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Fourth-order superlinear elliptic problems interacting with high eigenvalues

T.R.Cavalcante [1] and **E.D. Silva** [2]

[1]Universidade Federal do Tocantins-UFT,Tocantins-TO, 77330-000, Brazil
[2]Universidade Federal de Goiás - UFG,Goiás-GO, 74001-970, Brazil

Abstract. It is established existence of weak solutions for a class of superlinear elliptic involving a fourth-order elliptic problem under Navier conditions on the boundary. Here we do not apply the well known Ambrosetti-Rabinowitz condition at infinity. Instead of we assume that the nonlinear term is a nonlinear function satisfying the well known nonquadraticity condition at infinity. Using a Local Linking Theorem we get our main results without any restrictions on the first eigenvalue for the linear problem. Namely, the first eigenvalue can be negative or positive. Furthermore, we consider nonlinear terms interacting at high eigenvalues.

Keywords: Fourth-order elliptic problems, Local Linking Theorem, Superlinear elliptic problems, Nonquadraticity condition.

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1 Introduction

In this work we consider the fourth-order elliptic problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta^2 = \Delta \circ \Delta$ is the biharmonic operator, $N \geq 4, \Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\alpha > 0, \beta \in \mathbb{R}$. The problem (1.1) is named fourth-order elliptic problem under Navier boundary conditions. Throughout this work, λ_1 denotes the first eigenvalue for the linear eigenvalue problem associated to Laplacian operator. The nonlinear term f is a continuous function which is superlinear at infinity and at the origin. Later on, we shall consider the assumptions on the nonlinear term f.

Semilinear elliptic problems involving operators of fourth-order have been considered, see [20, 21, 15, 5]. In these works were modeled nonlinear oscillations for suspensions bridges. It is worthwhile to mention that problem (1.1) models static deflection of an elastic plate in a fluid. The same problems can be used to describe the static form change of beam or the motion of rigid body. Notice that this class of fourth-order elliptic problems have been extensively studied during the last years, see for instance [3, 6, 9, 12, 22, 28] and references therein. In these papers existence and multiplicity of solutions have been considered using some assumptions on the nonlinear term f. Most of them considered the case $f(x,t) = b[(t+1)^+ - 1]$ or f satisfying the well known Ambrosetti-Rabinowitz condition. From a mathematical point of view there exist some nonlinear functions such that the Ambrosetti-Rabinowitz does not work anymore. For example we cite $f(t) = tln(1+|t|), t \in \mathbb{R}$.

The main feature in this work is to consider fourth-order elliptic problems without the Ambrosetti-Rabinowitz condition introduced in [1]. The main difficulty arises from the fact that Palais-Smale sequences are not necessary bounded under our assumptions. In order to overcome this difficulty we apply the nonquadraticity condition introduced by Costa-Magalhães [7] proving that any Cerami sequences are necessary bounded, see Section 2 ahead. It is important to emphasize that compactness results such as Cerami condition is a powerful tool in order to use variational methods. At the same time, we consider problem (1.1) assuming that the first eigenvalue for the linear eigenvalue problem is negative or positive. Under these conditions the left side in equation (1.1) can be positive or negative which does not provide a norm in the Sobolev space. Using a local linking theorem and taking into account the linear eigenvalue problem for (1.1) we prove an existence result without any restrictions on the sign for the first eigenvalue. Later on, we shall give the conditions for the function f and some technical conditions on the linear eigenvalue problem for the problem (1.1) which allow us to consider the nonlinear term f interacting at high eigenvalues.

1.1 Assumptions and main theorems

As was mentioned before we shall consider our main problem (1.1) under some extra assumptions on α, β and f. Throughout this work, (λ_i) denotes a sequence of eigenvalues for the linear problem associated to the Laplacian operator. Consider the eigenvalue problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = \mu u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is easy to verify that $\mu \in \mathbb{R}$ is an eigenvalue for the problem (1.2) if only if $\mu_i = \lambda_i (\alpha\lambda_i - \beta), i \in \mathbb{N}$. Furthermore, the eigenfunction for the eigenvalue problem (1.2) denoted by (φ_k) which are the eigenfunctions for the linear eigenvalue problem associated to Laplacian operator. Here we observe that there exist at most a finite number of negative eigenvalues for the linear problem (1.2).

It is important to mention that fourth-order elliptic problems are modeled in the functional space $\mathcal{H} = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Throughout this work we consider the following norm

$$\|u\| = \sqrt{\int |\Delta u|^2 dx}, u \in \mathcal{H} \tag{1.3}$$

and the inner product given by

$$\langle u, v\rangle = \int_\Omega \Delta u \Delta v dx, u, v \in \mathcal{H}.$$

Notice that the operator $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ given by

$$B(u, v) = \int_\Omega (\alpha \Delta u \Delta v - \beta \nabla u \nabla v)\, dx, u, v \in \mathcal{H}, \tag{1.4}$$

is bilinear. Moreover, the map $u \to B(u, u)$ is negative definite in some subspace $\mathcal{H}^1$ and positive definite in another subspace $\mathcal{H}^2$. It is important to emphasize that $\mathcal{H} = \mathcal{H}^1 \oplus \mathcal{H}^2$ where $\mathcal{H}^1 = \langle \varphi_1, \dots, \varphi_k\rangle$ and $\mathcal{H}^2 = \langle \varphi_{k+1}, \dots\rangle$ for some $k \in \mathbb{N}$. Using the usual norm given in (1.3) we know that $\mathcal{H}^1$ and $\mathcal{H}^2$ are orthogonal.

The weak solutions for problem (1.1) are precisely the critical points for the functional of C^1 class $I : \mathcal{H} \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} B(u, u) - \int_\Omega F(x, u)\, dx, \tag{1.5}$$

where the primitive for f is denoted by $F(x, t) = \int_0^t f(x, s) ds$, $x \in \Omega,\ t \in \mathbb{R}$. More specifically, given $u \in \mathcal{H}$, we mention that

$$I'(u)v = B(u, v) - \int_\Omega f(x, u) v dx,\ u, v \in \mathcal{H}$$

where $I'(u)v$ is standard for the duality product between $\mathcal{H}$ and $\mathcal{H}'$. Furthermore, $u \in \mathcal{H}$ is a critical point for I if and only if u is a weak solution to the elliptic problem (1.1).

In this work we shall consider that $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Moreover, we assume the following hypotheses:

(f_0) There exist $a_1 > 0$ and $p \in (2, 2_*)$ such that

$$|f(x, t)| \le a_1(1 + |t|^{p-1}), \quad \text{for any} \quad (x, t) \in \Omega \times \mathbb{R}$$

where $2_* = \dfrac{2N}{N-4}$ for each $N \ge 5$ and $2_* = \infty$ for $N = 4$.

(f_1) It holds

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t} = +\infty$$

uniformly in Ω.

(f_2) There exist $k \in \mathbb{N}$ and $f_0 \in (\mu_k, \mu_{k+1})$ such that

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = f_0 \quad \text{uniformly in} \quad \Omega.$$

Here we emphasize that μ_k and μ_{k+1} are two consecutive eigenvalues for the problem (1.2). Under these conditions problem (1.1) is superlinear at infinity and asymptotically linear at the origin.

It is worthwhile to mention that hypothesis (f_2) implies that $f(x,0) = 0, \ \forall \ x \in \Omega$. As a consequence $u \equiv 0$ is a trivial solution to the elliptic problem (1.1). Hence, taking into account hypotheses $(f_0) - (f_2)$, the main objective in the present work is to verify the existence of nontrivial solutions for (1.1). It is also important to mention that $2_* = 2N/(N-4)$ is the critical Sobolev exponent for fourth-order elliptic equations for any $N \geq 4$. More precisely, we have that $\mathcal{H}$ is continuously embedded into $L^s(\Omega)$ for any $s \in [1, 2_*]$ where $N \geq 5$. For the case $N = 4$ we have that $\mathcal{H}$ is not included in $L^\infty(\Omega)$. For this case we consider $p \in [1, 2_*)$ in the following form $1 \leq p < \infty$ proving also that $\mathcal{H}$ is continuously embedded in $L^s(\Omega)$ for every $p \in [1, 2_\star)$. Notice also that the embedding $\mathcal{H} \subset L^s(\Omega)$ is compact for any $s \in [1, 2_*)$ whenever $N \geq 4$.

In [22, 24] the authors considered the fourth-order elliptic problem where the nonlinear term is a function that satisfies the well known Ambrosetti-Rabinowitz condition, in short (AR) condition, says that: There are $\theta > 2$ and $R > 0$ in way such that

$$0 < \theta F(x,t) \leq t f(x,t), |t| \geq R, x \in \Omega.$$

It is not hard to verify that (AR) condition implies that

$$F(x,t) \geq C_1 |t|^\theta - C_2, t \in \mathbb{R}, x \in \Omega \tag{1.6}$$

holds true for some $C_1, C_2 > 0$. However, there exists superlinear functions f where (1.6) is not verified as was quoted in the introduction.

The main novel in this work is to find existence of solutions for fourth-order elliptic problems given by problem (1.1) where the nonlinear term is nonquadratic at infinity. As were mentioned before we do not require that f satisfies the (AR) condition. For further results on elliptic problem without the (AR) we infer the reader to [8, 13, 14, 16, 23, 26, 27] and references therein. More precisely, we consider the nonquadraticity condition introduced in [7] as follows:

(NQ) setting $H(x,t) := f(x,t)t - 2F(x,t)$, we have that

$$\lim_{|t|\to\infty} H(x,t) = +\infty, \quad \text{uniformly for } x \in \Omega.$$

At this stage, we define $I_k := (\mu_k, \mu_{k+1})$ and $\overline{I}_k := [\mu_k, \mu_{k+1}]$ for some $k \geq 1$ where the eigenvalues μ_k, μ_{k+1} are obtained from hypothesis (f_2). Now we are stay in position in order to consider the Local Linking Theorem, under the Cerami condition, writing our main result in the following form:

Theorem 1.1. *Suppose that f satisfies $(f_0), (f_1), (f_2)$ and (NQ). Assume also that one of the following conditions*

i) the first eigenvalue μ_1 is positive;

ii) the first eigenvalue μ_1 is negative and $0 \in I_k$;

iii) the first eigenvalue μ_1 is negative and $0 \notin \overline{I}_k$.

Then problem (1.1) admits at least one nontrivial solution.

Remark 1.2. Here we assume that $N = 1, 2, 3$ and $p \in (1, \infty)$ where the function f is subcritical according to hypothesis (f_0). Using assumptions $(f_0) - (f_2)$ the energy functional I admits also the local linking geometry introduced in [17]. Furthermore, using hypothesis (NQ), we also mention that I verifies the Cerami condition. Hence our main results remains true in this setting.

Remark 1.3. Let $k \in \mathbb{N}$ be fixed. Consider also the following elliptic problem

$$\begin{cases} \alpha\Delta^{2k}u + \displaystyle\sum_{j=1}^{k-1} \beta_j \Delta^{2j} u + \beta\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = \ldots = \Delta^k u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\alpha > 0$ and $\beta, \beta_j \in \mathbb{R}$ with $j = 1, \ldots, k-1$. Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 1$. The nonlinearity is a subcritical continuous functions which is asymptotically linear at the origin and superlinear at infinity. The problem (1.7) is named polyharmonic elliptic problem under Navier conditions at the boundary, see [9, 10]. The functional space for this case is defined by

$$\mathcal{H}_k = \left\{ u \in H^k(\Omega) : u = \Delta u = \ldots = \Delta^{(k-1)} u = 0 \text{ on } \partial\Omega \right\}.$$

The energy functional $I : \mathcal{H}_k \to \mathbb{R}$ is given by

$$I(u) = \frac{1}{2}\int_\Omega \left(\alpha|\Delta^k u|^2 + \sum_{j=1}^{k-1} \beta_j |\Delta^j u|^2 - \beta|\nabla u|^2 \right) dx - \int_\Omega F(x,u)dx$$

where $u \in \mathcal{H}_k$ and $F(x,t) = \int_0^t f(x,s)ds, x \in \Omega, t \in \mathbb{R}$. After some minor modifications we emphasize that our main results remains true for the elliptic problem (1.7).

Remark 1.4. Notice that our main results holds true also for Dirichlet conditions at the boundary. More specifically, we consider the elliptic problem

$$\begin{cases} \alpha\Delta^2 u + \beta\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0, \ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\dfrac{\partial u}{\partial \eta}$ denotes the normal derivative on the boundary. Here we also have $\alpha > 0, \beta \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded smooth domain. For this problem the energy functional $I : H_0^2(\Omega) \to \mathbb{R}$ is given by

$$I(u) = \frac{1}{2}\int_\Omega \left(\alpha|\Delta u|^2 - \beta|\nabla u|^2\right) dx - \int_\Omega F(x,u)dx, \ u \in H_0^2(\Omega).$$

Here the first eigenvalue $\mu_1 > 0$ can be characterized by

$$\mu_1 = \inf \left\{ \int_\Omega \left(\alpha|\Delta u|^2 - \beta|\nabla u|^2\right) dx, \|u\|_2 = 1, u \in H_0^2(\Omega) \right\}.$$

For more details on this subject we refer the reader to [9, 10]. Furthermore, we can also consider different boundary conditions such as Steklov conditions at the boundary. As a consequence, we can prove existence of solutions for fourth-order elliptic problems under several boundary conditions. The main novel here concerns on the fact that the first eigenvalue can be negative or positive. At the same time, we can also consider the problem (1.7) under Dirichlet boundary conditions or Steklov conditions proving existence of solutions via the Local Linking Theorem where the first eigenvalue for the linear eigenvalue problem is negative or positive.

It is important to worthwhile that fourth-order elliptic problem have been widely studied in the last years. In [3] were considered concave-convex nonlinearities with a critical term where the authors were obtained existence and multiplicity of solutions. In [6] the authors studied fourth-order elliptic problems in $\mathbb{R}^N$ where the nonlinear term is critical. In [9] were studied a fourth-order ordinary differential equation which blows up in finite time with wide oscillations. In [18] were explored existence of solutions for a fourth-order biharmonic problem involving the $p(x)$-biharmonic with singular weights. In [22] were studied problem (1.1) where the nonlinear term f is a powerlike function or f satisfies the well known Ambrosetti-Rabinowitz condition. The problem (1.1) were also considered in [24] assuming that $\alpha = 1, \beta < \lambda_1$ and f superlinear or asymptotically linear at infinity and at the origin. Furthermore, the authors in [24] considered the following hypothesis:

$$\liminf_{|t|\to\infty} \frac{tf(x,t) - 2F(x,t)}{|t|^\sigma} \geq a > 0, t \in \mathbb{R}, x \in \Omega \tag{1.8}$$

uniformly in $x \in \Omega$ for some $\sigma > \frac{N}{4}(p-1)$ where $a > 0$ is a suitable constant. Here we mention also that hypothesis in the spirt of (1.8) was introduced by Costa-Magalhães in [7]. Under these conditions in [24] were proved several results concerns on existence and multiplicity of solutions for the biharmonic problem. It is worthwhile to mention that hypothesis (1.8) is usually changed by the following assumption:

$$t \to \frac{f(t)}{|t|} \text{ is an increasing function for each } t \neq 0. \tag{1.9}$$

It is important to mention that hypothesis (1.9) is crucial in order to ensure that any Palais-Smale sequence is bounded, see for instance [23]. This assumption is related also to find existence of ground state solutions where is usually used the Nehari method. For further results on this subject we refer the reader to [14, 29]. However, there are several superlinear functions f in such way that (1.9) does not hold.

In the present work the main feature is to consider the extremal case putting $\sigma = 0$ in assertion (1.8). Hence, applying hypothesis (NQ) which is weaker than (1.8), we get our main result without any kind of monotonicity or homogeneity in the nonlinear term f. Furthermore, we consider the case where the monotonicity condition given in (1.9) is not verified. Hence our results extend and complement the early results aforementioned. As an example for our setting we consider the following functions

a) $f_1(t) = \mu t + tln(1 + |t|) + \sin t,\ t \in \mathbb{R}$ where $\mu \in (\mu_k - 1, \mu_{k+1} - 1)$,

b) $f_2(t) = \mu t + tln(1 + |t|) + at\sin t,\ t \in \mathbb{R}$ where we take $a \in (0, 1/2)$ and $\mu \in (\mu_k, \mu_{k+1}), k \in \mathbb{N}$.

The eigenvalues μ_k, μ_{k+1} in the last examples can be positive or negative. Here we mention that f_1, f_2 does not satisfy neither (AR) condition nor assumption (1.9). In fact, the functions just above does not verify estimate give in (1.6). Moreover, taking into account the periodic term for each function given just above, we observe that

$$\frac{d}{dt}\left[\frac{f_1(t)}{t}\right] \text{ and } \frac{d}{dt}\left[\frac{f_2(t)}{t}\right]$$

are sign changing functions. Furthermore, the functions f_1, f_2 satisfy our hypotheses $(f_0) - (f_2)$ and (NQ).

The paper is organized as follows: In Section 2 we give the variational framework to the elliptic problem (1.1) proving the compactness condition required in variational methods. More specifically, we shall prove that the functional I satisfies the Cerami condition. In Section 3 we shall prove that I admits the local linking at the origin under several conditions on the first eigenvalue μ_1 for the linear problem (1.2). Section 4 is devoted to the proof of our main result. In the Appendix we prove that any critical point u for the functional I in the space $\mathcal{H} = H_0^1(\Omega) \cap H^2(\Omega)$ satisfies $u = \Delta u = 0$ on $\partial\Omega$.

Throughout this work $C, C_1, C_2, \ldots$ denote positive constants. The norm in $L^p(\Omega)$ is denoted by $\|\|_p$ for each $p \in [1, \infty)$. The norm in $\mathcal{H}$ is denoted by $\|\|$. The eigenfunctions for the linear problem (1.2) are denoted by φ_k which are precisely the eigenfunctions for the linear eigenvalue problem for the Laplacian operator. These functions are normalized in the $\mathcal{H}$ norm, i.e, we put $\|\varphi_k\| = 1$ for any $k \in \mathbb{N}$. The interval $I_k = (\mu_k, \mu_{k+1}) \subset \mathbb{R}$ comes from the hypothesis (f_2). The set $\overline{I}_k$ denotes the closure set for I_k, i.e, $\overline{I}_k = [\mu_k, \mu_{k+1}]$.

2 The Cerami Condition

In this section we give the variational framework to the elliptic problem (1.1) proving the compactness condition which is essential in variational methods. Namely, we shall prove that the Cerami condition holds which is a powerful tool. Let X be a Banach space and $I : X \to \mathbb{R}$ a functional of C^1 class. Recall that a sequence $(u_n) \in X$ is said to be a Cerami sequence at the level $c \in \mathbb{R}$, in short $(Ce)_c$ sequence, whenever $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\|_{X'} \to 0$ as $n \to \infty$. The functional I satisfies the Cerami condition at the level $c \in \mathbb{R}$, in short $(Ce)_c$ condition, whenever any Cerami sequence at the level c possesses a convergent subsequence. Furthermore, when I satisfies the Cerami condition at any level $c \in \mathbb{R}$ we say, in short, that I satisfies the Cerami condition. The Cerami condition was introduced in [4] and used in a more general setting in [2]. On this subject we refer the interested reader to [19].

In the next result we shall prove that the energy functional I given in (1.5) satisfies the (Ce) condition. This can be done using the fact that f is nonquadratic at infinity thanks to hypothesis (NQ).

Proposition 2.1. *Suppose that f satisfies $(f_0), (f_2)$ and (NQ). Then the functional I satisfies the Cerami condition at any level $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset \mathcal{H}$ be a sequence in such way that

$$I(u_n) \to c, \qquad \|I'(u_n)\|_{\mathcal{H}'}(1 + \|u_n\|) \to 0 \text{ as } n \to \infty,$$

where $c \in \mathbb{R}$. Since f has subcritical growth we need to prove only that (u_n) is a bounded sequence.

The proof follows arguing by contradiction. Suppose that, up to a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. Setting $v_n := u_n/\|u_n\|$ we obtain $||v_n|| = 1$ and there exits $v \in \mathcal{H}$ in such way that $v_n \rightharpoonup v$ in $\mathcal{H}$. In this case, along a subsequence, we obtain

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } \mathcal{H}, \\ v_n \to v \text{ strongly in } L^q(\Omega), \\ v_n(x) \to v(x), \text{ a. e. in } \Omega, \\ |v_n(x)| \leq h(x),\, h \in L^q(\Omega), \end{cases} \tag{2.1}$$

holds true for any $1 \leq q < 2_*$.

At this stage we claim that $v \neq 0$. The proof for this claim follows arguing by contradiction assuming that $v \equiv 0$. It follows from (f_0) that

$$|F(x,u)| \leq C|u|^2 + C|u|^p, \ \forall\ (x,u) \in \ \Omega \times \mathbb{R} \tag{2.2}$$

holds for some $C > 0$. Fix $m > 0$ any arbitrary constant. Taking into account (2.2) we see that

$$\int_\Omega |F(x, \sqrt{4m} v_n)|\, dx \leq C \int_\Omega |v_n|^p\, dx + C \int_\Omega |v_n|^2\, dx$$

holds for some $C = C(\Omega, f, m, p) > 0$. Using the strong convergence in (2.1) we know that

$$\int_\Omega F(x, \sqrt{4m} v_n) \to 0$$

as $n \to +\infty$ for any fixed $m > 0$.

Using the generality of constant m, for any $n \in \mathbb{N}$ large enough, we assume that $\sqrt{4m} < \|u_n\|$. Let $t_n \in [0,1]$ be in such way that

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n). \tag{2.3}$$

Using the definition of t_n in (2.3) and choosing $t = \sqrt{4m}/\|u_n\|$ we infer that

$$I(t_n u_n) \geq I\left(\frac{\sqrt{4m}}{||u_n||} u_n\right) = 2m - \int_\Omega F(x, \sqrt{4m} v_n)\, dx \geq m \tag{2.4}$$

holds true for any $n \in \mathbb{N}$ large enough. It is important to emphasize that if $t_n = 0$ holds for any $n \in \mathbb{N}$ we obtain $0 = I(t_n u_n) \geq m > 0$ which is a contradiction. Furthermore, assuming that $t_n = 1$ holds for any $n \in \mathbb{N}$ we also obtain $m \leq I(t_n u_n) = I(u_n) \to c$ which does not make sense for any $m > c$. Hence, up to a subsequence, we suppose that $t_n \in (0, 1)$.

Now we shall divide the proof in two steps. In the first one, up to a subsequence, we suppose that $t_n < (2/\|u_n\|)$. Notice that

$$H(x,t) = t f(x,t) - 2F(x,t), (x,t) \in \Omega \times \mathbb{R} \tag{2.5}$$

and $F(x,t) = \int_0^t f(x,s) ds, (x,t) \in \Omega \times \mathbb{R}$. In this case we use hypothesis (f_0) and Sobolev embedding to obtain c_1, $c_2 > 0$ verifying the following estimates

$$\begin{aligned} \left| \int_\Omega H(x, t_n u_n)\, dx \right| &\leq \int_\Omega |H(x, t_n u_n)|\, dx \leq c_1 (t_n \|u_n\|)^2 + c_2 (t_n \|u_n\|)^p \\ &\leq 4c_1 + c_2 2^p < \infty. \end{aligned} \tag{2.6}$$

Using the fact that $t_n \in (0,1)$ it follows from the identity $I'(t_n u_n)(t_n u_n) = 0$ that

$$0 = t_n^2 \|u_n\|^2 - \int_\Omega f(x, t_n u_n)(t_n u_n)\, dx = 2I(t_n u_n) - \int_\Omega H(x, t_n u_n)\, dx.$$

As a consequence, by using (2.6), we obtain

$$I(t_n u_n) = \frac{1}{2}\int_\Omega H(x, t_n u_n)\, dx \leq c_3$$

where $c_3 > 0$. This give us an absurd with (2.4) due the fact that $m > 0$ in that expression is arbitrary.

It remains to focus in the second step assuming that $t_n \geq (2/\|u_n\|)$. Consider $s_n := \frac{1}{\|u_n\|} < t_n$. Using the energy functional I we obtain the following identity

$$\frac{I(t_n u_n)}{t_n^2 ||u_n||^2} - \frac{I(s_n u_n)}{s_n^2 ||u_n||^2} = -\frac{1}{||u_n||^2}\int_\Omega \left(\frac{F(x, t_n u_n)}{t_n^2} - \frac{F(x, s_n u_n)}{s_n^2}\right) dx.$$

At this moment, we mention also that

$$\frac{I(t_n u_n)}{t_n^2 \|u_n\|^2} - \frac{I(s_n u_n)}{s_n^2 \|u_n\|^2} \quad = \quad -\int_\Omega \int_{s_n}^{t_n} \frac{d}{d\tau}\left(\frac{F(x, \tau u_n)}{\tau^2 \|u_n\|^2}\right) \mathrm{d}\tau\, \mathrm{d}x.$$

Now we shall consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) = e^{-1/t^2}, t \neq 0$ and $\phi(0) = 0$. It is not hard to see that $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ and $0 \leq \phi(t) \leq 1$ for any $t \in \mathbb{R}$. Furthermore, we observe that $\phi^{(j)}(0) = 0$ for any $j \in \mathbb{N}$. Using the nonquadraticity condition (NQ) we observe that

$$H(x, t) \geq R\phi(t), \quad \forall |t| \geq M, \ x \in \Omega \tag{2.7}$$

is satisfied for any $R > 0$ where $M = M(R) > 0$. On the other hand, by using hypothesis (f_0), we mention that

$$|H(x, t)| \leq C|t|, \ \forall \ \delta \leq |t| \leq M, \ x \in \Omega \tag{2.8}$$

holds for each $\delta > 0$ and some positive constant $C > 0$. Here was used the fact that

$$\lim_{t \to 0} \frac{H(x, t)}{|t|} = 0$$

Using the last assertion together with (2.7) and (2.8) we deduce that there exists $C > 0$ such that

$$H(x, t) \geq R\phi(t) - C|t|, \forall\, (x, t) \in \ \Omega \times \mathbb{R}. \tag{2.9}$$

It is not hard to see that

$$\frac{I(t_n u_n)}{t_n^2||u_n||^2} - \frac{I(s_n u_n)}{s_n^2||u_n||^2} = -\frac{1}{||u_n||^2}\int_\Omega \left(\frac{F(x,t_n u_n)}{t_n^2} - \frac{F(x,s_n u_n)}{s_n^2}\right) dx$$

In view of the Fundamental Theorem of Calculus we infer that

$$\begin{aligned}\frac{I(t_n u_n)}{t_n^2\|u_n\|^2} - \frac{I(s_n u_n)}{s_n^2\|u_n\|^2} &= -\int_\Omega \int_{s_n}^{t_n} \frac{d}{d\tau}\left(\frac{F(x,\tau u_n)}{\tau^2\|u_n\|^2}\right) \mathrm{d}\tau\,\mathrm{d}x \\ &= -\int_\Omega \int_{s_n}^{t_n} \frac{H(x,\tau u_n)}{\tau^3\|u_n\|^2}\,\mathrm{d}\tau\,\mathrm{d}x.\end{aligned}$$

Hence, by using the last assertion and (2.9), we obtain the following estimate

$$\frac{I(t_n u_n)}{t_n^2||u_n||^2} - \frac{I(s_n u_n)}{s_n^2||u_n||^2} \le I_1 + I_2, \tag{2.10}$$

where I_1 and I_2 are defined by

$$I_1 := -\frac{R}{||u_n||^2}\int_\Omega \left[\int_{s_n}^{t_n} \frac{\phi(\tau u_n)}{\tau^3} d\tau\right] dx$$

and

$$I_2 = \frac{C}{||u_n||^2}\int_\Omega \left[\int_{s_n}^{t_n} \frac{|\tau u_n|}{\tau^3} d\tau\right] dx.$$

Now, looking for the integral I_2, we observe that

$$I_2 \le C\int_\Omega \left(\frac{1}{s_n} - \frac{1}{t_n}\right)\frac{|u_n|}{||u_n||^2}\,dx.$$

Now, by using the fact that $s_n = 1/\|u_n\|$, taking into account that $v_n \rightharpoonup 0$ in $\mathcal{H}$ the integral I_2 is estimated as follows

$$I_2 \le C\int_\Omega |v_n|\,dx \to 0 \text{ as } n \to \infty.$$

It remains to estimate the integral I_1. Notice that $t \mapsto \phi(t)$ is an increasing function. According to Mean Value Theorem for integrals [25, Theorem 7.2] there exists $c_n \in (s_n, t_n)$ in such way that

$$\int_{s_n}^{t_n} \frac{\phi(\tau u_n)}{\tau^3}\,d\tau = \phi(c_n u_n)\int_{s_n}^{t_n} \frac{1}{\tau^3}\,d\tau = -\phi(c_n u_n)\left(1/2t_n^2 - 1/2s_n^2\right).$$

As a consequence, we obtain

$$I_1 \leq \frac{R}{2||u_n||^2}\left(\frac{1}{t_n^2} - \frac{1}{s_n^2}\right)\int_\Omega \phi(c_n u_n)\,dx.$$

The last identity implies that

$$I_1 \leq -\frac{3R}{8}\int_\Omega \phi(c_n u_n)\,dx. \tag{2.11}$$

Now we define the following set

$$\Omega_{n,m} = \left\{x \in \Omega : |(c_n u_n)(x)| \geq \frac{1}{m}\right\}.$$

The strategy here is to find a subsequence $(u_{n_k}) \in \mathcal{H}$ in such way that

$$|\Omega_{n_k,m}| \geq \delta_0 > 0, \ \forall\, k \in \mathbb{N} \tag{2.12}$$

holds true for some $\delta_0 > 0$ and for some $m \in \mathbb{N}$ fixed. Hence, by using (2.11), we deduce that

$$I_1 \leq -\frac{3R}{8}\int_{\Omega_{n_k,m}} e^{-m^2}\,dx \leq -\frac{3R}{8}\delta_0 e^{-m^2} < 0.$$

According to (2.10) and the last estimate we deduce that

$$\frac{I(t_n u_n)}{t_n^2||u_n||^2} \leq \frac{I(s_n u_n)}{s_n^2||u_n||^2} + o_n(1) - \frac{3R}{8}\delta_0 e^{-m^2}.$$

Since $v_n \rightharpoonup 0$ in $\mathcal{H}$ the last inequality implies that

$$\frac{I(t_n u_n)}{t_n^2||u_n||^2} \leq \frac{1}{2}\|v_n\|^2 + o_n(1) - \frac{3R}{8}\delta_0 e^{-m^2}.$$

As a consequence, for each $R > 4e^{m^2}/(3\delta_0)$, we obtain that

$$\limsup_{n\to\infty} \frac{I(t_n u_n)}{t_n^2||u_n||^2} \leq \frac{1}{2} - \frac{3R}{8}\delta_0 e^{-m^2} < 0.$$

This is a contradiction with (2.4) due the fact that $m > 0$ is arbitrary.

Now we proceed for the proof of inequality (2.12). Suppose by contradiction that for any subsequence $(u_{n_k}) \in \mathcal{H}$ and for each $\delta > 0$ and for each $m > 0$ we have $|\Omega_{n_k,m}| \leq \delta$. Recall also that $\Omega_{n_k,m} \subset \Omega_{n_k,m+1}$ holds for each $m \in \mathbb{N}$. Under these conditions we know that $|\Omega_{n_k,m}| = 0$ for any $k, m \in \mathbb{N}$. Consider the measurable set $\Omega_k = \bigcup_{m=1}^{\infty} \Omega_{n_k,m}$. It is easy to verify that

$$|\Omega_k| = \left| \bigcup_{m=1}^{\infty} \Omega_{n_k,m} \right| \leq \sum_{m=1}^{\infty} |\Omega_{n_k,m}| = 0. \tag{2.13}$$

Note also that $\Omega = \Omega_k \dot{\cup} (\Omega \setminus \Omega_k)$. It is important to emphasize that

$$I(t_{n_k} u_{n_k}) = \frac{1}{2} \int_{\Omega} H(x, t_{n_k} u_{n_k})\, dx + o_k(1)$$

Using the previous identity and (2.13) we see that

$$\begin{aligned} I(t_{n_k} u_{n_k}) &= \frac{1}{2} \left(\int_{\Omega_k} H(x, t_{n_k} u_{n_k})\, dx + \int_{\Omega \setminus \Omega_k} H(x, t_{n_k} u_{n_k})\, dx \right) + o_k(1) \\ &= \frac{1}{2} \int_{\Omega \setminus \Omega_k} H(x, t_{n_k} u_{n_k})\, dx + o_k(1) \end{aligned} \tag{2.14}$$

Analyzing the elements in the set $\Omega \setminus \Omega_k$ and using the Morgan's Law we get $\Omega \setminus \Omega_k = \{x \in \Omega; |(c_{n_k} u_{n_k})(x)| = 0\}, k \in \mathbb{N}$. Recall also that $0 < s_{n_k} = 1/\|u_{n_k}\| \leq c_{n_k} \leq t_{n_k}$. In particular, we obtain that $\Omega \setminus \Omega_k = \{x \in \Omega; |u_{n_k}(x)| = 0\}, k \in \mathbb{N}$. From now on, using the estimate given in (2.14) and the last assertion just above, we mention that

$$I(t_{n_k} u_{n_k}) = \frac{1}{2} \int_{\Omega \setminus \Omega_k} H(x, t_{n_k} u_{n_k})\, dx + o_k(1) = o_k(1),$$

which implies that $\lim_{k \to +\infty} I(t_{n_k} u_{n_k}) = 0$. This is a contradiction with (2.4) for each $m > 0$. Therefore the assertion given in (2.12) is now verified. Hence, we deduce that $v \neq 0$, i.e, the set $\widehat{\Omega} = \{x \in \Omega : v(x) \neq 0\}$ has positive Lebesgue measure. Furthermore, we know that $u_n(x) = v_n(x)\|u_n\|$ for each $x \in \widehat{\Omega}$ and $v_n(x) \to v(x) \neq 0$ a.e in $\widehat{\Omega}$. Since $\|u_n\| \to \infty$ it follows that $|u_n(x)| \to \infty$ a.e in $\widehat{\Omega}$. Notice also that hypothesis (NQ) implies that $H(x,t) \geq -C$ holds for any $(x,t) \in \Omega \times \mathbb{R}$

where $C > 0$. Under these conditions it follows from Fatou's Lemma that

$$c = \liminf_{n\to\infty} \left\{ I(u_n) - \frac{1}{2} I'(u_n)u_n \right\} = \frac{1}{2} \liminf_{n\to\infty} \int_\Omega H(x, u_n)dx \geq \frac{1}{2} \int_\Omega \liminf_{n\to\infty} H(x, u_n)dx = \infty.$$

This is a contradiction proving that $(u_n) \in \mathcal{H}$ is now a bounded sequence in $\mathcal{H}$. This finishes the proof. □

3 Linking Geometry

In this section we consider some geometric properties for the functional I which are related to the Local Linking Theorem. This is the key point in order to prove that that problem (1.1) has at least one nontrivial weak solution. It is important to recall some inequalities for the linear problem (1.2). Recall also that

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 dx, \int_\Omega u^2 dx = 1 \right\}.$$

Hence, for the first eigenvalue we deduce the following inequalities:

$$\int_\Omega |\nabla u|^2 dx \geq \lambda_1 \int_\Omega u^2 dx, \int_\Omega |\Delta u|^2 dx \geq \lambda_1 \int_\Omega |\nabla u|^2 dx, u \in \mathcal{H} \qquad (3.1)$$

and

$$\int_\Omega |\Delta u|^2 dx \geq \lambda_1^2 \int_\Omega u^2 dx, u \in \mathcal{H}. \qquad (3.2)$$

These inequalities follows from the variational formulation for the first eigenvalue problem for the Laplacian operator. For an easy reference, we prove the estimates (3.1) and (3.2) in the Proposition 5.1, see Appendix ahead.

Now we define $\mathcal{H}^1 = \langle \varphi_1, \ldots, \varphi_k \rangle$ and $\mathcal{H}^2 = \langle \varphi_{k+1}, \ldots \rangle$ for each $k \geq 2$. Under these conditions we obtain the following inequalities:

$$\int_\Omega |\nabla u|^2 dx \geq \lambda_{k+1} \int_\Omega u^2 dx, \int_\Omega |\Delta u|^2 dx \geq \lambda_{k+1} \int_\Omega |\nabla u|^2 dx, u \in \mathcal{H}^2 \qquad (3.3)$$

and
$$\int_\Omega |\Delta u|^2 dx \geq \lambda_{k+1}^2 \int_\Omega u^2 dx, u \in \mathcal{H}^2. \tag{3.4}$$
Furthermore, we also mention the following inequalities
$$\int_\Omega |\nabla u|^2 dx \leq \lambda_k \int_\Omega u^2 dx, \int_\Omega |\Delta u|^2 dx \leq \lambda_k \int_\Omega |\nabla u|^2 dx, u \in \mathcal{H}^1 \tag{3.5}$$
and
$$\int_\Omega |\Delta u|^2 dx \leq \lambda_k^2 \int_\Omega u^2 dx, u \in \mathcal{H}^1. \tag{3.6}$$
Notice that inequalities (3.1) - (3.6) can be checked using the variational formulation for (λ_k) where (λ_k) denotes the sequence of eigenvalues for the Laplacian operator. Namely, the proof of these inequalities follows the same ideas discussed in the proof of Proposition 5.1, see Appendix ahead.

Now we show consider an abstract result. Let X be a Banach space written as $X = X^1 \oplus X^2$. Here we define the sequences
$$X_0^1 \subset X_1^1 \subset \ldots \subset X^1, \, X_0^2 \subset X_1^2 \subset \ldots \subset X^2$$
satisfying
$$X^j = \overline{\bigcup_{n \in \mathbb{N}} X_n^j}, \quad j = 1, 2.$$
A functional $J \in C^1(X, \mathbb{R})$ has a local linking at the origin with respect (X^1, X^2) when for some $r > 0$ there holds
$$\begin{aligned} J(u) \geq 0, \, u \in X^2, ||u|| \leq r, \\ J(u) \leq 0, \, u \in X^1, ||u|| \leq r. \end{aligned}$$
Here we assume that $\dim X^1$ is finite and $\dim X_i^n$ is finite for each $i = 1, 2$ and $n \in \mathbb{N}$. Hence we can state the following result:

Theorem 3.1 (Local Linking Theorem [17])**.** *Suppose that $J : X \to \mathbb{R}$ is a functional of C^1 class satisfying the following assumptions*

(L_1) *J has a local linking at 0 with $X^1 \neq \{0\}$;*

(L_2) *J satisfies (Ce) condition;*

(L_3) *J maps bounded sets into bounded sets;*

(L_4) *for every $m \in \mathbb{N}$, $J(u) \to -\infty$, $||u|| \to \infty$, $u \in X^1 \oplus X^2_m$.*

Then J has at least two critical points.

At this stage we consider $X = \mathcal{H}$ and $J = I$. Hence the main aim here is to ensure that I satisfies the assumptions $(L_1) - (L_4)$. The main feature in this work concerns on the fact that the first eigenvalue of the problem (1.2) denoted by μ_1 can be negative or positive. Under these conditions the functional I satisfies also the local linking geometry which is essential in our arguments. For the reader convenience we shall divide the proof for local linking geometry in the following cases $\mu_1 > 0$ or $\mu_1 < 0$. Initially, we shall consider the following result:

Proposition 3.2. *Suppose that f satisfies $(f_0), (f_1), (f_2)$. Moreover, we assume that $\mu_1 > 0$. Then the functional I has a local linking at the origin.*

Proof. Using the fact that $\mu_1 > 0$ we observe that $\beta < \alpha\lambda_1$. Here we divide the proof into two distinct cases. Namely, we consider the following cases $\beta \geq 0$ and $\beta < 0$.

At this stage, we assume that $\beta \geq 0$. Define $\mathcal{H}^1 = \langle \varphi_1, \varphi_2, \ldots, \varphi_k \rangle$ and $\mathcal{H}^2 = (\mathcal{H}^1)^\perp$ where (φ_j) are the eigenfunctions for the eigenvalue problem associated to the Laplacian operator. Let $u \in \mathcal{H}^1$ be a fixed function. In view of (3.5) we obtain

$$I(u) = \frac{1}{2}B(u,u) - \int_\Omega F(x,u)\,dx \leq \frac{1}{2}\left(\alpha - \frac{\beta}{\lambda_k}\right)||u||^2 - \int_\Omega F(x,u)\,dx.$$

Using hypotheses $(f_0) - (f_2)$ and the fact that $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ we get

$$| F(x,t) | \leq \frac{(\varepsilon + f_0)}{2} | t |^2 + C | t |^p, \ \forall\ (x,t) \in \ \Omega \times \mathbb{R}, \tag{3.7}$$

holds true for any $\varepsilon > 0$ and for some $C = C(\varepsilon) > 0$. As a consequence, by using (3.6), we infer that

$$\begin{aligned} I(u) &\leq \frac{1}{2}\left(\alpha - \frac{\beta}{\lambda_k}\right)||u||^2 - \frac{(f_0-\varepsilon)}{2}\int_\Omega |u|^2\,dx + C\int_\Omega |u|^p\,dx \\ &\leq \frac{||u||^2}{2\lambda_k^2}\left[\lambda_k^2\left(\alpha - \frac{\beta}{\lambda_k}\right) - f_0 + \varepsilon\right] + C||u||^p \\ &\leq \frac{||u||^2}{2}\left[\frac{\mu_k - f_0 + \varepsilon}{\lambda_k^2} + C||u||^{p-2}\right]. \end{aligned}$$

Now we consider

$$r_0 = \left(\frac{\varepsilon_0}{4\lambda_k^2 C}\right)^{1/(p-2)}$$

where $\varepsilon_0 = \dfrac{f_0 - \mu_k}{2} > 0$. Under these conditions, for any $\varepsilon \in (0, \varepsilon_0)$, we deduce that $I(u) \leq 0, \forall\, u \in \mathcal{H}^1$; $\|u\| \leq r_0$. Moreover, choosing $u \in \mathcal{H}^2$ and using (3.3), we obtain

$$I(u) = \frac{1}{2}B(u,u) - \int_\Omega F(x,u)\,dx \geq \frac{1}{2}\left(\alpha - \frac{\beta}{\lambda_{k+1}}\right)||u||^2 - \int_\Omega F(x,u)\,dx.$$

Using the same ideas discussed just above we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2}\left(\alpha - \frac{\beta}{\lambda_{k+1}}\right)||u||^2 - \frac{(\varepsilon + f_0)}{2}\int_\Omega |\, u \,|^2 \;\; dx - C\int_\Omega |\, u\,|^p \;\; dx \\ &\geq \frac{||u||^2}{2}\left(\frac{\mu_{k+1} - f_0 - \varepsilon}{\lambda_{k+1}^2} + C||u||^{p-2}\right). \end{aligned}$$

Moreover, we choose

$$r_1 = \left(\frac{\varepsilon_0}{4\lambda_{k+1}^2 C}\right)^{1/(p-2)}$$

where $\varepsilon_1 = \dfrac{\mu_{k+1} - f_0}{2} > 0$. Under these conditions, for any $\varepsilon \in (0, \varepsilon_1)$, we obtain that $I(u) \geq 0, \forall\, u \in \mathcal{H}^2$; $\|u\| \leq r_1$. Now we define $r = \min\{r_0, r_1\}$ proving the linking geometry for case $\beta \geq 0$.

It remains to focus on the case $\beta < 0$. Let $u \in \mathcal{H}^1$ be a fixed function. It is easy to verify that (3.5) and (3.6) imply $B(u,u) \leq \mu_k ||u||_2^2;\;\; \forall u \in \mathcal{H}^1$.

As consequence, by using the last assertion and (3.7), (3.6), we obtain

$$
\begin{aligned}
I(u) &\le \frac{\mu_k}{2}||u||_2^2 - \int_\Omega F(x,u)\,dx \\
&\le \frac{||u||_2^2}{2}(\mu_k - f_0 + \varepsilon) + C||u||_p^p \le \frac{||u||^2}{2\lambda_k^2}(\mu_k - f_0 + \varepsilon) + C||u||^p.
\end{aligned}
$$

Now, using the same ideas discussed above, we are able to chose $r_0 > 0$ in such way that $I(u) \le 0, \forall\ u \in \mathcal{H}^1;\ \|u\| \le r_0$.

On the other hand, considering $u \in \mathcal{H}^2$ and using (3.3) and (3.4), we get $B(u,u) \ge \alpha||u||^2 - \beta\lambda_{k+1}||u||_2^2;\ \ \forall u \in \mathcal{H}^2$. Using the last estimate we observe that

$$
\begin{aligned}
I(u) &\ge \frac{\alpha}{2}||u||^2 - \frac{\beta}{2}\lambda_{k+1}||u||_2^2 - \int_\Omega F(x,u)\,dx \\
&\ge \frac{\alpha}{2}||u||^2 - \frac{\beta}{2}\lambda_{k+1}||u||_2^2 - \frac{(f_0+\varepsilon)}{2}||u||_2^2 - C||u||^p \\
&= \frac{\alpha}{2}||u||^2 - C||u||^p + \frac{||u||_2^2}{2}(-\beta\lambda_{k+1} - f_0 - \varepsilon).
\end{aligned}
$$

Now we suppose that $(-\beta\lambda_{k+1} - f_0 - \varepsilon) > 0$, i.e, we have $f_0 < -\beta\lambda_{k+1}$. Hence $I(u) \ge \frac{\alpha}{2}||u||^2 - C||u||^p$. Therefore we are able to consider $r_1 > 0$ in such way that $I(u) \ge 0, \forall\ u \in \mathcal{H}^2;\ ||u|| \le r_1$. Furthermore, assuming that $(-\beta\lambda_{k+1} - f_0 - \varepsilon) \le 0$, it follows from (3.4) that

$$
\begin{aligned}
I(u) &\ge \frac{\alpha}{2}||u||^2 - C||u||^p + \frac{||u||_2^2}{2}(-\beta\lambda_{k+1} - f_0 - \varepsilon) \\
&\ge \frac{\alpha}{2}||u||^2 - C||u||^p + \frac{||u||^2}{2\lambda_{k+1}^2}(-\beta\lambda_{k+1} - f_0 - \varepsilon) \\
&= ||u||^2\left(\frac{\mu_{k+1} - f_0 - \varepsilon}{2\lambda_{k+1}^2} - C||u||^{p-2}\right)
\end{aligned}
$$

Once again there exists $r_1 > 0$ small such that $I(u) \ge 0,\ \ \forall u \in \mathcal{H}^2;\ ||u|| \le r_1$. Under these conditions we take $r = \min\{r_0, r_1\}$ proving the linking geometry for case $\beta < 0$. This finishes the proof. □

In what follows we consider the case where the first eigenvalue μ_1 is negative. The main difficulty concerns on the fact that $u \to B(u,u), u \in \mathcal{H}$

is not positive anymore. However, we shall prove that I admits the local linking geometry at the origin whenever μ_1 is negative. Furthermore, taking into account hypothesis (f_2), we shall divide the proof of the local linking geometry in some cases. Initially, we consider the following case

Proposition 3.3. *Suppose that f satisfies $(f_0),(f_1)$, and (f_2). Moreover, assume that μ_1 is negative and $0 \in I_k = (\mu_k, \mu_{k+1})$ with $\mu_1 \leq \mu_k < 0 \leq f_0 < \mu_{k+1}$ for some $k \geq 1$. Then the functional I has a local linking at the origin.*

Proof. Once again we consider the following subspaces $\mathcal{H}^1 = \langle \varphi_1, \varphi_2, \ldots, \varphi_k \rangle$ and $\mathcal{H}^2 = (\mathcal{H}^1)^{\perp}$. Let $u \in \mathcal{H}^1$ be a fixed function. Using the fact that $\beta > 0$ and (3.5) we obtain that

$$B(u,u) \leq \left(\alpha - \frac{\beta}{\lambda_k}\right) ||u||^2, \quad \forall\, u \in \mathcal{H}^1.$$

Here we assume that $f_0 > 0$. The proof for case $f_0 = 0$ can be done using similar arguments. As a consequence, using the estimate (3.7) we observe that

$$I(u) \leq \frac{1}{2}\left(\alpha - \frac{\beta}{\lambda_k}\right) ||u||^2 - \frac{(f_0 - \varepsilon)}{2} \int_\Omega |u|^2\, dx + C \int_\Omega |u|^p\, dx$$

Then using the Sobolev embedding and (3.6) there exists a constant $C > 0$ such that

$$\begin{aligned} I(u) &\leq \frac{||u||^2}{2\lambda_k^2}\left[\lambda_k^2\left(\alpha - \frac{\beta}{\lambda_k}\right) - (f_0 - \varepsilon)\right] + C||u||^p \\ &= \frac{||u||^2}{2\lambda_k^2}(\mu_k - f_0 + \varepsilon) + C||u||^p. \end{aligned}$$

Now we take

$$r_0 = \left(\frac{f_0 - \mu_k - \varepsilon}{4\lambda_k^2 C}\right)^{1/(p-2)}.$$

Under these conditions, for any $\varepsilon > 0$ small, we deduce that $I(u) \leq 0, \forall\, u \in \mathcal{H}^1;\ \|u\| \leq r_0$. On the other hand, fore each $u \in \mathcal{H}^2$ fixed, taking into account $\beta > 0, f_0 > 0$ and the Sobolev embedding, (3.7) and (3.3) we obtain that

$$\begin{aligned} I(u) &\geq \frac{||u||^2}{2\lambda_{k+1}^2}\left[\lambda_{k+1}^2\left(\alpha - \frac{\beta}{\lambda_{k+1}}\right) - (f_0+\varepsilon)\right] - C||u||^p \\ &= \frac{||u||^2}{2\lambda_{k+1}^2}(\mu_{k+1} - f_0 - \varepsilon) - C||u||^p. \end{aligned}$$

Therefore, taking

$$r_1 = \left(\frac{\mu_{k+1} - f_0 - \varepsilon}{4\lambda_{k+1}^2 C}\right)^{1/(p-2)},$$

we infer that $I(u) \geq 0, \forall\, u \in \mathcal{H}^2;\ \|u\| \leq r_1$. According to the last assertion and setting $r = \min\{r_0, r_1\}$ the proof for the local linking geometry is now verified. □

Proposition 3.4. *Suppose that f satisfies $(f_0), (f_1)$ and (f_2). Moreover, we assume μ_1 is negative and $0 \in I_k = (\mu_k, \mu_{k+1})$ with $\mu_1 \leq \mu_k < f_0 \leq 0 < \mu_{k+1}$ holds true for some $k \geq 1$. Then the functional I has a local linking at the origin.*

Proof. Once again we consider the same spaces $\mathcal{H}^1$ and $\mathcal{H}^2 = (\mathcal{H}^1)^\perp$ as was done in the proof of Proposition 3.3. The proof for the case $f_0 = 0$ can be done using similar estimates discussed in the previous proposition. Here we consider the proof of local linking geometry assuming that $f_0 < 0$ holds. For the case $f_0 < 0$ the proof of local linking geometry at the origin follows arguing by contradiction.

Let $u \in \mathcal{H}^1$ be a fixed function. Here we shall prove that there exist $\delta_0, r_0 > 0$ small enough in such way that

$$I(u) \leq -\delta_0||u||^2, \quad \forall u \in \mathcal{H}^1;\ ||u|| \leq r_0. \tag{3.8}$$

The proof follows arguing by contradiction. Under these conditions there exists $(u_n) \in \mathcal{H}^1$ such that

$$I(u_n) > -\delta_n||u_n||^2;\ ||u_n|| \leq \delta_n = \frac{1}{n}.$$

Thus, we mention that

$$\frac{I(u_n)}{||u_n||^2} \geq -\frac{1}{n}; \quad \|u_n\| \leq \frac{1}{n}. \tag{3.9}$$

Setting $v_n := u_n/\|u_n\|$ we obtain $||v_n|| = 1$, $v_n \in \mathcal{H}^1$ and there exists $v \in \mathcal{H}^1$ in such way that $v_n \rightharpoonup v$ in $\mathcal{H}^1$. In this case, along a subsequence, we obtain

$$\begin{cases} v_n \rightharpoonup v \text{ weakly in } \mathcal{H}^1, \\ v_n \to v \text{ strongly in } L^q(\Omega), \\ v_n(x) \to v(x), \text{ a. e. in } \Omega, \\ |v_n(x)| \leq h(x),\ h \in L^q(\Omega), \end{cases}$$

holds true for any $1 \leq q < 2_*$. Using the operator B given in (1.4) and (3.9) we see that

$$\frac{B(v_n, v_n)}{2} - \frac{1}{2}\int_{\Omega\cap[u_n\neq 0]} \frac{2F(x,u_n)(v_n)^2}{(u_n)^2} \geq -\frac{1}{n}, \quad v_n \in \mathcal{H}^1.$$

Using the terms in the previous estimate, since $\mathcal{H}^1$ is a finite-dimensional, we apply the growth condition (3.7) and the Lebesgue Convergence Theorem proving that

$$B(v,v) \geq f_0\|v\|_2^2 \text{ and } \|v\|_2 = 1. \tag{3.10}$$

It is not hard to verify that $\mu_k = \sup\left\{B(u,u) : u \in \mathcal{H}^1, \|u\|_2^2 = 1\right\}$. As a consequence, the last assertion implies that

$$B(v,v) \leq \mu_k\|v\|_2^2, v \in \mathcal{H}^1. \tag{3.11}$$

Hence (3.10) and (3.11) imply that $f_0 \leq \mu_k$. This is a contradiction proving that the statement in (3.8) is now verified for some $\delta_0, r_0 > 0$ small enough.

Now we shall prove the linking geometry at the origin in $\mathcal{H}^2$. Let $u \in \mathcal{H}^2$ be a fixed function. The main objective here is to ensure that

$$I(u) \geq \delta_1||u||^2, \quad \forall u \in \mathcal{H}^2; \, ||u|| \, \leq \, r_1 \tag{3.12}$$

holds true for $\delta_1, r_1 > 0$ small enough. Arguing by contradiction we assume that there exists $(u_n) \in \mathcal{H}^2$ in such way that

$$\frac{I(u_n)}{||u_n||^2} \leq \frac{1}{n}; \quad \|u_n\| \leq \frac{1}{n}.$$

Setting $v_n := u_n/\|u_n\|$ we obtain $||v_n|| = 1, \; v_n \in \mathcal{H}^2$. As a consequence,

$$\frac{B(v_n, v_n)}{2} - \frac{1}{2}\int_\Omega \frac{2F(x,u_n)(v_n)^2}{(u_n)^2} \leq \frac{1}{n}. \tag{3.13}$$

Arguing along the same lines employed in the previous case, we deduce that $B(v,v) \leq f_0\|v\|_2^2$. Here we mention that $\mu_{k+1} = \inf\left\{B(v,v) : v \in \mathcal{H}^2, \|v\|_2 = 1\right\}$. Hence, using the assertion just above, we have the following estimate

$$\mu_{k+1}\|v\|_2^2 \leq B(v,v) \leq f_0\|v\|_2^2. \tag{3.14}$$

At this stage, we claim that v is an eigenfunction associated with the eigenvalue λ_{k+1}. In particular, v is not zero and estimate (3.14) give us $\mu_{k+1} \leq f_0$. This is a contradiction proving the assertion (3.12).

It remains to prove the claim just above. Here we shall prove that $v = \pm\phi_{k+1}$ where ϕ_{k+1} denotes the eigenfunction associated to the eigenvalue μ_{k+1}. In order to do that, we rewrite $(v_n) \in \mathcal{H}^2$ in the following form $v_n = v_{n,1} + v_{n,2}$, where $v_{n,1} \in \langle \varphi_{k+1}\rangle$ and $v_{n,2} \in \langle \varphi_{k+2}, \ldots\rangle$. Furthermore, the estimate (3.13) can be rewritten in the following form

$$\begin{aligned}\frac{B(v_{n,1}, v_{n,1})}{2} + \frac{B(v_{n,2}, v_{n,2})}{2} \quad &\leq \quad \frac{1}{n} + \frac{1}{2}\int_\Omega \frac{2F(x,u_n)v_{n,1}^2}{u_n^2}\, dx \\ &+ \quad \frac{1}{2}\int_\Omega \frac{2F(x,u_n)v_{n,2}^2}{u_n^2}\, dx.\end{aligned}$$

In what follows we recall that $\beta > 0$ and $\mu_{k+1} > 0$. Using (3.3) we have that

$$\begin{aligned}0 \quad &\leq \quad \frac{\mu_{k+1}}{\lambda_{k+1}^2}\left(||v_{n,1}||^2 + ||v_{n,2}||^2\right) \leq \frac{2}{n} + \int_\Omega \frac{2F(x,u_n)v_{n,1}^2}{u_n^2}\, dx - f_0\int_\Omega v_{n,1}^2\, dx \\ &+ \quad f_0\int_\Omega v_{n,1}^2\, dx + \int_\Omega \frac{2F(x,u_n)v_{n,2}^2}{u_n^2}\, dx - f_0\int_\Omega v_{n,2}^2\, dx + f_0\int_\Omega v_{n,2}^2\, dx\end{aligned} \tag{3.15}$$

Here we mention that

$$\lim_{n\to\infty}\int_\Omega \frac{2F(x,u_n)v_{n,1}^2}{u_n^2}\,dx = f_0\int_\Omega v_1^2 dx \tag{3.16}$$

and

$$\liminf_{n\to\infty}\int_\Omega \frac{2F(x,u_n)v_{n,2}^2}{u_n^2}\,dx \geq f_0\int_\Omega v_2^2\,dx, v = v_1 + v_2. \tag{3.17}$$

The proof of inequality (3.16) follows by using the fact that $\mathcal{H}^1$ is finite dimensional. Hence any norm in $\mathcal{H}^1$ is equivalent to the norm $\|\cdot\|$. Under these conditions, there exist constants $C_1, C_2 > 0$ such that $C_1\|u\|_\infty \leq \|u\| \leq C_2\|u\|_\infty$ holds true for each $u \in \mathcal{H}^1$. In particular, we obtain that $|u_n(x)| \leq \|u_n\|_\infty \leq C_1^{-1}\|u_n\| \to 0$ a. e. in Ω as $n \to \infty$. Since Ω has finite measure the last assertion together with hypothesis (f_2) and the Dominated Convergence Theorem imply that

$$\begin{aligned}\int_\Omega \frac{2F(x,u_n)v_{n,1}^2}{u_n^2}\,dx - f_0\int_\Omega v_1^2 dx &= \int_\Omega \left(\frac{2F(x,u_n)}{u_n^2} - f_0\right) v_{n,1}^2\,dx \\ &+ f_0\int_\Omega (v_{n,1}^2 - v_1^2)dx \to 0. \end{aligned} \tag{3.18}$$

Here was used also the fact that $v_{n,1} \to v_1$ in $L^q(\Omega)$ for each $q \in [1, 2_*)$ which says that $v_{n,1} \to v_1$ a.e. in Ω and $|v_{n,1}| \leq h$ for some $h \in L^q(\Omega)$. Therefore, the inequality (3.16) is now verified.

The proof of inequality (3.17) follows by using the Fatou's Lemma together with hypotheses (f_1) and (f_2). Namely, the hypothesis (f_1) implies that

$$\lim_{|t|\to\infty} \frac{2F(x,t)}{t^2} = +\infty \tag{3.19}$$

holds uniformly $x \in \Omega$. Furthermore, by using hypothesis (f_2), we also mention that

$$\lim_{|t|\to 0} \frac{2F(x,t)}{t^2} = f_0 \tag{3.20}$$

holds uniformly $x \in \Omega$. As a consequence, taking into account (3.19) and (3.20), there exists C > 0 such that $F(x,t)/t^2 \geq -C$ for each $(x,t) \in \Omega\times\mathbb{R}$.

It is not hard to verify that $v_{n,2} \rightharpoonup v_2$ in $\mathcal{H}$. Up to a subsequence we obtain

$$\begin{cases} v_{n,2} \to v_2 \text{ strongly in } L^2(\Omega), \\ v_{n,2}(x) \to v_2(x), \text{ a. e. in } \Omega, \\ |v_{n,2}(x)| \leq h_2(x), \; h_2 \in L^2(\Omega). \end{cases}$$

Under these conditions, we also obtain that

$$\frac{F(x,u_n)v_{n,2}^2}{u_n^2} \geq -Cv_{n,2}^2 \geq -Ch_2^2 \in L^1(\Omega). \tag{3.21}$$

In particular, we know that

$$h_n(x) := \frac{F(x,u_n(x))v_{n,2}^2(x)}{u_n^2(x)} + Ch_2^2(x) \geq 0, x \in \Omega, n \in \mathbb{N}.$$

Recall also that $u_n \to 0$ in $\mathcal{H}$ as $n \to \infty$. Hence, by using the Fatou's Lemma for the sequence (h_n) given just above and taking into account hypothesis (f_2) and (3.20), we infer that

$$\liminf_{n\to\infty} \int_\Omega \frac{2F(x,u_n)v_{n,2}^2}{u_n^2}\, dx \geq \int_\Omega \liminf_{n\to\infty} \frac{2F(x,u_n)v_{n,2}^2}{u_n^2}\, dx = f_0 \int_\Omega v_2^2\, dx.$$

The last assertion ends the proof of inequality (3.17). Now, taking into account (3.15), (3.16) and (3.17), we obtain that

$$\lim_{n\to\infty}\left(f_0||v_{n,1}||_2^2 - \frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,1}||^2\right) \geq \lim_{n\to\infty}\left(\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,2}||^2 - f_0||v_{n,2}||_2^2\right)$$

Furthermore, by using the fact that $f_0 < 0$, we deduce

$$\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,2}||^2 - f_0||v_{n,2}||_2^2 \geq \frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,2}||^2.$$

Therefore, using that $\mu_{k+1} > 0$, we obtain

$$\begin{aligned} \lim_{n\to\infty}\left(f_0||v_{n,1}||_2^2 - \frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,1}||^2\right) &\geq \lim_{n\to\infty}\left(\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,2}||^2 - f_0||v_{n,2}||_2^2\right) \\ &\geq \lim_{n\to\infty}\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,2}||^2 \geq 0. \end{aligned} \tag{3.22}$$

As a consequence, putting the previous estimates together we obtain

$$\lim_{n\to\infty}\left(f_0||v_{n,1}||_2^2-\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,1}||^2\right)\geq 0. \tag{3.23}$$

On the other hand, using more time that $f_0 < 0$ and $\mu_{k+1} > 0$, we see that

$$f_0||v_{n,1}||_2^2-\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,1}||^2\leq 0. \tag{3.24}$$

Taking into account (3.22),(3.23) and (3.24) we conclude that

$$\lim_{n\to\infty}\frac{\mu_{k+1}}{\lambda_{k+1}^2}||v_{n,2}||^2=0.$$

Hence $v_{n,2}\to 0$ in $\mathcal{H}^2$ and $v_{n,1}\to v$ in $\mathcal{H}$ where $v\in\langle\phi_{k+1}\rangle$. In particular, we have $v=a_{k+1}\phi_{k+1}$ for some constant $a_{k+1}\in\mathbb{R}$. In particular, using the fact that $\|v\|=1$, we obtain $a_{k+1}=\pm 1$. This finishes the proof. □

Proposition 3.5. *Suppose that f satisfies $(f_0),(f_1),(f_2)$. Moreover, we assume that μ_1 is negative and $0\notin\overline{I}_k=[\mu_k,\mu_{k+1}]$ with $\mu_1<0<\mu_k<f_0<\mu_{k+1}$ for some $k\geq 1$. Then the functional I has a local linking at the origin.*

Proof. Firstly, there exists a positive integer $l\in\mathbb{N}$ satisfying $l+1<k$ in such way that $\mu_1\leq\ldots<\mu_l\leq 0<\mu_{l+1}<\mu_k<f_0<\mu_{k+1}$. Under these conditions, we consider the following subspaces $\mathcal{H}^1=\langle\varphi_1,\varphi_2,\ldots,\varphi_k\rangle$ and $\mathcal{H}^2=(\mathcal{H}^1)^\perp$. Let $u\in\mathcal{H}^1$ be a fixed function. Using the fact that $\beta>0$ and (3.5) we deduce that

$$B(u,u)\leq\left(\alpha-\frac{\beta}{\lambda_k}\right)||u||^2\quad\forall\ u\in\mathcal{H}^1.$$

As a consequence, by using the growth condition (3.7), we mention that

$$I(u)\leq\frac{1}{2}\left(\alpha-\frac{\beta}{\lambda_k}\right)||u||^2-\frac{(f_0-\varepsilon)}{2}\int_\Omega|u|^2\,dx+C\int_\Omega|u|^p\,dx$$

Since $f_0 > 0$ and $\varepsilon > 0$ is small enough using (3.6) there exists a constant $C > 0$ in such way that

$$\begin{aligned} I(u) &\leq \frac{||u||^2}{2\lambda_k^2}\left[\lambda_k^2\left(\alpha - \frac{\beta}{\lambda_k}\right) - (f_0 - \varepsilon)\right] + C||u||^p \\ &= \frac{||u||^2}{2\lambda_k^2}(\mu_k - f_0 + \varepsilon) + C||u||^p. \end{aligned}$$

Now we take

$$r_0 = \left(\frac{f_0 - \mu_k - \varepsilon}{4\lambda_k^2 C}\right)^{1/(p-2)}.$$

Under these conditions, for any $\varepsilon > 0$ small, we observe that $I(u) \leq 0, \forall\ u \in \mathcal{H}^1;\ \|u\| \leq r_0$.

Let $u \in \mathcal{H}^2$ be a fixed function. Using the fact that $\beta > 0, f_0 > 0$ it follows from the (3.3) and Sobolev embedding that

$$\begin{aligned} I(u) &\geq \frac{||u||^2}{2\lambda_{k+1}^2}\left[\lambda_{k+1}^2\left(\alpha - \frac{\beta}{\lambda_{k+1}}\right) - (f_0 + \varepsilon)\right] - C||u||^p \\ &= \frac{||u||^2}{2\lambda_{k+1}^2}(\mu_{k+1} - f_0 - \varepsilon) - C||u||^p. \end{aligned}$$

In this way, we take

$$r_1 = \left(\frac{\mu_{k+1} - f_0 - \varepsilon}{4\lambda_{k+1}^2 C}\right)^{1/(p-2)}.$$

As a consequence, we obtain that $I(u) \geq 0, \forall\ u \in \mathcal{H}^2;\ \|u\| \leq r_1$. The desired result follows taking $r = \min\{r_0, r_1\}$. □

Proposition 3.6. *Suppose that f satisfies $(f_0), (f_1), (f_2)$. Moreover, we assume that μ_1 is negative and $0 \notin \overline{I}_k = [\mu_k, \mu_{k+1}]$ with $\mu_1 \leq \mu_k < f_0 < \mu_{k+1} < 0$ holds for some $k \geq 1$. Then the functional I has a local linking at the origin.*

Proof. Initially, there exists a positive integer $l \in \mathbb{N}$ satisfying $l + 1 > k$ in such way that $\mu_1 < \mu_2 \leq \ldots \leq \mu_k < f_0 < \mu_{k+1} < \mu_l \leq 0 < \mu_{l+1}$. Here we consider the following subspaces $\mathcal{H}^1 = \langle \varphi_1, \varphi_2, \ldots, \varphi_k \rangle$ and $\mathcal{H}^2 = (\mathcal{H}^1)^\perp = \langle \varphi_{k+1}, \varphi_{k+2}, \ldots \rangle$. The proof follows arguing as in the proof of Proposition 3.4. Here we omit the details. □

4 The proof of Theorem 1.1

Initially, we shall ensure conditions $(L_3) - (L_4)$ given in Theorem 3.1 assuming that $\mu_1 > 0$. Here we consider $\overline{B_\rho(0)} \subset \mathcal{H}$ the closed ball of radius $\rho > 0$ centered at the origin. Let $u \in \overline{B_\rho(0)}$ be a fixed function. Hence, using Sobolev embedding, we deduce that

$$I(u) \leq \rho^2(C + \overline{C}K^p\rho^{p-2}) + C|\Omega| < \infty, u \in \overline{B_\rho(0)}$$

where $\rho > 0$. Hence the functional I maps bounded sets into bounded sets. Now we shall prove that

$$\lim_{||u|| \to +\infty} I(u) = -\infty, \quad \forall\, u \in \mathcal{H}^1 \oplus \mathcal{H}^2_m \tag{4.1}$$

holds true for any $m \in \mathbb{N}$ where $\mathcal{H}^2_m = \langle \varphi_{k+1}, \varphi_{k+2}, \ldots, \varphi_{k+m} \rangle$ for $m \in \mathbb{N}$ fixed. At this stage, using hypothesis (f_1), for each $R > 0$ there exists $C_R > 0$ in such way that

$$F(x,u) \geq \frac{Ru^2}{2} - C_R, \quad \forall\, (x,u) \in \ \Omega \times \mathbb{R}.$$

Therefore, the last assertion yields

$$I(u) \leq \frac{B(u,u)}{2} + C_R|\Omega| - R\int_\Omega |u|^2\, dx.$$

Now we observe that $\mu_1\|u\|_2^2 \leq B(u,u), u \in \mathcal{H}$. Under these conditions, using the fact that $\mu_1 > 0$, the bilinear form $u \to B(u,u), u \in \mathcal{H}$ defines a new norm which is equivalent to the usual norm (1.3). Using the equivalence for any norms in finite dimensional spaces and the last assertion there exists a constant $K > 0$ in such way that

$$I(u) \leq B(u,u)\left(\frac{1}{2} - RK\right) + C_R|\Omega|.$$

Hence the assertion in (4.1) is now verified for any $R > 1/2K$ since $B(u,u) \to +\infty$ as $\|u\| \to +\infty$. Furthermore, using Proposition 3.2, the functional I admits the local linking geometry. Therefore the energy functional I satisfies hypotheses $(L_1) - (L_4)$ of Theorem 3.1. As a consequence, the functional I admits a nontrivial critical point $u \in \mathcal{H}$. Moreover, we observe that $u = \Delta u = 0$ on $\partial\Omega$, see Appendix ahead. Hence problem (1.1) has at least one nontrivial weak solution. This ends the proof for the case $\mu_1 > 0$.

At this moment we shall consider the proof when μ_1 is negative in such way that $0 \in I_k = (\mu_k, \mu_{k+1})$ or $0 \notin \overline{I}_k = [\mu_k, \mu_{k+1}]$ for some $k \geq 1$. It is important to emphasize that I admits also the local linking geometry, see Propositions 3.3, 3.4 and Propositions 3.5, 3.6. Using the same ideas discussed just above the functional I applies bounded sets into bounded sets. Moreover, the functional I satisfies $I(u) \to -\infty, \|u\| \to \infty$ with $u \in \mathcal{H}^1 \oplus \mathcal{H}^2_m$. Here we use one more time that for any finite dimensional space $E \subset \mathcal{H}$ any norm is equivalent to the usual norm. Hence the functional I satisfies the properties $(L_1)-(L_4)$ of Theorem 3.1 proving that I admits at least one nontrivial critical point. This ends the proof. □

5 Appendix

The main objective in this appendix is to ensure that any critical point $u \in \mathcal{H}$ for the energy functional I satisfies $u = 0$ and $\Delta u = 0$ on $\partial\Omega$. More specifically, we shall prove that any critical point of I give us a weak solution for the problem (1.1) satisfying the Navier boundary conditions.

Let $u \in \mathcal{H} = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ be a critical point for I, i.e, we have that

$$\alpha \int_\Omega \Delta u \Delta \phi \, dx + \beta \int_\Omega \phi \Delta u \, dx = \int_\Omega g(x,u)\phi \, dx.$$

holds for any function $\phi \in \mathcal{H}$. From a standard point of view the trace theorem implies that $u = 0$ on $\partial\Omega$. It remains to show that $\Delta u = 0$ on $\partial\Omega$. In order to do that we consider $v = -\Delta u$ and $h(x) = g(x,u)$. It is not hard to verify that $v \in L^2(\Omega)$ and $h \in L^2(\Omega)$. Furthermore, we also have

$$\int_\Omega v[-\alpha\Delta\phi - \beta\phi] \, dx = \int_\Omega h(x)\phi \, dx, \quad \forall \phi \in \mathcal{H}. \tag{5.1}$$

Let $w \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ be the unique weak solution for the elliptic problem

$$\begin{cases} -\alpha \Delta w - \beta w &= h(x) \quad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

In order to ensure the existence of solutions to the elliptic problem (5.2) for each $h \in L^2(\Omega)$ we can apply the Ekeland Variational Principle. For the uniqueness of solutions to the elliptic problem (5.2) we take w_1 and w_2 two weak solutions proving that

$$\begin{cases} -\alpha \Delta w - \beta w &= 0 \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

where $w = w_1 - w_2$. Taking w as testing function we get

$$\int_\Omega \alpha |\nabla w|^2 - \beta w^2 \, dx = 0.$$

On the other hand, using the hypothesis $\mu_1 > 0$ we deduce that $-\infty < \beta < \alpha\lambda_1$. In particular, for each $\beta \leq 0$, it follows from the last estimate that

$$\alpha \int_\Omega |\nabla w|^2 \, dx \leq 0.$$

This implies that $w \equiv 0$. Furthermore, choosing $0 < \beta < \alpha\lambda_1$, we obtain that

$$0 \leq \left(\alpha - \frac{\beta}{\lambda_1} \right) \int_\Omega |\nabla w|^2 \, dx \leq 0$$

holds true. Hence the last estimate implies also that $w \equiv 0$. As a consequence, $w_1 \equiv w_2$ in Ω proving that the elliptic problem (5.2) admits exactly one weak solution for each $h \in L^2(\Omega)$.

On the other hand, using the weak formulation for (5.2) we obtain

$$\int_\Omega -\alpha w \Delta\phi - \beta w \phi \, dx = \int_\Omega h(x) \phi \, dx \tag{5.3}$$

holds for any $\phi \in \mathcal{H}$. Putting together the identities (5.1) and (5.3) we deduce that

$$\int_\Omega (v - w)(\alpha \Delta \phi - \beta \phi) \, dx = 0, \quad \forall \phi \in C_0^\infty(\Omega).$$

At this stage, using Du Bois Raymond's Lemma [11], we infer that $v = w$ a.e. in Ω. As a consequence $v = w$ in $\mathcal{H}$ which says that $v = -\Delta u = 0$ on $\partial\Omega$.

Now, we shall prove an auxiliary results used in the present work. Namely, we consider the following result:

Proposition 5.1. *Let $u \in \mathcal{H}$ be a fixed function and λ_1 the first positive eigenvalue on $(-\Delta, H_0^1(\Omega))$. Then*

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda_1 \int_\Omega u^2 \, dx \tag{5.4}$$

$$\int_\Omega |\Delta u|^2 \, dx \geq \lambda_1 \int_\Omega |\nabla u|^2 \, dx \tag{5.5}$$

$$\int_\Omega |\Delta u|^2 \, dx \geq \lambda_1^2 \int_\Omega u^2 \, dx. \tag{5.6}$$

Proof. Firstly, the estimate (5.4) follows from the definition of $\lambda_1 > 0$ which is given by

$$\lambda_1 = \inf_{u \in \mathcal{H}\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Now, we shall prove the estimate (5.5). Consider a function $\phi \in C_0^\infty(\Omega)$. Hence the derivative $\phi_{x_i} = \dfrac{\partial \phi}{\partial x_i}$ satisfies

$$\int_\Omega |\nabla(\phi_{x_i})|^2 \, dx \geq \lambda_1 \int_\Omega (\phi_{x_i})^2 \, dx.$$

Therefore, by using the previous identity, we have that

$$\sum_{i=1}^N \int_\Omega |\nabla(\phi_{x_i})|^2 \, dx \geq \lambda_1 \int_\Omega (\phi_{x_1})^2 + \ldots + (\phi_{x_N})^2 \, dx. \tag{5.7}$$

Recall that $\nabla \phi = \left(\dfrac{\partial \phi}{\partial x_1}, \ldots \dfrac{\partial \phi}{\partial x_N} \right)$. As a consequence, we obtain

$$|\nabla \phi|^2 = \frac{\partial^2 \phi}{\partial x_1^2} + \ldots + \frac{\partial^2 \phi}{\partial x_N^2} = (\phi_{x_1})^2 + \ldots + (\phi_{x_N})^2.$$

Combining this identity with (5.7) we get

$$\sum_{i=1}^{N}\int_{\Omega}|\nabla(\phi_{x_i})|^2\,dx \geq \lambda_1\int_{\Omega}|\nabla\phi|^2\,dx. \tag{5.8}$$

Now, by using the Green's identity on the left hand side of (5.8), we infer that

$$\sum_{i=1}^{N}\int_{\Omega}(\nabla\phi_{x_i})(\nabla\phi_{x_i})\,dx = -\lambda_1\int_{\Omega}(\phi_{x_i})\Delta\phi_{x_i}\,dx.$$

As a consequence, we obtain

$$-\sum_{i=1}^{N}\int_{\Omega}(\phi_{x_i})\Delta\phi_{x_i}\,dx \geq \lambda_1\int_{\Omega}|\nabla\phi|^2\,dx. \tag{5.9}$$

It is not difficult to show that

$$\begin{aligned}\Delta(\phi_{x_i}) &= \frac{\partial}{\partial x_1}\left(\frac{\partial^2\phi}{\partial x_1\partial x_i}\right)+\ldots+\frac{\partial}{\partial x_N}\left(\frac{\partial^2\phi}{\partial x_N\partial x_i}\right)\\ &= \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N}\frac{\partial^2\phi}{\partial x_j^2}\right) = (\Delta\phi)_{x_i}.\end{aligned}$$

It follows from the estimate (5.9) that

$$-\sum_{i=1}^{N}\int_{\Omega}(\Delta\phi)_{x_i}(\phi_{x_i})\,dx \geq \lambda_1\int_{\Omega}|\nabla\phi|^2\,dx. \tag{5.10}$$

Furthermore, by using the Green's identity once again, we deduce that

$$\int_{\Omega}(\Delta\phi)_{x_i}(\phi_{x_i})\,dx = -\int_{\Omega}(\Delta\phi)(\phi_{x_i})_{x_i}\,dx$$

Using the last assertion together with (5.10) we obtain

$$\int_{\Omega}|\Delta\phi|^2\,dx \geq \lambda_1\int_{\Omega}|\nabla\phi|^2\,dx, \quad \forall\phi\in C_0^\infty(\Omega)$$

In particular, by using the fact that $C_0^\infty(\Omega)$ is dense in $\mathcal{H}$, we obtain that (5.5) is now verified. The inequality (5.6) follows by using the estimates (5.4) and (5.5). This ends the proof. □

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