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Strictly minimal linearly topologized rings

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Abstract. Strictly minimal linearly topologized rings are introduced and it is shown that every discrete valuation ring is strictly minimal. Necessary and sufficient conditions for a Hausdorff linearly topologized ring to be strictly minimal are obtained, as well as necessary and sufficient conditions for a complete Hausdorff linearly topologized ring to be strictly minimal.

Keywords: discrete valuation rings, strictly minimal linearly topologized rings, linearly topologized modules.

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1 Introduction

Hausdorff [4] proved that every n-dimensional normed space over the field $\mathbb{R}$ of real numbers is isomorphic to $\mathbb{R}^n$, a result generalized to arbitrary n-dimensional Hausdorff topological vector spaces over $\mathbb{R}$ by Tychonoff [9]. The decisive contribution in this direction was given by Nachbin [5]. As a matter of fact, he introduced the notion of a strictly minimal topological

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division ring and showed that, for a given Hausdorff topological division ring K to be strictly minimal, it is necessary and sufficient that every not identically zero linear form on an arbitrary topological vector space over K, whose kernel is closed, be continuous. Nachbin's approach has also been considered in the framework of topological rings [6].

In the spirit of the above-mentioned work of Nachbin, the notion of a strictly minimal linearly topologized ring is introduced in this paper, where it is proved that every discrete valuation ring is strictly minimal. A characterization of strictly minimal linearly topologized rings by means of universal properties is established. In the same vein, a characterization of strictly minimal complete linearly topologized rings by means of universal properties is also established. A few consequences of the just-mentioned results are presented.

In this work ring will mean commutative ring with an identity element $1 \neq 0$ and module will mean unitary module.

2 The notion of a strictly minimal linearly topologized ring

Let R be a topological ring. A *linearly topologized R-module E* is a topological R-module whose origin admits a fundamental system of neighborhoods consisting of submodules of E. R is said to be a *linearly topologized ring* if R is a linearly topologized R-module, R being endowed with its canonical R-module structure (thus the origin of R admits a fundamental system of neighborhoods consisting of ideals of R). Linearly topologized modules and linearly topologized rings play an important role in Algebraic Geometry [3], Commutative Algebra [2] and Number Theory [1, 8].

It is obvious that any ring endowed with the discrete topology is a complete Hausdorff linearly topologized ring.

Before we proceed let us recall the following important concept ([7], Chap. I).

Definition 2.1. A principal ring R is said to be a *discrete valuation ring* if the set M of non-invertible elements of R is a non-trivial ideal of R (hence R is not a field).

Then M is a maximal ideal of R and

$$M^1 = M \supset M^2 \supset \ldots \supset M^n \supset M^{n+1} \supset \ldots$$

is a decreasing sequence of ideals of R such that $\bigcap_{n \geq 1} M^n = \{0\}$. By Theorem 11.4 of [10], there is a unique ring topology τ_M on R for which $(M^n)_{n \geq 1}$ is a fundamental system of τ_M-neighborhoods of 0. Thus (R, τ_M) is a Hausdorff linearly topologized ring and τ_M does not coincide with the discrete topology on R.

Example 2.2. The following noteworthy examples of discrete valuation rings may be mentioned [1, 7]:
(a) The ring $\mathbb{Z}_p$ of p-adic integers (p a prime natural number), M being $p\mathbb{Z}_p$.
(b) The ring $\mathbb{K}[[X]]$ of formal power series with coefficients in an arbitrary field $\mathbb{K}$, M being $X\mathbb{K}[[X]]$.
(c) For each $z_0 \in \mathbb{C}$, the ring $\mathcal{H}_{z_0}$ of complex analytic mappings on an open ball (in $\mathbb{C}$) with center at z_0, M being $(z - z_0)\mathcal{H}_{z_0}$.

Discrete valuation rings satisfy an interesting property:

Proposition 2.3. *Let (R, τ_M) be a discrete valuation ring and let τ be a Hausdorff topology on R making R a linearly topologized (R, τ_M)-module. Then $\tau = \tau_M$.*

Proof. Since the mapping

$$(\lambda, \mu) \in (R \times R, \tau_M \times \tau) \longmapsto \lambda\mu \in (R, \tau)$$

is continuous, the mapping

$$\lambda \in (R, \tau_M) \longmapsto \lambda \in (R, \tau)$$

is continuous, and τ is coarser than τ_M.

In order to show that τ_M is coarser than τ, write $M = \pi R$ and let n be an arbitrary integer ≥ 1. Let v be a discrete valuation on the field of fractions K of R so that $R = \{\lambda \in K; v(\lambda) \geq 0\}$ and $M = \{\lambda \in K; v(\lambda) > 0\}$ ([7], p. 17), and let U be a τ-neighborhood of 0 in R such that U is an ideal of R and $\pi^n \notin U$. We claim that $U \subset M^n$. If not, there would exist a $\xi \in U$ so that $\xi \notin M^n$, that is, $v(\xi) \in \{0, 1, \ldots, n-1\}$. Therefore $\xi \neq 0$ and, since $0 = v(1) = v(\xi\xi^{-1}) = v(\xi) + v(\xi^{-1})$, $v(\xi^{-1}) \in \{-(n-1), \ldots, -1, 0\}$. Hence $\xi^{-1}\pi^n \in R$, because $v(\xi^{-1}\pi^n) = v(\xi^{-1}) + n > 0$. Consequently, $\pi^n = \xi(\xi^{-1}\pi^n) \in UR \subset U$, which does not occur. Thus $U \subset M^n$. $\square$

The central notion of our work reads:

Definition 2.4. A Hausdorff linearly topologized ring (R, τ_R) is said to be *strictly minimal* if, for every Hausdorff topology τ on R making (R, τ) a linearly topologized (R, τ_R)-module (which implies that τ is coarser than τ_R), one has $\tau = \tau_R$.

We have seen in Proposition 2.3 that every discrete valuation ring is strictly minimal. In the next example we shall furnish a Hausdorff linearly topologized ring which is not strictly minimal.

Example 2.5. Let (R, τ_M) be an arbitrary discrete valuation ring and let τ_R be the discrete topology on R. Then (R, τ_M) is a linearly topologized (R, τ_R)-module. In fact, since (R, τ_M) is an additive topological group, it is enough to show the continuity of the mapping

$$(\lambda, \mu) \in (R \times R, \tau_R \times \tau_M) \longmapsto \lambda\mu \in (R, \tau_M)$$

at an arbitrary element $(\lambda_0, \mu_0) \in R \times R$, which follows from the inclusion

$$\{\lambda_0\} \times (\mu_0 + M^n) \subset (\lambda_0\mu_0) + M^n,$$

valid for each integer $n \geq 1$.

Therefore (R, τ_R) is not strictly minimal.

3 A characterization of strictly minimal linearly topologized rings

The next result establishes necessary and sufficient conditions for a Hausdorff linearly topologized ring to be strictly minimal by means of universal properties.

Theorem 3.1. *For a Hausdorff linearly topologized ring (R, τ_R), the following conditions are equivalent:*
(a) (R, τ_R) is strictly minimal.
(b) For every Hausdorff linearly topologized (R, τ_R)-module F which is a free R-module with a basis of 1 element, every R-module isomorphism from R onto F is a homeomorphism from (R, τ_R) onto F.
(c) For every free R-module F with a basis of 1 element, there is only one Hausdorff topology making F a linearly topologized (R, τ_R)-module.
(d) For every linearly topologized (R, τ_R)-module E and for every Hausdorff linearly topologized (R, τ_R)-module F which is a free R-module with a basis of 1 element, every surjective R-linear mapping from E into F with a closed kernel is continuous.
(e) For every linearly topologized (R, τ_R)-module E and for every Hausdorff linearly topologized (R, τ_R)-module F which is a free R-module with a basis of 1 element, every R-linear mapping from E into F with a closed graph is continuous.

As a consequence of Proposition 2.3 and Theorem 3.1, one concludes that conditions $(b), (c), (d)$ and (e) are valid if the Hausdorff linearly topologized ring under consideration is a discrete valuation ring.

In order to prove Theorem 3.1 we shall need two auxiliary lemmas.

Lemma 3.2. *Let (E, τ) be a linearly topologized (R, τ_R)-module, let F be an R-module and suppose that $u : E \to F$ is a surjective R-linear mapping. Let τ_u be the direct image of τ under u (τ_u is the topology on F under which $Y \subset F$ is τ_u-open if $u^{-1}(Y) \subset E$ is τ-open). Then (F, τ_u) is a linearly* *topologized (R, τ_R)-module. Moreover, τ_u is a Hausdorff topology if and only if $Ker(u)$ (the kernel of u) is τ-closed.*

Proof. First let us observe that a subset Y of F is τ_u-open if and only if $Y = u(X)$ for some τ-open subset X of E. In fact, if Y is τ_u-open, $X = u^{-1}(Y)$ is τ-open and $u(X) = Y$. Conversely, if X is τ-open and $u(X) = Y$, then the equality

$$u^{-1}(Y) = Ker(u) + X = \bigcup_{t \in Ker(u)} (t + X)$$

implies that Y is τ_u-open. Consequently, since the direct image under u of a submodule of E is a submodule of F, it is easily seen that (F, τ_u) is a linearly topologized (R, τ_R)-module.

Finally, $Ker(u)$ is obviously τ-closed if τ_u is a Hausdorff topology. Conversely, if $Ker(u)$ is τ-closed, $u(E \backslash Ker(u))$ is τ_u-open. Therefore $\{0\} = F \backslash u(E \backslash Ker(u))$ is τ_u-closed, from which we conclude that τ_u is a Hausdorff topology. □

Lemma 3.3. *Let F be a linearly topologized (R, τ_R)-module. If for every linearly topologized (R, τ_R)-module E we have that every surjective R-linear mapping from E into F with a closed kernel is continuous, then for every linearly topologized (R, τ_R)-module E we have that every R-linear mapping from E into F with a closed graph is continuous.*

Proof. Let E be an arbitrary linearly topologized R-module and let $u : E \to F$ be an R-linear mapping whose graph $Gr(u)$ is closed. Consider $E \times F$ endowed with the product topology, which makes $E \times F$ a linearly topologized (R, τ_R)-module, and define $v : E \times F \to F$ by $v(x, y) = u(x) - y$. Then v is a surjective R-linear mapping such that $Ker(v) = Gr(u)$. Thus, by hypothesis, v is continuous. Therefore u is continuous, because $u(x) = v(x, 0)$ for all $x \in E$. □

Now, let us turn to the proof of Theorem 3.1.

Proof. First let us prove that (a) implies (b). Indeed, let F be as in (b) and let $u : R \to F$ be an R-module isomorphism. Since $u(\lambda) = \lambda u(1)$ for all $\lambda \in R$, u is a continuous R-linear mapping from (R, τ_R) onto F. If θ is the initial topology on R under u, it is easily seen that (R, θ) is a Hausdorff linearly topologized (R, τ_R)-module. Thus, by hypothesis, $\theta = \tau_R$, and hence $u^{-1} : F \to (R, \tau_R)$ is continuous. Therefore $u : (R, \tau_R) \to F$ is a homeomorphism.

Let us prove that (b) implies (c). Indeed, let F be as in (c) and let τ_1 and τ_2 be two Hausdorff topologies on F such that both (F, τ_1) and (F, τ_2) are linearly topologized (R, τ_R)-modules. Let $u : R \to F$ be an R-module isomorphism. Then $u : (R, \tau_R) \to (F, \tau_1)$ and $u : (R, \tau_R) \to (F, \tau_2)$ are homeomorphisms by hypothesis, from which we conclude that the identity mapping $1_F : (F, \tau_1) \to (F, \tau_2)$ is a homeomorphism. Therefore $\tau_1 = \tau_2$, proving (c).

Now, let us prove that (c) implies (d). Indeed, let E and F be as in (d) and let $u : E \to F$ be a surjective R-linear mapping whose kernel is closed. By Lemma 3.2, (F, τ_u) is a Hausdorff linearly topologized (R, τ_R)-module, where τ denotes the topology of E and τ_u denotes the direct image of τ under u. By hypothesis, τ_u coincides with the topology of F, which implies the continuity of u. This proves (d).

Finally, since Lemma 3.3 guarantees that (d) implies (e), it remains to prove that (e) implies (a). Indeed, let θ be a Hausdorff topology on R such that (R, θ) is a linearly topologized (R, τ_R)-module. Since we already know that θ is coarser than τ_R, it remains to prove that the identity mapping $1_R : (R, \theta) \to (R, \tau_R)$ is continuous. Notice that, if $\lambda, \mu \in R$ and $\lambda \neq \mu$, then there are a θ-neighborhood U of λ in R and a θ-neighborhood V of μ in R so that $U \cap V = \emptyset$. Since θ is coarser than τ_R, $U \times V$ is a $(\theta \times \tau_R)$-neighborhood of (λ, μ) in $R \times R$ with $(U \times V) \cap Gr(1_R) = \emptyset$, where $Gr(1_R)$ denotes the graph of 1_R. Therefore $Gr(1_R)$ is $(\theta \times \tau_R)$-closed, and the hypothesis guarantees the continuity of $1_R : (R, \theta) \to (R, \tau_R)$.

This completes the proof of the theorem. $\square$

We shall close this section with two consequences of Theorem 3.1.

Corollary 3.4. *Let (R, τ_R) be a strictly minimal linearly topologized ring and let E be a Hausdorff linearly topologized (R, τ_R)-module. Let M and N be two submodules of E such that N is a free R-module with a basis of 1 element, M is closed in E and $E = M \oplus N$. Then E is the topological direct sum of M and N.*

Proof. Let $p_N : E \to N$ be the projection of E onto N along M. Then $Ker(p_N) = M$ is closed by hypothesis. If N is endowed with the topology induced by that of E, then N is a Hausdorff linearly topologized (R, τ_R)-module. Therefore, by Theorem 3.1, p_N is continuous, as was to be shown. □

Corollary 3.5. *Let (R, τ_R) be a strictly minimal linearly topologized ring, let E be a linearly topologized (R, τ_R)-module and suppose that F is a Hausdorff linearly topologized (R, τ_R)-module which is a free R-module with a basis of 1 element. If $u : E \to F$ is a surjective continuous R-linear mapping, then u is open.*

Proof. Let $\pi : E \to E/Ker(u)$ be the canonical surjection and let $\tilde{u} : E/Ker(u) \to F$ be the unique R-module isomorphism so that $u = \tilde{u} \circ \pi$. If we endow $E/Ker(u)$ with the quotient topology, $E/Ker(u)$ is a Hausdorff linearly topologized (R, τ_R)-module by Lemma 3.2; moreover, $E/Ker(u)$ is a free R-module with a basis of 1 element. Consequently, by Theorem 3.1, $\tilde{u}$ is a homeomorphism. Therefore u is open, because π is open. □

4 A characterization of strictly minimal complete linearly topologized rings

The main result of this section establishes necessary and sufficient conditions for a complete Hausdorff linearly topologized ring to be strictly minimal by means of universal properties.

Theorem 4.1. *For a complete Hausdorff linearly topologized ring (R, τ_R), the following conditions are equivalent:*

(a) (R, τ_R) *is strictly minimal.*

(b) For every integer $n \geq 1$ *and for every Hausdorff linearly topologized* (R, τ_R)*-module* F *which is a free* R*-module with a basis of* n *elements, every* R*-module isomorphism from* R^n *onto* F *is a homeomorphism,* R^n *being endowed with the product topology.*

(c) For every integer $n \geq 1$ *and for every free* R*-module* F *with a basis of* n *elements, there is only one Hausdorff topology making* F *a linearly topologized* (R, τ_R)*-module.*

(d) For every integer $n \geq 1$*, for every linearly topologized* (R, τ_R)*-module* E *and for every Hausdorff linearly topologized* (R, τ_R)*-module* F *which is a free* R*-module with a basis of* n *elements, every surjective* R*-linear mapping from* E *into* F *with a closed kernel is continuous.*

(e) For every integer $n \geq 1$*, for every linearly topologized* (R, τ_R)*-module* E *and for every Hausdorff linearly topologized* (R, τ_R)*-module* F *which is a free* R*-module with a basis of* n *elements, every* R*-linear mapping from* E *into* F *with a closed graph is continuous.*

As a consequence of Proposition 2.3 and Theorem 4.1, one concludes that conditions (b), (c), (d) and (e) of Theorem 4.1 are valid if the complete Hausdorff linearly topologized ring under consideration is a complete discrete valuation ring. Now let us turn to the proof of Theorem 4.1.

Proof. In order to prove that (a) implies (b) we shall argue by induction on n, the case $n = 1$ being a consequence of Theorem 3.1. Let $n \geq 2$ and assume the result valid for $n-1$. Let F be a Hausdorff linearly topologized (R, τ_R)-module which is a free R-module with a basis of n elements and let $u : R^n \to F$ be an R-module isomorphism. If $e_1 = (1, 0, \ldots, 0) \in R^n, e_2 = (0, 1, 0, \ldots, 0) \in R^n, \ldots, e_n = (0, 0, \ldots, 0, 1) \in R^n$ and $d_i = u(e_i) \in F$ for $i = 1, \ldots, n$, $\{d_1, \ldots, d_n\}$ is a basis of F. Moreover, if M is the submodule of F generated by $\{d_1, \ldots, d_{n-1}\}$ and N is the submodule of F generated by $\{d_n\}$, F is the direct sum of M and N. Since the mapping

$$(\lambda_1, \ldots, \lambda_{n-1}) \in R^{n-1} \longmapsto \sum_{i=1}^{n-1} \lambda_i d_i \in M$$

is an R-module isomorphism, the induction hypothesis implies that it is a homeomorphism of R^{n-1} onto M (R^{n-1} endowed with the product topology, under which it is a complete Hausdorff linearly topologized (R, τ_R)-module; M endowed with the topology induced by that of F, under which it is a Hausdorff linearly topologized (R, τ_R)-module). Since R^{n-1} is complete, M is complete, and hence M is closed in F. Thus, by Corollary 3.1, F is the topological direct sum of M and N. Consequently, the mapping

$$(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) \in R^{n-1} \times R \longmapsto \sum_{i=1}^{n} \lambda_i d_i \in F$$

is a homeomorphism, proving that u is a homeomorphism. This proves (b).

Let us prove that (b) implies (c). In fact, let n be an integer ≥ 1 and let F be a free R-module with a basis of n elements, and let τ_1, τ_2 be Hausdorff topologies on F such that (F, τ_1), (F, τ_2) are linearly topologized (R, τ_R)-modules. If $u : R^n \to F$ is an R-module isomorphism and R^n is endowed with the product topology, the hypothesis ensures that $u : R^n \to (F, \tau_1)$ and $u : R^n \to (F, \tau_2)$ are homeomorphisms. Therefore the identity mapping $1_F : (F, \tau_1) \to (F, \tau_2)$ is a homeomorphism, that is, $\tau_1 = \tau_2$.

Now, let us prove that (c) implies (d). Indeed, let n, E and F be as in (d), and let $u : E \to F$ be a surjective R-linear mapping whose kernel is closed. By Lemma 3.2, (F, τ_u) is a Hausdorff linearly topologized (R, τ_R)-module, where τ denotes the topology of E and τ_u denotes the direct image of τ under u. By hypothesis, τ_u coincides with the topology of F, which furnishes the continuity of u.

Finally, (d) implies (e) in view of Lemma 3.3, and it follows immediately from Theorem 3.1 that (e) implies (a).

This completes the proof of the theorem. □

Corollary 4.2. *Let (R, τ_R) be a complete strictly minimal linearly topologized ring and let E be a Hausdorff linearly topologized (R, τ_R)-module. Suppose that n is an integer ≥ 1, N is a submodule of E which is a free*

R-module with a basis of n elements and M is a closed submodule of E so that $E = M \oplus N$. Then E is the topological direct sum of M and N.

Proof. Analogous to that of Corollary 3.4, by applying Theorem 4.1 instead of Theorem 3.1. □

Corollary 4.3. *Let (R, τ_R) be a complete strictly minimal linearly topologized ring and let E be a linearly topologized (R, τ_R)-module. Suppose that n is an integer ≥ 1 and F is a Hausdorff linearly topologized (R, τ_R)-module which is a free R-module with a basis of n elements. If $u : E \to F$ is a surjective continuous R-linear mapping, then u is open.*

Proof. Analogous to that of Corollary 3.5, by applying Theorem 4.1 instead of Theorem 3.1. □

References

[1] Y. Amice. *Les nombres p-adiques.* Presses Universitaires de France, 1975.

[2] N. Bourbaki. *Commutative Algebra.* Hermann and Addison-Wesley, Paris and Reading, Massachusetts, 1972.

[3] A. Grothendieck and J. A. Dieudonné. *Eléments de Géométrie Algébrique.* Die Grundlehren der mathematischen Wissenschaften 166, Springer-Verlag, Berlin - Heidelberg - New York, 1971.

[4] F. Hausdorff. Zur Theorie der linearen metrischen Räume. *J. reine angew. Math.*, 157: 294-311, 1932.

[5] L. Nachbin. On strictly minimal topological division rings. *Bull. Amer. Math. Soc.*, 55: 1128-1136, 1949.

[6] D. P. Pombo Jr. Topological modules over strictly minimal topological rings. *Comment. Math. Univ. Carolinae*, 44: 461-467, 2003.

[7] J. -P. Serre. *Corps locaux*. Quatrième édition, Actualités Scientifiques et Industrielles 1296, Hermann, Paris, 1968.

[8] J. -P. Serre. *A Course in Arithmetic*. Third printing, Graduate Texts in Mathematics 7, Springer-Verlag, Berlin - Heidelberg - New York, 1985.

[9] A. Tychonoff. Ein Fixpunktsatz. *Math. Ann.*, 111: 767-776, 1935.

[10] S. Warner. *Topological Fields*. Notas de Matemática 126, North-Holland, Amsterdam, 1989.