$

for any $y\in Q$, whereas if $\inf_{Q}\omega < 1$ then

$$
\frac{1}{|Q|}\int_Q \min\{\omega,1\}(x)\,dx \le \frac{1}{|Q|}\int_Q \omega(x)\,dx \le C(\omega,1)\operatorname{ess\,inf}_{Q}\omega,
$$

since $\omega \in A_1$. Put $\lambda = \operatorname{ess\,inf}_Q \omega < 1$ and assume by contradiction that $\inf_Q(\min\{\omega, 1\}) < \lambda$; then there exists $E \subset Q$, $|E| > 0$ such that $\min\{\omega, 1\} < \lambda' < \lambda$ in E and hence, since $\lambda < 1$, we obtain $\omega < \lambda'$ in E which is a contradiction. Thus we have proved the first part of (III), that is, $\min\{\omega, 1\} \in A_1$.
To prove the second part we note that if $x \in F_\alpha^+$ then

$$\omega_\alpha(x) = \omega(x) \geq \min\{\omega, 1\}(x).$$

If $x \in Q_j^\alpha$, for some j, then by definition and (CZ2), we get

$$\omega_\alpha(x) = \frac{1}{|Q_j^\alpha|} \int_{Q_j^\alpha} \omega(y)dy > \alpha \geq \min\{\omega, 1\}(x).$$

Analogously, if $x \in Q_j^\alpha$, for some j, then

$$\omega_\alpha(x) \leq C(\omega, 1) \operatorname{ess\,inf}_{Q_j^\alpha} \omega \leq C(\omega, 1)\omega(x).$$

Therefore, $\min\{\omega, 1\} \leq \omega_\alpha(x) \leq C(\omega, 1)\omega(x)$.

Finally, assertion (IV) follows straightforwardly from (III) by using (CZ1) and (CZ3).

We define the coefficients a_{ij}^α by

$$a_{ij}^\alpha(x) = \sum_{l=1}^{\infty} \left(\frac{1}{|Q_l^\alpha|} \int_{Q_l^\alpha} a_{ij}(y)dy \right) \chi_{Q_l^\alpha}(x) + a_{ij}(x)\chi_{F_\alpha^+}(x).$$

Case 2. Suppose now $\omega \in A_p$, for $p > 1$. Then by Jones's Factorization Theorem (Theorem 3.1) there exist ω_0, $\omega_1 \in A_1$ such that $\omega = \omega_0 \omega_1^{1-p}$, and $C(\omega_i, 1)$ $(i = 0, 1)$ depends only on $C(\omega, p)$. Choose α, $\beta > 1$ and define $\omega_{\alpha\beta}$ by

$$\omega_{\alpha\beta} = (\omega_0)_\alpha \left[(\omega_1)_{\beta^{1/(p-1)}} \right]^{1-p}.$$

We need to show that $\omega_{\alpha\beta}$ satisfies properties (i) - (v).
Obviously (i) follows from (IV).
(ii) follows from (III) and Jones's Factorization Theorem, with

$$\tilde{\omega}_1 = \min\{\omega_0, 1\}\omega_1^{1-p},$$

$$\tilde{\omega}_2 = \omega_0(\min\{\omega_1, 1\})^{1-p}.$$

(iii) follows from (I) and Jones's Factorization Theorem.

(v) follows from (II).

For (iv), we define $F_{\alpha\beta} = F^0_\alpha \cap F^1_{\beta^{1/(p-1)}}$, where $F^0_\alpha = F^+_\alpha$ for the weight ω_0 and $F^1_{\beta^{1/(p-1)}} = \left(F_{\beta^{1/(p-1)}}\right)^+$ for the weight ω_1. By definition, $(\omega_0)_\alpha \equiv \omega_0$ and $(\omega_1)_{\beta^{1/(p-1)}} \equiv \omega_1$ on $F_{\alpha\beta}$.

Hence, to prove that $\omega_{\alpha\beta} \sim \tilde{\omega}_1$, we can replace $\omega_{\alpha\beta}$ by ω (on $F_{\alpha\beta}$). We have $\min\{\omega_0, 1\} \sim \omega_0$ on F^0_α, since $\min\{\omega_0, 1\} \le \omega_0$ (by (III)). Moreover, if $x \in F^0_\alpha$, $\min\{\omega_0(x), 1\} = 1$. Hence, by (CZ3) we obtain

$$\omega_0(x) \le \alpha = \alpha \min\{\omega_0(x), 1\}.$$

Therefore, $\min\{\omega_0, 1\} \le \omega_0 \le \alpha \min\{\omega_0, 1\}$ on F^0_α .

An analogous argument shows that $\omega_1 \sim \min\{\omega_1, 1\}$ on $F^1_{\beta^{1/(p-1)}}$, and hence,

$$\omega_{\alpha\beta} \sim \min\left\{\omega_0, 1\right\}\left[(\omega_1)_{\beta^{1/(p-1)}}\right]^{1-p} = \tilde{\omega}_1 \quad \text{on } F_{\alpha\beta},$$

and analogously, we also have $\omega_{\alpha\beta} \sim \tilde{\omega}_2$.

Finally to prove (vi) and (vii), we define the coefficients $a^{\alpha\beta}_{ij}$ by

$$a^{\alpha\beta}_{ij}(x) = \left[(\omega_1)_{\beta^{1/(p-1)}}\right]^{1-p}(x)\left\{\sum_{k=1}^{\infty}\left(\frac{1}{|Q^\alpha_k|}\int_{Q^\alpha_k} a_{ij}(y)\omega_1^{p-1}(y)dy\right)\chi_{Q^\alpha_k}(x) + a_{ij}(x)\omega_1^{p-1}(x)\chi_{F^+_\alpha}(x)\right\},$$

where $\{Q^\alpha_k\}$ is the Calderon-Zygmund decomposition of ω_0. □

The main results of this paper are the following.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that*
(**H1**) $\omega \in A_2$;
(**H2***) $f/\omega \in L^2(\Omega, \omega) \cap L^2(\Omega, \omega^3)$;
(**H3**) $b_i/\omega \in L^\infty(\Omega)$ *(i=1,...,n) and* $g/\omega \in L^\infty(\Omega)$.
Then the unique solution $u \in W^{1,2}(\Omega, \omega)$ of problem (P) is the weak limit in $W^{1,2}(\Omega, \tilde{\omega}_1)$ of a sequence of solutions $u_m \in W^{1,2}(\Omega, \omega_m)$ of the problems

$$(P_m)\begin{cases} L_m u_m(x) = f_m(x), & \text{in } \Omega, \\ \langle A^m(x)\nabla u_m, \vec{\eta}(x)\rangle = 0, & \text{on } \partial\Omega, \end{cases}$$

with

$$L_m u_m = -\sum_{i,j=1}^{n} D_j(a^{mm}_{ij}\, D_i u_m) + \sum_{i=1}^{n} b_{mi}\, D_i u_m + g_m\, u_m + \theta\, u_m\, \omega_m,$$

$f_m = f\,(\omega_m/\omega)^{1/2}$, $g_m = g\,\omega_m/\omega$, $b_{mi} = b_i\,\omega_m/\omega$ and $\omega_m = \omega_{mm}$ (where ω_{mm}, a_{ij}^{mm} and $\tilde{\omega}_1$ are as Lemma 3.2 and $A^m(x) = \big(a_{ij}^{mm}(x)\big)$).

Proof. **Step 1.** First, if $f_m = f(\omega/\omega_m)^{-1/2}$, $g_m = g\,\omega_m/\omega$ and $b_{mi} = b_i\,\omega_m/\omega$, we note that

$$\left\|\frac{f_m}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)}, \quad \left\|\frac{g_m}{\omega_m}\right\|_{L^\infty(\Omega)} = \left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)}$$
$$\left\|\frac{b_{mi}}{\omega_m}\right\|_{L^\infty(\Omega)} = \left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}. \tag{3.3}$$

Then, if $u_m \in W^{1,2}(\Omega,\omega_m)$ is a solution of problem (P_m) we have (by (3.3))

$$\|u_m\|_{W^{1,2}(\Omega,\omega_m)} \le \frac{2}{\lambda}\left\|\frac{f_m}{\omega_m}\right\|_{L^2(\Omega,\omega_m)} = \frac{2}{\lambda}\left\|\frac{f}{\omega}\right\|_{L^2(\Omega,\omega)} = C_3.$$

Using Lemma 3.2, $\tilde{\omega}_1 \le \omega_m$, we obtain

$$\|u_m\|_{W^{1,2}(\Omega,\tilde{\omega}_1)} \le \|u_m\|_{W^{1,2}(\Omega,\omega_m)} \le C_3. \tag{3.4}$$

Consequently, $\{u_m\}$ is a bounded sequence in $W^{1,2}(\Omega,\tilde{\omega}_1)$. Therefore, there is a subsequence, again denoted by $\{u_m\}$, and $\tilde{u} \in W^{1,2}(\Omega,\tilde{\omega}_1)$ such that

$$u_m \rightharpoonup \tilde{u} \text{ in } L^2(\Omega,\tilde{\omega}_1), \tag{3.5}$$
$$\frac{\partial u_m}{\partial x_j} \rightharpoonup \frac{\partial \tilde{u}}{\partial x_j} \text{ in } L^2(\Omega,\tilde{\omega}_1), \tag{3.6}$$
$$u_m \to \tilde{u} \text{ a.e. in } \Omega, \tag{3.7}$$

where the symbol "$\rightharpoonup$" denotes weak convergence (see Theorem 1.31 in [11]).

Step 2. We have that $\tilde{u} \in W^{1,2}(\Omega,\omega)$. In fact, for $F_k = F_{kk}$ fixed (see Lemma 3.2), we have by (3.5) and (3.6), for all $\varphi \in W^{1,2}(\Omega,\tilde{\omega}_1)$, we obtain

$$\int_\Omega u_m \varphi\,\tilde{\omega}_1\,dx \to \int_\Omega \tilde{u}\,\varphi\,\tilde{\omega}_1\,dx,$$
$$\int_\Omega D_i u_m D_i\varphi\,\tilde{\omega}_1\,dx \to \int_\Omega D_i\tilde{u}\,D_i\varphi\,\tilde{\omega}_1\,dx.$$

If $\psi \in L^2(\Omega,\omega)$, then $\psi\,\chi_{F_k} \in L^2(\Omega,\tilde{\omega_1})$ (since $\omega \sim \tilde{\omega}_1$ in F_k). Consequently,

$$\int_\Omega u_m \varphi \chi_{F_k}\,\tilde{\omega}_1\,dx \to \int_\Omega \tilde{u}\,\varphi\,\chi_{F_k}\,\tilde{\omega}_1\,dx,$$
$$\int_\Omega D_i u_m\,\varphi\,\chi_{F_k}\,\tilde{\omega}_1\,dx \to \int_\Omega D_i\tilde{u}\,\varphi\,\chi_{F_k}\,\tilde{\omega}_1\,dx,$$

for all $\varphi \in L^2(\Omega,\omega)$, that is, the sequence $\{\dfrac{\partial u_m}{\partial x_i}\,\chi_{F_k}\}$ is weakly convergent to a function in $L^2(\Omega,\omega)$, again since $\omega \sim \tilde{\omega}_1$ on F_k. Therefore, we have

$$\|\,|\nabla \tilde{u}|\,\|^2_{L^2(F_k,\omega)} = \int_{F_k} |\nabla \tilde{u}|^2 \omega\,dx \leq \limsup_{m\to\infty} \int_{F_k} |\nabla u_m|^2\,\omega\,dx,$$

and for $m \geq k$ we have $\omega = \omega_m$ in F_k. Hence, by (3.4), we obtain

$$\begin{aligned}\|\,|\nabla \tilde{u}|\,\|^2_{L^2(F_k,\omega)} &\leq \limsup_{m\to\infty} \int_{F_k} |\nabla u_m|^2\,\omega\,dx \\ &= \limsup_{m\to\infty} \int_{F_k} |\nabla u_m|^2\,\omega_m\,dx \\ &\leq \limsup_{m\to\infty} \int_{\Omega} |\nabla u_m|^2 \omega_m\,dx \leq C_3^2.\end{aligned}$$

By the Monotone Convergence Theorem we obtain $\|\,|\nabla \tilde{u}|\,\|_{L^2(\Omega,\omega)} \leq C_3$. Analogously, $\|\tilde{u}\|_{L^2(\Omega,\omega)} \leq C_3$. Therefore, we have $\tilde{u} \in W^{1,2}(\Omega,\omega)$.

Step 3. We need to show that $\tilde{u}$ is a solution of problem (P), i.e, for every $\varphi \in W^{1,2}(\Omega,\omega)$ we have

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}\,D_i\tilde{u}\,D_j\varphi\;dx + \sum_{i=1}^{n} \int_{\Omega} b_i\,\varphi\,D_i\tilde{u}\,dx + \int_{\Omega} g\,\tilde{u}\,\varphi\,dx + \theta \int_{\Omega} \tilde{u}\,\varphi\,\omega\,dx = \int_{\Omega} f\,\varphi\,dx.$$

Using the fact that u_m is a solution of (P_m), we have

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}^{mm}\,D_iu_m\,D_j\varphi\,dx + \sum_{i=1}^{n} \int_{\Omega} b_{mi}\,\varphi\,D_iu_m\,dx + \int_{\Omega} g_m\,u_m\,\varphi\,dx + \theta \int_{\Omega} u_m\,\varphi\,\omega_m\,dx = \int_{\Omega} f_m\,\varphi\,dx,$$

for every $\varphi \in W^{1,2}(\Omega,\omega_m)$. Moreover, by Lemma 3.2 and (3.3), over $F_k = F_{kk}$ (for $m \geq k$) we have the following properties:

(i) $\omega = \omega_m$;

(ii) $f_m = f$, $g_m = g$ and $b_{mi} = b_i$;

(iii) $a_{ij}^{mm}(x) = a_{ij}(x)$.

For $\varphi \in W^{1,2}(\Omega,\omega)$ and $k > 0$(fixed), we define $G_1, G_2 : W^{1,2}(\Omega,\tilde{\omega}_1) \to \mathbb{R}$ by

$$G_1(u) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}\,D_iu\;D_j\varphi\,\chi_{F_k}\,dx,$$

$$G_2(u)=\sum_{i=1}^{n}\int_\Omega \varphi\, b_i\, D_i u\,\chi_{F_k}\,dx+\int_\Omega g\,u\,\varphi\,\chi_{F_k}\,dx+\theta\int_\Omega u\,\varphi\,\omega\,\chi_{F_k}\,dx.$$

(a) We have that G_1 is linear and continuous functional. In fact, we have (by Lemma 3.2(iv)) $\omega\sim\tilde{\omega}_1$ in F_k. And by (1.2) we obtain

$$\begin{aligned}
|G_1(u)| &\leq \int_{F_k} |\langle A\nabla u,\nabla\varphi\rangle|\,dx\\
&\leq \int_{F_k} (\langle A\nabla u,\nabla u\rangle)^{1/2}\,(\langle A\nabla\varphi,\nabla\varphi\rangle)^{1/2}\,dx\\
&\leq \left(\int_{F_k}\langle A\nabla u,\nabla u\rangle\,dx\right)^{1/2}\left(\int_{F_k}\langle A\nabla\varphi,\nabla\varphi\rangle\,dx\right)^{1/2}\\
&\leq \Lambda\left(\int_{F_k}|\nabla u|^2\,\omega\,dx\right)^{1/2}\left(\int_{F_k}|\nabla\varphi|^2\,\omega\,dx\right)^{1/2}\\
&\leq \Lambda\left(\int_{F_k}c\,|\nabla u|^2\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_{\Omega}|\nabla\varphi|^2\,\omega\,dx\right)^{1/2}\\
&\leq \Lambda\, c^{1/2}\|\varphi\|_{W^{1,2}(\Omega,\omega)}\,\|u\|_{W^{1,2}(\Omega,\tilde{\omega}_1)}.
\end{aligned}$$

(b) We have that G_2 is linear and continuous functional. In fact,

$$\begin{aligned}
&|G_2(u)|\\
&\leq\sum_{i=1}^{n}\int_{F_k}|\varphi|\,|b_i|\,|D_i u|\,dx+\int_{F_k}|g|\,|u|\,|\varphi|\,dx+\,\theta\int_{F_k}|u|\,|\varphi|\,\omega\,dx\\
&\leq\sum_{i=1}^{n}\left\|\frac{b_i}{\omega}\right\|_{L^\infty(F_k)}\left(\int_{F_k}|D_i u|^2\,\omega\,dx\right)^{1/2}\left(\int_{F_k}|\varphi|^2\,\omega\,dx\right)^{1/2}\\
&+\left\|\frac{g}{\omega}\right\|_{L^\infty(F_k)}\left(\int_{F_k}|u|^2\,\omega\,dx\right)^{1/2}\left(\int_{F_k}|\varphi|^2\,\omega\,dx\right)^{1/2}\\
&+\theta\left(\int_{F_k}|u|^2\,\omega\,dx\right)^{1/2}\left(\int_{F_k}|\varphi|^2\,\omega\,dx\right)^{1/2}\\
&\leq\left(\max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}\right)\left(c\int_{F_k}|D_i u|^2\,\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_\Omega|\varphi|^2\,\omega\,dx\right)^{1/2}\\
&+\left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)}\left(c\int_{F_k}|u|^2\,\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_\Omega|\varphi|^2\,\omega\,dx\right)^{1/2}\\
&+\theta\left(c\int_{F_k}|u|^2\,\tilde{\omega}_1\,dx\right)^{1/2}\left(\int_\Omega|\varphi|^2\,\omega\,dx\right)^{1/2}\\
&\leq\left(\max\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}+\left\|\frac{g}{\omega}\right\|_{L^\infty(\Omega)}+\theta\right)c^{1/2}\;\|u\|_{W^{1,2}(\Omega,\tilde{\omega}_1)}\|\varphi\|_{W^{1,2}(\Omega,\omega)}.
\end{aligned}$$

Using (a), (b), properties (i),(ii) and (iii), and that u_m is solution of (P_m), we obtain

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\int_{F_k} a_{ij}\, D_i\tilde{u}\, D_j\varphi\, dx + \sum_{i=1}^{n}\int_{F_k} \varphi\, b_i\, D_i\tilde{u}\, dx + \int_{F_k} g\,\tilde{u}\,\varphi\, dx + \theta\int_{F_k}\tilde{u}\,\varphi\,\omega\, dx\\
&= \lim_{m\to\infty}\big(G_1(u_m)+G_2(u_m)\big)\\
&= \lim_{m\to\infty}\Bigg(\sum_{i,j=1}^{n}\int_{F_k} a_{ij}^{mm}\, D_i u_m\, D_j\varphi\, dx + \sum_{i=1}^{n}\int_{F_k}\varphi\, b_{mi}\, D_i u_m\, dx\\
&+\int_{F_k} g_m\, u_m\,\varphi\, dx + \theta\int_{F_k} u_m\,\varphi\,\omega_m\, dx\Bigg)\\
&= \lim_{m\to\infty}\Bigg(\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}^{mm}\, D_i u_m\, D_j\varphi\, dx + \sum_{i=1}^{n}\int_{\Omega}\varphi\, b_i\, D_i u_m\, dx + \int_{\Omega} g_m\, u_m\,\varphi\, dx\\
&+\theta\int_{\Omega} u_m\,\varphi\,\omega_m\, dx\\
&-\sum_{i,j=1}^{n}\int_{\Omega\cap F_k^c} a_{ij}^{mm}\, D_i u_m\, D_j\varphi\, dx - \sum_{i=1}^{n}\int_{\Omega\cap F_k^c}\varphi\, b_i\, D_i u_m\, dx\\
&-\int_{\Omega\cap F_k^c} g_m\, u_m\,\varphi\, dx - \theta\int_{\Omega\cap F_k^c} u_m\,\varphi\,\omega_m\, dx\Bigg)\\
&= \lim_{m\to\infty}\Bigg(\int_{\Omega} f_m\,\varphi\, dx - \sum_{i,j=1}^{n}\int_{\Omega\cap F_k^c} a_{ij}^{mm}\, D_i u_m\, D_j\varphi\, dx\\
&-\sum_{i=1}^{n}\int_{\Omega\cap F_k^c}\varphi\, b_i\, D_i u_m\, dx - \int_{\Omega\cap F_k^c} g_m\, u_m\,\varphi\, dx - \theta\int_{\Omega\cap F_k^c} u_m\,\varphi\,\omega_m\, dx. \qquad (3.8)
\end{aligned}
$$

(I) By Lemma 3.2(v) we have $f_m = \dfrac{f\omega_m^{1/2}}{\omega^{1/2}} \to f$ a.e. in Ω. Since $\omega_m = \omega$ in F_k $(m \geq k)$ we also have

$$
\begin{aligned}
\int_{\Omega} f_m^2\,\omega\, dx &= \int_{\Omega} f^2\,\omega_m\, dx = \int_{F_k} f^2\,\omega_m\, dx + \int_{\Omega\,\cap\, F_k^c} f^2\,\omega_m\, dx\\
&= \int_{F_k} f^2\,\omega\, dx + \int_{\Omega\,\cap\, F_k^c} f^2\,\omega_m\, dx\\
&\leq \int_{\Omega} f^2\,\omega\, dx + \int_{\Omega\,\cap\, F_k^c} f^2\,\omega_m\, dx\\
&= \int_{\Omega}\left(\frac{f}{\omega}\right)^2\omega^3\, dx + \int_{\Omega\,\cap\, F_k^c} f^2\,\omega_m\, dx.
\end{aligned}
$$

By Lemma 3.2 (iv), we know that $|\Omega\cap F_k^c| \to 0$ when $k \to \infty$. Then, for sufficiently large k we have $\displaystyle\int_{\Omega\,\cap\,F_k^c} f^2\,\omega_m\,dx \leq 1$. Therefore, for sufficiently large m and (H2*), we obtain

$$\int_\Omega f_m^2\,\omega\,dx \leq \int_\Omega \left(\frac{f}{\omega}\right)^2 \omega^3\,dx + 1 < \infty.$$

Hence the sequence $\{f_m\}$ is bounded in $L^2(\Omega,\omega)$. Then there is a subsequence, still denoted by $\{f_m\}$, and a function $\tilde{f}$ such that

$$f_m \rightharpoonup \tilde{f} \text{ in } L^2(\Omega,\omega),$$
$$f_m \to \tilde{f} \text{ a.e. in } \Omega.$$

Since $f_m \to f$ a.e. in Ω, then $\tilde{f} = f$ a.e. in Ω. Therefore, for all $\varphi \in W^{1,2}(\Omega,\omega)$, we have

$$\int_\Omega f_m\,\varphi\,dx \to \int_\Omega f\,\varphi\,dx.$$

(II) Since the matrix $A^m(x) = (a_{ij}^{mm})(x)$ is symmetric, we have

$$|\langle A^m\nabla u_m, \nabla\varphi\rangle| \leq \langle A^m\nabla u_m, \nabla u_m\rangle^{1/2}\,\langle A^m\nabla\varphi, \nabla\varphi\rangle^{1/2}.$$

Then, by Lemma 3.2 (vi) and (3.4), we obtain

$$\begin{aligned}
&\left|\sum_{i,j=1}^n \int_{\Omega\cap F_k^c} a_{ij}^{mm} D_i u_m D_j\varphi\,dx\right| \\
&\leq \int_{\Omega\cap F_k^c} |\langle A^m\nabla u_m, \nabla\varphi\rangle|\,dx \\
&\leq \Lambda\bigg(\int_{\Omega\cap F_k^c} |\nabla u_m|^2\omega_m\,dx\bigg)^{1/2}\bigg(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\omega_m\,dx\bigg)^{1/2} \\
&\leq \Lambda\|u_m\|_{W^{1,2}(\Omega,\omega_m)}\bigg(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\,w_m\,dx\bigg)^{1/2} \\
&\leq \Lambda\,C_3\left(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2\,w_m\,dx\right)^{1/2}. \qquad (3.9)
\end{aligned}$$

(III) By (H3), (3.3) and (3.4) we have

$$\begin{aligned}
&\left|\int_{\Omega\cap F_k^c} \varphi\,b_{mi}\,D_i u_m\,dx\right| \\
&\leq \int_{\Omega\cap F_k^c} |\varphi|\,|b_{mi}|\,|D_i u_m|\,dx
\end{aligned}$$

$$\leq \left\| \frac{b_{mi}}{\omega_m} \right\|_{L^\infty(\Omega)} \left(\int_{\Omega\cap F_k^c} |D_i u_m|^2 \omega_m \, dx \right)^{1/2} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \, \omega_m \, dx \right)^{1/2}$$

$$\leq \left\| \frac{b_i}{\omega} \right\|_{L^\infty(\Omega)} \|u_m\|_{W^{1,2}(\Omega,\omega_m)} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m \, dx \right)^{1/2}$$

$$\leq C_3 \left\| \frac{b_i}{\omega} \right\|_{L^\infty(\Omega)} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m \, dx \right)^{1/2}, \tag{3.10}$$

and analogously

$$\left| \int_{\Omega\cap F_k^c} g_m \, u_m \, \varphi \, dx \right| \leq C_3 \left\| \frac{g}{\omega} \right\|_{L^\infty(\Omega)} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m \, dx \right)^{1/2}, \tag{3.11}$$

and

$$\left| \int_{\Omega\cap F_k^c} u_m \, \varphi \, \omega_m \, dx \right| \leq C_3 \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m \, dx \right)^{1/2}. \tag{3.12}$$

Note now that $\omega_m \leq \tilde{\omega}_2$ and $\tilde{\omega}_2 \in A_2$ (by Lemma 3.2). Hence, by Remark 2.2(b), there exist $\delta > 0$ and $C > 0$ such that, if K_0 is a cube containing $\bar{\Omega}$ then

$$\begin{aligned} \mu_m(\Omega \cap F_k^c) &= \int_{\Omega\,\cap\, F_k^c} \omega_m(x) \, dx \\ &\leq \int_{\Omega\,\cap\, F_k^c} \tilde{\omega}_2(x) \, dx = \tilde{\mu}_2(\Omega \cap F_k^c) \\ &\leq C \, \tilde{\mu}_2(K_0) \left(\frac{|F_k^c|}{|K_0|} \right)^{\delta}, \end{aligned}$$

which is independent of m and tends to zero as $k \to \infty$ by Lemma 3.2(iv). Then

$$\lim_{k\to\infty} \left(\int_{\Omega\cap F_k^c} |\varphi|^2 \omega_m \, dx \right)^{1/2} = \lim_{k\to\infty} \left(\int_{\Omega\cap F_k^c} |\nabla\varphi|^2 \, \omega_m \, dx \right)^{1/2} = 0,$$

and we obtain in (3.9), (3.10), (3.11) and (3.12)

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} a_{ij}^{mm}(x) D_i u(x) \, D_j \varphi(x) \, dx = 0, \tag{3.13}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} \varphi \, b_{mi} \, D_i u_m \, dx = 0, \tag{3.14}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} g_m \, u_m \, \varphi \, dx = 0, \tag{3.15}$$

$$\lim_{k\to\infty} \int_{\Omega\cap F_k^c} u_m \, \varphi \, \omega_m \, dx = 0. \tag{3.16}$$

Therefore, by (3.8), (3.13), (3.14) and (3.16), we conclude, when $k \to \infty$ (and $m \geq k$),

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i\tilde{u}\, D_j\varphi\, dx + \sum_{i=1}^{n} \int_\Omega b_i\, \varphi\, D_i\tilde{u}\, dx + \int_\Omega g\, \tilde{u}\, \varphi\, dx + \theta \int_\Omega \tilde{u}\, \varphi\, \omega\, dx$$
$$= \int_\Omega f\, \varphi\, dx,$$

for all $\varphi \in W^{1,2}(\Omega,\omega)$, that is, $\tilde{u}$ is a solution of problem (P). Therefore, $u = \tilde{u}$ (by the uniqueness). □

Example 3.4. Consider the domain $\Omega = \{(x,y) \in \mathbb{R}^2 :\ x^2 + y^2 < 1\}$, the weight function $\omega(x,y) = (x^2+y^2)^{-1/2}$ and the coefficient matrix

$$A(x,y) = \begin{pmatrix} 2(x^2+y^2)^{-1/2} & 0 \\ 0 & 4(x^2+y^2)^{-1/2} \end{pmatrix}$$

We have for all $\xi \in \mathbb{R}^2$ and almost every $(x,y) \in \Omega$,

$$\frac{2}{(x^2+y^2)^{1/2}} |\xi|^2 \leq \langle A(x,y)\xi, \xi\rangle \leq \frac{4}{(x^2+y^2)^{1/2}} |\xi|^2.$$

If $(x,y) \in \partial\Omega = \{(x,y) \in \mathbb{R}^2 \ : x^2+y^2 = 1\}$, then $\vec{\eta}(x,y) = (x,y)$ is the unit outward normal to $\partial\Omega$. By Theorem 2.7 the Neumann problem

$$\begin{cases} Lu(x,y) = (x^2+y^2)^{-1/5}\cos(xy) \ \text{ on } \ \Omega, \\ \langle A(x,y)\nabla u, \vec{\eta}\rangle = 0, \ \text{ on } \ \partial\Omega, \end{cases}$$

where

$$\begin{aligned} Lu(x,y) &= -\left[\frac{\partial}{\partial x}\left(\frac{2}{(x^2+y^2)^{1/2}}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{4}{(x^2+y^2)^{1/2}}\frac{\partial u}{\partial y}\right)\right] \\ &+ \frac{\cos(xy)}{(x^2+y^2)^{1/3}}\frac{\partial u}{\partial x} + \frac{\sin(xy)}{(x^2+y^2)^{1/4}}\frac{\partial u}{\partial y} \\ &+ \frac{u(x,y)\sin(xy)}{(x^2+y^2)^{1/3}} + \theta\frac{u(x,y)}{(x^2+y^2)^{1/2}} \end{aligned}$$

has a unique solution $u \in W^{1,2}(\Omega,\omega)$ (if $\theta \geq 13/4$), and by Theorem 3.3 the solution u can be approximated by a sequence of solutions of non-degenerate elliptic equations.

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References

[1] D. Bresch, J. Lemoine and F. Guíllen-Gonzalez, A note on a degenerate elliptic equation with applications for lakes and seas, *Electron. J. Differential Equations*, **vol. 2004** (2004), no.42, 1-13

[2] A.C. Cavalheiro, Existence and uniqueness of solution for a class of nonlinear degenerate elliptic equations, *Anal. Theory Appl.*, **vol. 36** (2020), no.1, 69-88

[3] A.C. Cavalheiro, The existence of entropy solutions for nonlinear degenerate elliptic equations, *J. Nonlinear Funct. Anal.*, **vol. 2020** (2020), 1-14.

[4] V. Chiadò Piat and F. Serra Cassano, Relaxation of degenerate variational integrals, *Nonlinear Anal.*, **22**, (1994), 409-429.

[5] M. Colombo, Flows of Non-Smooth Vector Fields and Degenerate Elliptic Equations: With Applications to the Vlasov-Poisson and Semigeostrophic Systems, *Publications of the Scuolla Normale Superiore Pisa,* **22**, Pisa, (2017).

[6] P. Drábek, A. Kufner and F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, Walter de Gruyter, Berlin, 1997.

[7] J.C.Fernandes and B.Franchi,Existence and properties of the Green function for a class of degenerate parabolic equations, *Rev. Mat. Iberoam.*, **12** (1996), 491-525.

[8] B. Franchi and R. Serapioni, Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approch, *Ann. Sc. Norm. Sup. Pisa Cl.Sci.*, **14** (1987), 527-568.

[9] J. Garcia-Cuerva and J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, *North-Holland Mathematics Studies*, **116**, 1985.

[10] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second order*, 2nd Ed., Springer, New York, 1983.

[11] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, 1993.

[12] T. Kilpeläinen, Weighted Sobolev spaces and capacity, *Ann. Acad. Sci. Fenn. Math.*, **19** (1994), 95-113.

[13] A. Kufner, *Weighted Sobolev Spaces*, **Vol.31**, Teubner-Texte zur Mathematik, John Willey & Sons, Leipzig (1985).

[14] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Am. Math. Soc.*, **165** (1972), 207-226.

[15] A. Torchinsky, *Real-variable Methods in Harmonic Analysis*, Academic Press, San Diego, 1986.

[16] B.O. Turesson, *Nonlinear Potential Theory and Weighted Sobolev Spaces*, Lectures Notes in Math., vol. 1736, Springer-Verlag, 2000.