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On the real multipolynomial Bohnenblust-Hille inequality

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Abstract. We exhibit an extension of the real Bohnenblust-Hille inequality to multipolynomials and briefly state the main points of a method, motivated by Jamilson et al. and Diniz et al., for finding lower bounds for its constants.

Keywords: Bohnenblust-Hille inequality, Multilinear mappings, Homogeneous polynomials, Multipolynomials.

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1 Definitions, notations and conventions

Given positive integers m and $n_1, \ldots, n_m$, we say that a mapping $P : E^m \to \mathbb{R}$ is an $(n_1, \ldots, n_m)$-*homogeneous polynomial* if, for each i with $1 \leq i \leq m$, the mapping

$$P\left(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_m\right) : E \to \mathbb{R}$$

is an n_i-homogeneous polynomial for all fixed $x_j \in E$ with $j \neq i$. This kind of map is often referred to as a *multipolynomial*. Note that it reduces to an m-linear form when $m \geq 1$ and $n_1 = \ldots = n_m = 1$, and to an

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n_1-homogeneous polynomial when $m = 1$. Continuous multipolynomials are all those bounded over products of the unit ball B_E of E. In that case,

$$\|P\| := \sup \{|P(x_1, \ldots, x_m)| : x_1, \ldots, x_m \in B_E\}$$

defines a norm on the space of all continuous $(n_1, \ldots, n_m)$-homogeneous polynomials from E^m into $\mathbb{R}$. As for the basics of the theory of multipolynomials between Banach spaces, we refer to [2, 4, 5].

Henceforth c_0 will stand for the classical space of all real-valued sequences which vanish at ∞. The set $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. For fixed m and $n_1, \ldots, n_m$ positive integers, we shall write $M := \sum_{i=1}^{m} n_i$. We shall denote by $\mathbb{M}_{m\times\infty}(\mathbb{N}_0)$ and $\mathbb{M}_{m\times\infty}(\mathbb{R})$ the set of all $m \times \infty$ semi-infinite matrices with entries in $\mathbb{N}_0$ and $\mathbb{R}$, respectively. If $x_i := (x_{ij})_j$ and $\alpha_i := (\alpha_{ij})_j$ are the ith row of $x \in \mathbb{M}_{m\times\infty}(\mathbb{R})$ and $\alpha \in \mathbb{M}_{m\times\infty}(\mathbb{N}_0)$, respectively, such that the summation of the entries of α_i is $|\alpha_i| = n_i \in \mathbb{N}_0$, we shall write $x_i^{\alpha_i} := \prod_j x_{ij}^{\alpha_{ij}}$ for each i with $1 \leq i \leq m$.

Similar to the polynomial case (see, e.g. [1, p. 392]), with the aid of the above notation, one may show that every continuous $(n_1, \ldots, n_m)$-homogeneous polynomial $P : c_0 \times \cdots \times c_0 \to \mathbb{R}$ can be written as

$$P(x_1, \ldots, x_m) = \sum c_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

for all $x_1, \ldots, x_m \in c_0$, where $c_\alpha \in \mathbb{R}$ and where the summation is taken over all matrices $\alpha \in \mathbb{M}_{m\times\infty}(\mathbb{N}_0)$ such that $|\alpha_i| = n_i$, for each i with $1 \leq i \leq m$.

2 Main Results

The multipolynomial Bohnenblust-Hille inequality [4] for real scalars asserts that for all positive integers m and $n_1, \ldots, n_m$ there exists a constant $C_M \geq 1$ such that

$$\left(\sum_{|\alpha_1|=n_1, \ldots, |\alpha_m|=n_m} |c_\alpha|^{\frac{2M}{M+1}} \right)^{\frac{M+1}{2M}} \leq C_M \|P\| \tag{2.1}$$

for all continuous $(n_1, \ldots, n_m)$-homogeneous polynomials $P : c_0 \times \cdots \times c_0 \to \mathbb{R}$.

By fixing $m > 1$ and choosing $n_1 = \ldots = n_m = 1$ in the above inequality, it reduces to the Bohnenblust-Hille inequality for m-linear forms (see [3, Inequality 1.1]). At the other extremity, by setting $m = 1$ and then choosing any positive integer $n_1 = m$, we recover the Bohnenblust-Hille inequality for m-homogeneous polynomials (see [1, Inequality 1.2]).

In [3, Sec. 5], the best lower bound for the constants in the Bohnenblust-Hille inequality for m-linear forms is given by

$$C_m \geq 2^{\frac{m-1}{m}} \tag{2.2}$$

for every $m \geq 2$. As for the constants in the Bohnenblust-Hille inequality for m-homogeneous polynomials, the best-obtained estimate in [1, Theorem 2.2] is given by

$$D_{\mathbb{R},m} \geq \frac{\left(3^{\frac{m}{2}}\right)^{\frac{m+1}{2m}}}{\left(\frac{5}{4}\right)^{\frac{m}{2}}} \qquad \text{if } m \text{ is even}$$

and

$$D_{\mathbb{R},m} \geq \frac{\left(4 \cdot 3^{\frac{m-1}{2}}\right)^{\frac{m+1}{2m}}}{2 \cdot \left(\frac{5}{4}\right)^{\frac{m-1}{2}}} \qquad \text{if } m \neq 1 \text{ is odd.}$$

In any case, we have

$$D_{\mathbb{R},m} > (1.17)^m, \tag{2.3}$$

which holds, therefore, for every positive integer $m > 1$.

In [6], we adapt the techniques due to [1] and [3] aiming to yield non-trivial lower bounds for C_M in (2.1). To do so, we let f and g denote the real-valued functions defined by means of the equations

$$f(n_i) = \begin{cases} 1 & \text{, if } n_i = 1 \\ 3^{\frac{n_i}{2}} & \text{, if } n_i \text{ is even} \\ 4 \cdot 3^{\frac{n_i-1}{2}} & \text{, if } n_i \neq 1 \text{ is odd} \end{cases}$$

and

$$g(n_i) = \begin{cases} 1 & , \text{ if } n_i = 1 \\ \left(\frac{5}{4}\right)^{\frac{n_i}{2}} & , \text{ if } n_i \text{ is even} \\ 2 \cdot \left(\frac{5}{4}\right)^{\frac{n_i - 1}{2}} & , \text{ if } n_i \neq 1 \text{ is odd} \end{cases}$$

for each i with $1 \leq i \leq m$.

The following result proved in [6] provides proper lower bounds for C_M and, in extreme cases, recovers the estimates (2.2) and (2.3).

Theorem 2.1. *Let $C_M \geq 1$ be a constant as in the inequality (2.1). Then*

$$C_M \geq \frac{\left(4^{m-1} f(n_1) \cdots f(n_m)\right)^{\frac{M+1}{2M}}}{2^{m-1} g(n_1) \cdots g(n_m)}$$

for all positive integers m and $n_1, \ldots, n_m$.

As we mentioned, the classical multilinear and polynomial estimates can be derived from this result. Indeed, it reduces to the best estimate (2.2) for the m-linear constants C_m when $m > 1$ and $n_1 = \ldots = n_m = 1$. An application of the theorem by assuming $m = 1$ and then $n_1 = m$, on the other hand, yields the more accurate lower bound (2.3) for $D_{\mathbb{R},m}$.

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