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# Generalized warped products and the $\kappa$-nullity of Riemannian curvature

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*Dedicated to Renato de Azevedo Tribuzy*
*on the occasion of his 75th anniversary.*

**Abstract.** In this short survey, we show how two (classes of) known examples of inhomogeneous, curvature homogeneous Riemannian manifolds with nontrivial $\kappa$-nullity can be seen as deformations of homogeneous metrics along the vertical distribution of an integrable Riemannian submersion. We also pose two open questions.

**Keywords:** warped product, nullity, riemannian submersion, curvature homogeneous.

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# 1 Introduction

This is a survey discussing two known examples of complete Riemannian manifolds with nontrivial $\kappa$-nullity from the stance of generalized warped products. There are essentially no original results herein, perhaps only the viewpoint is slightly new.

The idea of nullity was introduced in case $\kappa = 0$ by Chern and Kuiper in [CK52], and for general $\kappa$ by Otsuki [Ôts54], and later reformulated and studied by different authors (see e.g. [Gra66, Mal72] and, for more recent work, [DOV20] and the references therein). Let $M$ be a connected Riemannian manifold, and consider the curvature tensor $R$ of its Levi-Civita connection $\nabla$ with the sign convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for vector fields $X$, $Y$, $Z \in \Gamma(TM)$. For $\kappa \in \mathbb{R}$, the *$\kappa$-nullity distribution* of $M$ is the variable rank distribution $\mathcal{N}_\kappa$ on $M$ defined for each $p \in M$ by

$$\mathcal{N}_\kappa|_p = \{z \in T_pM : R_p(x,y)z = -\kappa(\langle x,z\rangle_p y - \langle y,z\rangle_p x) \text{ for all } x,\, y \in T_pM\}.$$

The number $\nu_\kappa(p) := \dim \mathcal{N}_\kappa|_p$ is called the *index of $\kappa$-nullity* at $p$. We call the orthogonal complement of $\mathcal{N}_\kappa$ the *$\kappa$-conullity distribution* of $M$, and its dimension at a point $p \in M$ the *index of $\kappa$-conullity* at $p$, or simply, the $\kappa$-conullity at $p$.

In case $\kappa = 0$ we obtain trivial examples of manifolds with positive $\nu_0$ simply by taking a Riemannian product with an Euclidean space, but similar product examples do not occur if $\kappa \neq 0$. It is easily seen that $\nu_\kappa(p)$ is nonzero for at most one value of $\kappa$. For general $M$, $\nu_\kappa$ is nonnecessarily constant if nonzero, but it is an upper semicontinuous function, so there is an open and dense set of $M$ where $\nu_\kappa$ is locally constant, and there is an open subset $\Omega$ of $M$ where $\nu_\kappa$ attains its minimum value. It is known that $\mathcal{N}_\kappa$ is an autoparallel distribution on any open set where $\nu_\kappa$ is locally constant and, in case $M$ is a complete Riemannian manifold, the leaves of

$\mathcal{N}_\kappa$ in $\Omega$ are *complete* totally geodesic submanifolds of constant curvature $\kappa$ [Mal72].

Having nontrivial $\kappa$-nullity distribution per se is usually not a very strong restriction for a Riemannian manifold, but, when combined with other conditions (e.g. homogeneity, irreducibility, control of the scalar curvature, finite volume, etc.), it can led to interesting classification/rigidity/non-existence results (see e.g. [Bro19, FZ20, GG22]). Further, manifolds of (maximal) 0-conullity 2 are closely related to semi-symmetric spaces. Recall that a *semi-symmetric* space is a Riemannian manifold whose curvature tensor is, at each point, orthogonally equivalent to the curvature tensor of a symmetric space; the symmetric space may depend on the point. Conversely, Z. I. Szabó [Sza85] showed that every complete irreducible semi-symmetric space is either locally symmetric or has 0-conullity at most 2 on an open and dense subset. In a related vein, a Riemannian manifold is called *curvature homogeneous* if the curvature tensors at any two points are orthogonally equivalent. A Riemannian manifold is said to be *modelled* on a given algebraic curvature tensor $T$ if its curvature tensor is, at each point, orthogonally equivalent to $T$. A curvature homogeneous semi-symmetric space is exactly a Riemannian manifold whose curvature tensor is modelled on the curvature tensor of a fixed symmetric space (see [BKV96] for more details).

## 2 A class of generalized warped products and their $\kappa$-nullity

In this section we present formulae for the curvature of a class of generalized warped products, with a view toward the description of two known examples of inhomogeneous manifolds with non-trivial $\kappa$-nullity. The calculation is straightforward and follows the lines of [GW09, Ch. 2], but it differs from that book in two respects: herein the function $\varphi$ does not need to be *basic* (that is, constant along the fibers), and the Riemannian submersion is assumed *integrable*, that is, its horizontal distribution is integrable.

Let $\pi : (M, \langle , \rangle) \to B$ be an integrable Riemannian submersion. Denote its Levi-Civita connection and curvature tensor by $\nabla$, $R$, respectively. For each smooth function $\varphi : M \to \mathbb{R}$, we define a new metric on $M$ by

$$\langle u, v\rangle_\varphi = \langle u^h, v^h\rangle + e^{2\varphi(p)}\langle u^v, v^v\rangle$$

for $u$, $v \in T_pM$, where the superscripts denote the horizontal and vertical components. Denote the associated Riemannian manifold by $M_\varphi$, and its Levi-Civita connection and curvature tensor by $\tilde\nabla$, $\tilde R$, respectively. Note that $\pi : M_\varphi \to B$ is still an integrable Riemannian submersion. Denote by $X$, $Y, \ldots$ horizontal vector fields, and by $U$, $V, \ldots$ vertical ones. The derivation of $\tilde\nabla$ in terms of $\nabla$ is a straightforward calculation using the Koszul formula:

$$\begin{aligned}
\tilde\nabla_X Y &= \nabla_X Y,\\
(\tilde\nabla_V X)^v &= (\nabla_V X)^v + X(\varphi)V,\\
(\tilde\nabla_V X)^h &= [V, X]^h \quad (= 0, \text{ if } X \text{ is basic}),\\
(\tilde\nabla_X V)^v &= (\nabla_X V)^v + X(\varphi)V\\
(\tilde\nabla_X V)^h &= 0,\\
(\tilde\nabla_U V)^h &= e^{2\varphi}\{(\nabla_U V)^h - \langle U, V\rangle(\nabla\varphi)^h\},\\
(\tilde\nabla_U V)^v &= (\nabla_U V)^v + U(\varphi)V + V(\varphi)U - \langle U, V\rangle(\nabla\varphi)^v,
\end{aligned} \tag{2.1}$$

where $\nabla\varphi$ denotes the gradient of $\varphi$ with respect to $\langle , \rangle$.

Regarding $\tilde R$, we need only the following formulae, whose derivation, which uses (2.1), we skip:

$$\begin{aligned}
\tilde{R}(X,Y) &= R(X,Y) \\
\tilde{R}(X,V)Y &= R(X,V)Y - X(\varphi)S_Y V - Y(\varphi)S_X V \\
&\qquad + \{\mathrm{Hess}_\varphi(X,Y) + X(\varphi)Y(\varphi)\}V \\
e^{-2\varphi}\tilde{R}^h(X,U)V &= R^h(X,U)V + X(\varphi)\sigma(U,V) + \langle X, \sigma(U,V)\rangle(\nabla\varphi)^h \\
&\qquad - \langle U,V\rangle\left\{(\nabla_X\nabla\varphi)^h + X(\varphi)(\nabla\varphi)^h\right\} \\
\tilde{R}^v(U,V)X &= R^v(U,V)X - U(\varphi)S_X V + V(\varphi)S_X U \\
&\qquad + \mathrm{Hess}_\varphi(U,X)V - \mathrm{Hess}_\varphi(V,X)U.
\end{aligned} \tag{2.2}$$

Here $S$ denotes the shape operators of the fibers, $\sigma$ denotes their second fundamental forms, $\mathrm{Hess}_\varphi$ denotes the Hessian of $\varphi$, and $(\mathrm{Hess}_\varphi)^h$ is its restriction to the horizontal distribution.

As an easy consequence of (2.2), we deduce the following result.

**Theorem 2.1** (Nullity of warping)**.** *Let $\pi : (M, \langle,\rangle) \to B$ be an integrable Riemannian submersion and consider $\varphi : M \to \mathbb{R}$.*

*(i) If $\pi$ has umbilic fibers (in particular, 1-dimensional fibers) and $\nabla\varphi$ is vertical, then the horizontal vectors in the $\kappa$-nullity of $M$ lie in the $\kappa$-nullity of $M_\varphi$ (here $\kappa$ is arbitrary).*

*(ii) If $S \equiv 0$ (so that $\pi$ splits, or $M$ is a local product) then the horizontal lifts of vectors in the 0-nullity of $B$, which in addition lie in* $\ker d\varphi \cap \ker(\mathrm{Hess}_\varphi)^h$*, belong to the 0-nullity of $M_\varphi$.*

# 3 Inhomogeneous examples of metrics with $(-1)$-conullity 2

In [TV89] Tricerri and Vanhecke quote a problem posed by Gromov, namely, whether isometry classes of germs of Riemannian metrics modelled on the curvature tensor of a given “irreducible” homogeneous Riemannian manifold depend on a finite number of parameters. Schmidt and Wolfson constructed in [SW15] complete metrics, continuous deformations of a certain left-invariant metric on the unimodular group $G = SL(2,\mathbb{R})$, which have constant $(-1)$-conullity 2 and constant scalar curvature $-2$; equivalently, these metrics are curvature homogeneous and are all modelled on the curvature tensor of the left-invariant metric on $SL(2,\mathbb{R})$. Hence they answered in the negative Tricerri and Vanhecke's question. In this section we discuss those examples in light of Theorem 2.1(i).

Start with left-invariant vector fields associated to the factors of the Iwasawa decomposition $G = NAK$, namely,

$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ T = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we have the bracket relations

$$[X,Y] = 2T, \ [T,X] = -X + 2Y, \ [T,Y] = Y. \tag{3.1}$$

Note that $Y$ and $T$ span an involutive distribution. Consider the left-invariant metric obtained by declaring the frame $X$, $Y$, $T$ orthonormal. Then one easily computes that

$$\begin{gathered} \nabla_T T = \nabla_T X = \nabla_T Y = 0, \ \nabla_X T = X - 2Y, \ \nabla_Y T = -Y, \\ \nabla_X X = -T, \ \nabla_Y Y = T, \ \nabla_X Y = 2T, \ \nabla_Y X = 0. \end{gathered} \tag{3.2}$$

We deduce that the distribution $\mathcal{D}$ spanned by $Y$ and $T$ is autoparallel, its leaves are isometric to real hyperbolic planes $\mathbb{R}H^2(-1)$ of curvature $-1$, and $G$ has $(-1)$-nullity distribution of rank 1, spanned by $T$. We note that the sectional curvature of the plane spanned by $X$, $Y$ is 1, which implies constancy of the scalar curvature of $G$, equal to $-2$.

Consider now the projection $\pi : G \to G/K = NA = \mathbb{R}H^2(-1)$. The vector field $X$ is Killing transversal, that is, its flow is isometric on $\mathcal{D}$ (this follows from (3.2)), so its flow lines induce a Riemannian foliation of $G$ and $\pi$ is the projection onto the space of leaves. In particular, $\pi$ is a Riemannian submersion, the horizontal distribution being spanned by $Y$, $T$ and thus integrable, and we can apply Theorem 2.1(i). For every smooth

$\varphi : G \to \mathbb{R}$ with a vertical gradient, $G_\varphi$ also has $(-1)$-nullity distribution of rank 1, spanned by $T$. It is easily seen that the scalar curvature of $G_\varphi$ is likewise $-2$ (this also follows from the last equation in (3.4)). Since $\nabla\varphi$ is vertical, $\varphi$ is actually a function on the compact group $K$ and hence bounded. This easily implies that every divergent curve in $G_\varphi$ has infinite length and hence $G_\varphi$ is complete as a Riemannian manifold. We remark that inhomogeneous Schmidt-Wolfson metrics do not cover a manifold of finite volume [SW14, Thm. 1.1].

How general are these examples? Namely, assume $M$ is a given Riemannian 3-manifold with minimal $(-1)$-nullity 1 and constant scalar curvature. Around any point of minimal nullity, we can find a local orthonormal frame $T$, $X$, $Y$, where $T$ spans the nullity distribution, satisfying (we refer to [GG22, subsection 3.3] for this construction, which is not difficult):

$$\begin{gathered}\nabla_T T = \nabla_T X = \nabla_T Y = 0,\ \nabla_X T = X - 2FY,\ \nabla_Y T = -Y,\\ \nabla_X X = -T + \alpha Y,\ \nabla_Y Y = T + \beta X,\ \nabla_X Y = 2FT - \alpha X,\ \nabla_Y X = -\beta Y.\end{gathered} \tag{3.3}$$

for some locally defined smooth functions $\alpha$, $\beta$, $F$. The bracket relations follow:

$$[X, Y] = 2FT - \alpha X + \beta Y,\ [T, X] = -X + 2FY,\ [T, Y] = Y.$$

Next, the curvature relations

$$\langle R(X, Y)X, Y\rangle = -k,$$

where $k$ is the sectional curvature of the plane spanned by $X$, $Y$, and

$$\langle R(X, Y)X, T\rangle = \langle R(T, Y)X, Y\rangle = \langle R(X, Y)Y, T\rangle = 0,$$

yield the equations

$$\begin{aligned} \alpha &= -\beta F,\\ T(\beta) &= \beta,\\ Y(F) &= -\beta(1 + F^2),\\ X(\beta) - FY(\beta) &= k - 1. \end{aligned} \tag{3.4}$$

Note that $Y$ and $T$ span an involutive distribution, with integral manifolds isometric to $\mathbb{R}H^2(-1)$. Further, in view of (3.3) these hyperbolic planes are totally geodesic if and only if $\beta = 0$ if and only if the foliation by integral curves of $X$ is Riemannian (indeed a *polar foliation*, see e.g. [Thr22]) if and only if the projection of $SL(2,\mathbb{R})$ onto the space of leaves is a Riemannian submersion; equivalently, $Y(F) = 0$. *Assume this is the case*; we work on the open set $U$ defined by $F \neq 0$, where we can take a new frame $\tilde{T} = T$, $\tilde{X} = F^{-1}X$, $\tilde{Y} = Y$. The new frame satisfies

$$[\tilde{X}, \tilde{Y}] = 2\tilde{T},\ [\tilde{T}, \tilde{X}] = -\tilde{X} + 2\tilde{Y},\ [\tilde{T}, \tilde{Y}] = \tilde{Y}.$$

Comparing with (3.1), by Lie's third theorem, the metric defined on $U$ that declares $\tilde{X}$, $\tilde{Y}$, $\tilde{T}$ orthonormal is locally isometric to a left-invariant metric on $SL(2,\mathbb{R})$. We deduce that the original metric on $U$ is locally isometric to a Schmidt-Wolfson metric on $SL(2,\mathbb{R})$, that is, a deformation as above.

**Question 3.1.** *Are there complete solutions to (3.4) with $\beta \neq 0$?*

**Question 3.2.** *Are there other examples of polar foliations of dimension one (or with umbilic leaves) on non-symmetric spaces, other than $SL(2,\mathbb{R})$?*

## 4 Inhomogeneous examples of metrics with 0-conullity 2

In 1968, Nomizu conjectured that every complete irreducible semi-symmetric space of dimension greater than or equal to three is locally symmetric. His conjecture was refuted by Takagi and Sekigawa in 1972, who constructed counterexamples. Here we are concerned with another two of Sekigawa's counterexamples [Sek75], later generalized by Kowalski, Tricerri and Vanhecke [KTV92] (these were later further generalized, see [BKV96] for the full range of generalizations).

We start with an almost Abelian Lie group (that is, a Lie group with a codimension one ideal) $G = S^1 \ltimes_\rho V$, where the representation $\rho : S^1 \to SO(V)$ is orthogonal. Its Lie algebra is $\mathfrak{g} = \mathbb{R} \ltimes_{d\rho} V$. We fix $\xi \in \mathbb{R}$ and consider $d\rho(\xi) \in \mathfrak{so}(V)$. We also equip $G$ with a left-invariant Riemannian metric that makes $\xi$ unit, and $\xi$ and $V$ orthogonal. Koszul's formula for the Levi-Civita connection immediately yields:

$$\nabla_\xi \xi = \nabla_X \xi = \nabla_X Y = 0, \ \nabla_\xi X = d\rho(\xi)X, \tag{4.1}$$

for all $X$, $Y \in V$. It immediately follows that the metric is flat.

Next we are going to deform the metric on $G$, as in Theorem 2.1(ii). The projection $\pi : G \to V$ is obviously an integrable Riemannian submersion. We need to construct the function $\varphi$. Let $Z \in V$ be a fixed unit vector, and consider the associated height function $h(X) = \langle X, Z\rangle$ for $X \in V$. Extend $h$ to a smooth function $t : G \to \mathbb{R}$ by setting $t(g) = h(\rho(e^{-i\theta})Y)$, for $g = (e^{i\theta}, Y) \in G$. Below we are going to use the following calculation: for any $X \in V$, we have

$$g \exp(sX) = (e^{i\theta}, Y)(1, sX) = (e^{i\theta}, s\rho(e^{i\theta})X + Y),$$

so

$$\begin{aligned} X_g(t) &= \frac{d}{ds}\Big|_{s=0} t(g \exp(sX)) \\ &= \frac{d}{ds}\Big|_{s=0} \langle sX + \rho(e^{-i\theta})Y, Z\rangle \\ &= \langle X, Z\rangle, \end{aligned}$$

and hence $(\nabla t)^h = Z$. Finally, define $\varphi = \log f$, where

$$f = c_1 e^{at} + c_2 e^{-at}$$

for constants $c_1$, $c_2 \geq 0$ and $a > 0$.

Back to Theorem 2.1(ii): of course $\pi$ splits, and the 0-nullity of the base $V$ is $V$ itself. We have

$$(\nabla\varphi)^h = \varphi_t (\nabla t)^h = \varphi_t Z,$$

so the horizontal part of $\ker d\varphi$ is the left-invariant, horizontal hyperplane distribution $Z^\perp$. For $X \in Z^\perp$ and $Y \in V$, we have

$$\begin{aligned}\mathrm{Hess}_\varphi(Y,X) &= YX(\varphi) - \nabla_Y X(\varphi) \\ &= YX(\varphi) \quad (\text{since } \nabla_Y X = 0) \\ &= 0 \quad (\text{since } X(\varphi) = 0),\end{aligned}$$

that is, $(\ker \mathrm{Hess}_\varphi)^h \supset \ker d\varphi$. It follows from the theorem that the 0-nullity distribution of $G_\varphi$ is $\mathcal{N}_0 = Z^\perp$. In particular, $G_\varphi$ has constant 0-conullity 2.

Consider the unit vector field $\tilde{\xi} = \frac{1}{f}\xi$ in the metric $\langle,\rangle_\varphi$. From formulae (2.2), we obtain:

$$\begin{aligned}\tilde{R}(X,Y) &= 0, \\ \tilde{R}(\tilde{\xi},X)Y &= -a^2\langle X,Z\rangle\langle Y,Z\rangle\tilde{\xi}, \\ \tilde{R}(\tilde{\xi},X)\tilde{\xi} &= a^2\langle X,Z\rangle Z,\end{aligned} \tag{4.2}$$

for $X$, $Y \in V$. From here we see that the sectional curvature $\tilde{K}(\tilde{\xi},Z) = -a^2$, and hence $G_\varphi$ has constant (negative) scalar curvature $-2a^2$. It follows that $G_\varphi$ is curvature homogeneous, semi-symmetric, and modelled on $\mathbb{R}H^2(-a^2) \times \mathbb{R}^{m-1}$, where $m = \dim V$.

Next we prove completeness of $G_\varphi$. A Riemannian manifold is complete if and only if every divergent curve has infinite length. Let $\gamma$ be a divergent curve in $G_\varphi$. Since $S^1$ is compact, also $\pi\circ\gamma$ is divergent, so by completeness of the base $V$ we have that the length $L(\gamma) \geq L(\pi\circ\gamma) = \infty$. This proves that $G_\varphi$ is complete.

Finally, we state a sufficient condition for $G_\varphi$ to be irreducible as a Riemannian manifold. Let $\hat{V} = \mathbb{R}\oplus V$ denote the tangent space to $G_\varphi$ at the basepoint. We are going to prove that if $Z$ is a cyclic vector for the representation $d\rho$, then the identity component of the holonomy group of $G_\varphi$ at the basepoint is the full rotation group $SO(\hat{V})$; in particular, this holonomy group acts irreducibly on $\hat{V}$.

It is convenient to use the standard identification $\bigwedge^2\hat{V} \cong \mathfrak{so}(\hat{V})$ given by $u\wedge v \mapsto \langle u,\cdot\rangle_\varphi v - \langle v,\cdot\rangle_\varphi u$. Then the Lie bracket $[v\wedge w, u\wedge v] = u\wedge w$, for $u$, $v$, $w \in \hat{V}$. Recall that $m = \dim V$, let $T = d\rho(\tilde{\xi})$, and denote by $\hat{V}_k$ the span of $\tilde{\xi}, Z, \ldots, T^k Z$, where $k = 0, \ldots, m-1$.

It is enough to prove that, under the assumption that $Z$ is a cyclic vector, the Lie algebra of the infinitesimal holonomy at the basepoint is $\mathfrak{so}(\hat{V})$; indeed we will show by induction on $k_0$ that the Lie subalgebra generated by

$$\{\tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, Z) \in \mathfrak{so}(\hat{V}) : k = 0, \ldots, k_0\}$$

contains $\mathfrak{so}(\hat{V}_{k_0})$, where $k_0 = 0, \ldots, m-1$. In turn, this result will follow from the following claims, where a decomposable element of $\bigwedge^2 \hat{V}$ is said to have order $< k$ if it only involves $\tilde{\xi}$, $Z, \ldots, T^{k-1} Z$:

Claim 1. $\tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, Z) = a^2(\tilde{\xi} \wedge T^k Z) +$ terms of order $< k$.

Claim 2. $\tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, X) =$ sum of terms of order $< k$, where $X \in Z^\perp$.

Claim 3. $\tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(X, Y) =$ sum of terms of order $< k$, where $X$, $Y \in V$.

We combine (2.1) with (4.1) to write (by the way, a left-invariant $X \in V$ is generally not basic with respect to $\pi$):

$$\begin{aligned} \tilde{\nabla}_X Y &= 0, \\ \tilde{\nabla}_{\tilde{\xi}} \tilde{\xi} &= -\varphi_t Z, \\ \tilde{\nabla}_{\tilde{\xi}} X &= TX + \varphi_t \langle X, Z \rangle \tilde{\xi}, \end{aligned} \tag{4.3}$$

where $X$, $Y \in V$. Also, from (4.2) we see that

$$\begin{aligned} \tilde{R}(\tilde{\xi}, Z) &= a^2 \tilde{\xi} \wedge Z, \\ \tilde{R}(\tilde{\xi}, X) &= 0, \quad \text{if } X \in Z^\perp, \\ \tilde{R}(X, Y) &= 0, \quad \text{for } X,\, Y \in V. \end{aligned}$$

This already gives the initial case $k = 0$ in claims 1 through 3; their proof now follows from a straightforward induction on $k$, using (4.3) in

$$\begin{aligned} \tilde{\nabla}^{k+1}_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, Z) &= \tilde{\nabla}_{\tilde{\xi}} (\tilde{\nabla}^k_{\tilde{\xi}} \tilde{R})(\tilde{\xi}, Z) \\ &= \tilde{\nabla}_{\tilde{\xi}} (\tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, Z)) - \tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(\tilde{\nabla}_{\tilde{\xi}} \tilde{\xi}, Z) - \tilde{\nabla}^k_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, \tilde{\nabla}_{\tilde{\xi}} Z), \end{aligned}$$

and in similar identities for $\tilde{\nabla}^{k+1}_{\tilde{\xi}} \tilde{R}(\tilde{\xi}, X)$ (where $X \in Z^\perp$) and $\tilde{\nabla}^{k+1}_{\tilde{\xi}} \tilde{R}(X, Y)$ (where $X$, $Y \in V$).

This completes the proof of irreducibility of $G_\varphi$ under the assumption that $Z$ is a cyclic vector, and the construction of the examples. We finish with two remarks: first, it is elementary to see that cyclic vectors exist in $V$ if and only if $\rho$ is multiplicity free (that is, it has pairwise different weights as a representation of $S^1$), and in such a case, the cyclic vectors are exacly those vectors with nonzero components in all irreducible components; and second, in [KTV92] it was further shown that cyclicity of the vector $Z$ is also a necessary condition for the irreducibility of $G_\varphi$.

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