Matemática
Contemporânea
Vol. 50, 38–53
http://doi.org/10.21711/231766362022/rmc503

Index bounds for closed minimal surfaces in 3-manifolds with the Killing property

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. Let Σ be a closed minimal surface immersed in a Riemannian 3-manifold carrying an orthonormal Killing frame. This class of ambient spaces includes Lie groups with a bi-invariant metric. In this paper, we prove that the sum of the Morse index and the nullity of Σ is bounded from below by a constant times its genus.

Keywords: Minimal surfaces, Morse index, harmonic forms.

2020 Mathematics Subject Classification: 53A10, 49Q05, 35P15.

1 Introduction

Minimal hypersurfaces are critical points for the area functional, while constant mean curvature (CMC) hypersurfaces are critical points for the area functional for volume preserving variations. If such a hypersurface minimizes area up to the second order, then we say that it is *stable*.

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More generally, we may define the *index* of a minimal or a CMC hypersurface as the dimension of the maximal subspace in the respective space of variations, where the second variation of the area functional is negative definite. It is well known that the second variation of the area functional is given by a quadratic form associated to the Jacobi operator J (see Section 2 for details and definitions) and the index of Σ equals the number of negative eigenvalues of J, counting with multiplicities. Geometrically, the index indicates the number of distinct variations which decrease area. In particular, a hypersurface is stable if and only if the index is zero. In this paper, let us assume that all manifolds are oriented.

It is well known that there exists a strong connection between topology and index of minimal and CMC hypersurfaces. For instance, in [24], Schoen and Yau proved that if Σ^2 is a closed stable minimal surface embedded in a 3-manifold (M^3, g) with nonnegative scalar curvature R_g, then either Σ is a sphere or Σ is a torus. In the Euclidean 3-space, if $\Sigma \subset \mathbb{R}^3$ is a complete minimal stable surface, then Σ is a flat plane. This result is a nice generalization of the classical Bernstein theorem and was proved independently by do Carmo and Peng [10], Fischer-Colbrie and Schoen [13], and Pogorelov [21]. Very recently, Chodosh and Li, proved in [6] this is still true for complete stable minimal hypersurface in $\mathbb{R}^4$. More generally, $\Sigma \subset \mathbb{R}^3$ has finite index if, and only if, Σ is conformally equivalent to a compact Riemann surface with finitely many points removed, see [12] and [16]. In fact, denoting by $Ind(\Sigma)$ the index of Σ, we have that

$$Ind(\Sigma) \geq \frac{1}{3}(2g + 4k - 5),$$

where g and k denote the genus and the number of ends of Σ, respectively. This nice estimate was proved by Chodosh and Maximo [7, 8], and other important previous estimates were obtained by Ros in [22], and by Grigor'yan, Netrusov and Yau in [15].

For closed minimal hypersurfaces in the unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, Savo used harmonic vector fields to prove in [23] that the index is bounded from below by a linear function of the first Betti number. This technique was generalized by Ambrozio, Carlotto and Sharp in [2], for a class of ambient spaces that admit a special embedding in the Euclidean space. In both works, the authors make use of the parallel canonical orthonormal frame in an appropriate $\mathbb{R}^d$. Related results were obtained by Mendes Radeschi [19], Gorodski, Mendes and Radeschi [14], and Chao Li [17]. These results strongly support Schoen and Marques-Neves conjecture [18, 20], which claims that if the ambient space M^{n+1} is complete and has positive Ricci curvature, then there exists a positive constant C, depending only on M, such that for any closed minimal hypersurface Σ in M it holds that

$$Ind(\Sigma) \geq C(b_1(\Sigma) + \dots b_n(\Sigma)), \tag{1.1}$$

where $b_i(\Sigma)$ denotes the i-th Betti number of Σ. This conjecture is still opened, but an important contribution was recently made by Song [25].

For CMC hypersurfaces, a classical result due to Barbosa, do Carmo, and Eschenburg [3] , asserts that geodesic spheres are the only closed stable CMC hypersurfaces in simply connected space forms. In particular, small geodesic spheres in $\mathbb{T}^3$ are examples of stable CMC surfaces.

The first result estimating the index of a CMC surface by the topology was obtained by the first and the second authors in [5]. It was proved that the index of a closed CMC surface immersed in $\mathbb{R}^3$ or in $\mathbb{S}^3$ is bounded from below by an explicit constant times the genus. This result was recently generalized in [1] for the same class of ambient spaces considered in [2].

In this present paper, we consider closed minimal surfaces immersed in Riemannian 3-manifolds with the *Killing property*, that is, supporting a global orthonormal frame of Killing vector fields. This class of ambient space was studied by D'Atri and Nickerson in [9] and Tanno in [26] and contains all Lie groups endowed with a bi-invariant metric. See Section 2.1 for more details.

We also recall that the *nullity* $Null(\Sigma)$ of a minimal hypersurface is the dimension of the space of L^2 solutions of the Jacobi operator. In other words $Null(\Sigma) = \dim\{u \in W^{1,2} : Ju = 0\}$. In [17], Chao Li proved that the nullity is also related with the topology of the hypersurface.

In our first result we use a novel approach to prove that the sum of the index and the nullity of minimal surfaces immersed in this class of spaces is bounded from by a multiple of its genus. Since index and nullity are non negative integers, we can use the ceiling function $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$ in our estimates.

Theorem 1.1. *If Σ^2 is a closed minimal surface immersed in a Riemannian 3-manifold M^3 with the Killing property, then*

$$Ind(\Sigma) + Null(\Sigma) \geq \left\lceil \frac{g(\Sigma)}{3} \right\rceil.$$

Moreover, if M has positive Ricci curvature, then

$$Ind(\Sigma) \geq \left\lceil \frac{g(\Sigma)}{3} \right\rceil.$$

As a consequence of the proof we have

Corollary 1.2. *If Σ^2 is a closed CMC surface immersed in a Riemannian 3-manifold M^3 with the Killing property, then*

$$Ind(\Sigma) \geq \left\lceil \frac{g(\Sigma)}{3} - 1 \right\rceil.$$

We point out that our bounds can be found already in the literature, sometimes better ones, when applied to the list of known examples of 3-manifolds with the Killing property.

2 Preliminaries

In this section we present some preliminaries definitions and notations we will use to prove our results.

2.1 The Killing property

We say that a Riemannian manifold M has the *Killing property* if, in some neighborhood of each point of M, there exists an orthonormal Killing frame, that is, an orthonormal frame $\{X_1, \ldots, X_n\}$ such that each X_i is a Killing vector field, in the sense that it generates infinitesimal isometries.

This concept was introduced in [9] by D'Atry and Nickerson in 1968, and among other things they proved that Riemannian manifolds with Killing property have nonnegative sectional curvature. Next, Tanno in [26] proved that in the 3-dimensional case, Killing property implies that M has constant (nonnegative) sectional curvature.

In this paper we will consider Riemannian manifolds endowed with a *global* orthonormal Killing frame as ambient spaces for minimal hypersurfaces.

The most simple example is the Euclidean space. More generally, we can check that Lie groups endowed with a bi-invariant metric have a global orthonormal frame of Killing vector fields. In fact, fixed an orthonormal basis in the Lie algebra, the corresponding left (or equivalently right) invariant frame is an orthonormal Killing frame. In the 3-dimensional case such spaces were classified in [11]. They are $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$, $\mathbb{S}^1 \times \mathbb{R}^2$, and the real projective space $\mathbb{RP}^3$ ($=SO(3)$).

It is an interesting fact that, among the canonical spheres, only $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ have a global orthonormal Killing frame (see [9]). Finally, it is easy to see that the Riemannian product of manifolds with the Killing property also have the Killing property.

2.2 Stability of closed hypersurfaces

Let M^{n+1} be a Riemannian manifold, and let Ric denote its Ricci tensor. If Σ^n is a closed, two-sided, minimal hypersurface in M, we denote by N a globally defined unit normal vector along Σ, and by A its second fundamental form. For simplicity of notation, we also use A to denote the shape operator of Σ. In this case, the second variation of the area functional of Σ is given by the quadratic form

$$Q(u,u) = -\int_\Sigma uJu\,dv,$$

where $J = \Delta + \|A\|^2 + \mathrm{Ric}(N)$ is the Jacobi operator acting in the Sobolev space $W^{1,2}(\Sigma)$. We can easily see that J is a self-adjoint elliptic operator and its spectrum is given by a sequence of eigenvalues diverging to $+\infty$, say

$$\lambda_1 < \lambda_2 \leq \ldots \lambda_k \leq \ldots$$

In particular, if $\{\phi_1, \phi_2, \ldots, \phi_k, \ldots\}$ is an orthonormal basis of eigenfunctions of J, and if we denote by $\mathcal{S}_k = \langle \phi_1, \ldots, \phi_k \rangle^{\perp}$ the subspace of $W^{1,2}(\Sigma)$ orthogonal to the first k eigenfunctions of J, then the min-max characterization of eigenvalues implies that

$$\lambda_k = \inf_{u \in \mathcal{S}_{k-1}} \frac{\int_\Sigma uJu\, dv}{\int_\Sigma u^2\, dv}.$$

The *index* $Ind(\Sigma)$ of Σ is defined as the number of negative eigenvalues of J in $W^{1,2}(\Sigma)$. Equivalently, the index of Σ is the maximal dimension of a subspace of $W^{1,2}(\Sigma)$ where Q is negative defined. The *nullity* of Σ, denoted by $Nul(\Sigma)$, is the dimension of the subspace of eigenfunctions corresponding the eigenvalue $\lambda = 0$.

If Σ is a CMC hypersurface, that is, $H \neq 0$ and constant, the index and the nullity are defined in the same way, but considering the space of test functions as the subspace of $W^{1,2}(\Sigma)$ orthogonal to constants. In fact, the test functions may have zero mean integral in order to generate volume preserving variations.

2.3 Test functions and harmonic vector fields

We denote by $\Omega^p(\Sigma)$ the space of p-forms on Σ and by $\mathcal{H}^p(\Sigma)$ the subspace of harmonic p-forms, that is,

$$\mathcal{H}^p(\Sigma) = \{\omega \in \Omega^p(\Sigma) : \Delta\omega = d\delta\omega + \delta d\omega = 0\}.$$

We recall from Hodge-de Rham theorem that $\mathcal{H}^p(\Sigma)$ is isomorphic to the p-th de Rham cohomology group $H^p(\Sigma)$, and so $\dim \mathcal{H}^p(\Sigma) = b_p(\Sigma)$ is the p-th Betti number of Σ. In particular, if Σ is 2-dimensional, then $b_1(\Sigma) = 2g(\Sigma)$, where $g(\Sigma)$ denotes the topological genus of Σ.

The Riemannian metric of Σ induces the so called *musical isomorphism* between 1-forms $\omega \in \Omega^1(\Sigma)$ and vector fields $\xi \in T\Sigma$ given by the following identity

$$\omega(X) = \langle \xi, X \rangle,$$

for all $X \in T\Sigma$. In this case, we say that ω and ξ are dual, and we write $\xi = \omega^\sharp$ or $\omega = \xi^\flat$. For instance, using this isomorphism we have that

$$\delta\omega = \operatorname{div} \xi = -tr\nabla\xi. \tag{2.1}$$

Moreover, we define the *Hodge Laplacian* of ξ as the vector field dual to the Hodge Laplacian of $\xi^\flat$, that is, $\Delta\xi = (\Delta\xi^\flat)^\sharp$. In particular, we say that ξ is a *harmonic vector field* if and only if ω is a harmonic form.

On the other hand, for vector fields on Σ we also have the so called *rough Laplacian*, which is defined as

$$\nabla^\star\nabla\xi = -\sum_{k=1}^{n}(\nabla_{e_k}\nabla_{e_k}\xi - \nabla_{\nabla_{e_k}e_k}\xi),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on Σ. These two Laplacians are related by the Bochner-Weitzenböck formula

$$\Delta\xi = \nabla^\star\nabla\xi + \mathrm{Ric}_\Sigma(\xi). \tag{2.2}$$

Now, let us assume that M^{n+1} has the Killing property, and let denote by $\{X_1, \ldots, X_{n+1}\}$ an orthonormal frame of Killing vectors. If $\Sigma \subset M$ is a closed minimal hypersurface, we denote by E_i the orthogonal projection of X_i onto the tangent space of Σ, that is,

$$E_i = X_i - g_i N,$$

where $g_i = \langle X_i, N \rangle$. Following ideas in [23] and [2], given a harmonic vector field $\xi = \omega^\sharp$ on Σ we will consider

$$w_i = \langle \xi, E_i \rangle,$$

as test functions to estimate index and nullity of Σ.

3 Minimal surfaces in 3-manifolds with Killing property

In this section, M^3 denotes a 3-dimensional Riemannian manifold endowed with a global orthonormal frame of Killing vectors, $\{X_1, X_2, X_3\}$. In the next lemma we use the notations presented in Section 2.

Lemma 3.1. *Let Σ be a closed oriented CMC surface immersed in M^3. Then we have the following assertions:*

1. $\langle \nabla_X E_i, Y\rangle + \langle \nabla_Y E_i, X\rangle = 2g_i\langle AX, Y\rangle$, *for all* $X, Y \in T\Sigma$.
2. $\langle \nabla E_i, \nabla \xi\rangle = g_i\langle A, \nabla\xi\rangle$.
3. $\mathrm{div} E_i = -2Hg_i$.

Proof. The first assertion follows taking the tangent part of the Killing equation for the vector fields X_i.

To prove the second assertion, we consider ∇E_i and $\nabla\xi$ as 2-forms using the musical isomorphism. So, $\nabla E_i(X, Y) = \langle \nabla_X E_i, Y\rangle$. It allow us to consider the algebraic dual of ∇E_i, defined by the following equation

$$(\nabla E_i)^t(X, Y) = \langle X, \nabla_Y E_i\rangle.$$

That is, $(\nabla E_i)^t(X, Y) = \nabla E_i(Y, X)$, and from Assertion 1, we have, $\nabla E_i + (\nabla E_i)^t = 2g_i A$.

On the other hand, since ξ is harmonic, $\nabla\xi$ is a symmetric tensor. Thus, if $\{e_1, e_2\}$ is a local orthonormal frame, geodesic at $p \in \Sigma$ we have

$$\begin{aligned}
\langle (\nabla E_i)^t, \nabla\xi\rangle &= \sum_{k,j} (\nabla E_i)^t(e_k, e_j)\cdot\nabla\xi(e_k, e_j)\\
&= \sum_{k,j} (\nabla E_i)(e_j, e_k)\cdot\nabla\xi(e_j, e_k)\\
&= \sum_k\sum_j \langle \nabla_{e_k} E_i, e_j\rangle\langle \nabla_{e_k}\xi, e_j\rangle\\
&= \sum \langle \nabla_{e_k} E_i, \nabla_{e_k}\xi\rangle\\
&= \langle \nabla E_i, \nabla\xi\rangle.
\end{aligned}$$

So,

$$\begin{aligned}
\langle \nabla E_i, \nabla \xi \rangle &= \frac{1}{2}\langle \nabla E_i + (\nabla E_i)^t, \nabla \xi \rangle + \frac{1}{2}\langle \nabla E_i - (\nabla E_i)^t, \nabla \xi \rangle \\
&= \frac{1}{2}\langle \nabla E_i + (\nabla E_i)^t, \nabla \xi \rangle \\
&= \frac{1}{2}\langle 2g_i A, \nabla \xi \rangle \\
&= g_i \langle A, \nabla \xi \rangle.
\end{aligned}$$

Finally, denoting by h the induced metric on Σ, we have

$$\begin{aligned}
\mathrm{div} E_i &= -\mathrm{tr}(\nabla E_i) \\
&= -\langle \nabla E_i, h \rangle \\
&= -\frac{1}{2}\langle \nabla E_i + (\nabla E_i)^t, h \rangle \\
&= -\frac{1}{2}\langle 2g_i A, h \rangle \\
&= -g_i \langle A, h \rangle \\
&= -g_i \mathrm{tr}(A) \\
&= -2Hg_i.
\end{aligned}$$

This concludes the lemma. □

In the next lemma we use auxiliary test functions associated to the Hodge star of ω in Σ. More precisely, we define

$$\bar{w}_i = \star\omega(E_i) = \langle \star\xi, E_i \rangle.$$

Moreover, we point out that due to our convention in (2.1) the Green's formula for closed manifolds reads as

$$\int_\Sigma w \,\mathrm{div} X \, dv = \int_\Sigma \langle \nabla w, X \rangle dv,$$

for $w \in C^1(\Sigma)$ and $X \in T\Sigma$.

Lemma 3.2.

$$\sum_i \Big(\int_\Sigma w_i J w_i dv + \int_\Sigma \bar{w}_i J \bar{w}_i dv \Big) = -2\int_\Sigma \mathrm{Ric}(N,N)|\xi|^2 - 4H^2 \int_\Sigma |\xi|^2 dv.$$

Proof. In this proof we will use Einstein sum notation. Recall that $\{e_1, e_2\}$ denotes a local orthonormal frame, geodesic at $p \in \Sigma$.

From the first assumption of Lemma 3.1 we have

$$\begin{aligned} \nabla w_i &= \nabla_{E_i}\xi + 2g_i\langle A\xi, e_k\rangle e_k - \langle e_k, \nabla_\xi E_i\rangle e_k \\ &= [E_i, \xi] + 2g_i A\xi. \end{aligned}$$

On the other hand, since $\operatorname{div}\xi = 0$, computing the divergence of the 2-form $\xi^\flat \wedge E_i^\flat$ we obtain

$$\begin{aligned} \delta(\xi^\flat \wedge E_i^\flat) &= \operatorname{div}(\xi)E_i - \operatorname{div}(E_i)\xi - [\xi, E_i] \\ &= 2Hg_i\xi + [E_i, \xi]. \end{aligned}$$

That is,

$$\nabla w_i = \delta(\xi^\flat \wedge E_i^\flat) - 2Hg_i\xi + 2g_i A\xi.$$

Now we compute the energy of w_i as follows

$$\begin{aligned} \int_\Sigma |\nabla w_i|^2\, dv &= \int_\Sigma \langle \nabla w_i, \delta((\xi^\flat \wedge E_i^\flat) - 2Hg_i\xi + 2g_i A\xi\rangle dv \\ &= -2H\int_\Sigma \langle \nabla w_i, g_i\xi\rangle dv + 2\int_\Sigma \langle \nabla w_i, g_i A\xi\rangle dv \\ &= -2H\int_\Sigma w_i \operatorname{div}(g_i\xi) dv + 2\int_\Sigma w_i \operatorname{div}(g_i A\xi) dv \\ &= 2H\int_\Sigma w_i\langle \nabla g_i, \xi\rangle dv - 2\int_\Sigma w_i\langle \nabla g_i, A\xi\rangle dv, \end{aligned}$$

since $\operatorname{div} A\xi$ also vanishes. Similarly, we have

$$\int_\Sigma |\nabla \bar{w}_i|^2\, dv = 2H\int_\Sigma \bar{w}_i\langle \nabla g_i, \star\xi\rangle dv - 2\int_\Sigma \bar{w}_i\langle \nabla g_i, A\star\xi\rangle dv.$$

Thus,

$$\int_\Sigma |\nabla w_i|^2\, dv + \int_\Sigma |\nabla \bar{w}_i|^2\, dv = 2H\int_\Sigma \langle E_i \nabla g_i\rangle |\xi|^2 dv - 2\int_\Sigma \langle E_i, A\nabla g_i\rangle |\xi|^2 dv.$$

Now let us compute these terms separately. Using the harmonicity of ξ we have

$$\begin{aligned}
\int_\Sigma \langle E_i, A\nabla g_i\rangle |\xi|^2 dv &= \int_\Sigma g_i \operatorname{div}(|\xi|^2 AE_i) dv \\
&= \int_\Sigma g_i |\xi|^2 \operatorname{div}(AE_i) dv \\
&= -\int_\Sigma g_i |\xi|^2 \langle \nabla_{e_k}(AE_i), e_k\rangle dv \\
&= -\int_\Sigma g_i |\xi|^2 \langle (\nabla_{e_k} A)E_i + A(\nabla_{e_k} E_i), e_k\rangle dv \\
&= -\int_\Sigma g_i |\xi|^2 \langle E_i, (\nabla_{e_k} A)e_k\rangle - \int_\Sigma g_i |\xi|^2 \langle A, \nabla E_i\rangle dv \\
&= -\int_\Sigma g_i |\xi|^2 \langle A, \nabla E_i\rangle dv \\
&= -\frac{1}{2}\int_\Sigma g_i |\xi|^2 \langle A, \nabla E_i + (\nabla E_i)^t\rangle dv \\
&= -\frac{1}{2}\int_\Sigma g_i |\xi|^2 \langle A, 2g_i A\rangle dv \\
&= -\int_\Sigma g_i^2 |\xi|^2 \|A\|^2 dv.
\end{aligned}$$

Finally, using Assertion 3 of Lemma 3.1, we have

$$\begin{aligned}
H\int_\Sigma \langle E_i \nabla g_i\rangle |\xi|^2 dv &= H\int_\Sigma g_i \operatorname{div}(|\xi|^2 E_i) dv \\
&= H\int_\Sigma g_i\, |\xi|^2 \operatorname{div}(E_i) dv \\
&= -2H^2 \int_\Sigma g_i^2\, |\xi|^2 dv.
\end{aligned}$$

Therefore, we get

$$\sum_i \Big(\int_\Sigma |\nabla w_i|^2\, dv + \int_\Sigma |\nabla \bar{w}_i|^2\, dv \Big) = 2\int_\Sigma \|A\|^2 |\xi|^2 dv - 4H^2 \int_\Sigma |\xi|^2 dv,$$

and the lemma follows.

3.1 Proof of Theorem 1.1

Let us recall the notation fixed in Section 2.2. So, let $\lambda_1 < \lambda_2 \leq \ldots \lambda_k \leq \ldots$ be the sequence of eigenvalues of the Jacobi operator J, and let $\{\phi_1, \phi_2, \ldots, \phi_k, \ldots\}$ be an orthonormal basis of eigenfunctions of J. We also set $\mathcal{S}_k = \langle \phi_1, \ldots, \phi_k \rangle^{\perp}$.

Let us consider a harmonic vector field $\xi \in \mathcal{H}^1(\Sigma)$ such that the test functions w_i and $\bar{w}_i$ belong in the space $\mathcal{S}_{k-1}$, for some $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. This is equivalent to find $\xi \in \mathcal{H}^1(\Sigma)$ such that the system of $6(k-1)$ homogeneous linear equations

$$\int_M w_i \phi_j dv = \int_M \bar{w}_i \phi_j dv = 0, \quad i \in \{1, 2, 3\}, \, j \in \{1, \ldots, k-1\}. \tag{3.1}$$

Since $\dim \mathcal{H}^1(\Sigma) = b_1(\Sigma) = 2g(\Sigma)$, we have a non trivial solution if $2g(\Sigma) > 6(k-1)$. In this case, we have from Lemma 3.2 and the min-max characterization that

$$\lambda_k \leq 0,$$

that is $Ind(\Sigma) + Nul(\Sigma) \geq k$. On the other hand, since $2g(\Sigma) > 6(k-1)$, have that $k \geq g(\Sigma)/3$, and we have done.

3.2 Proof of Corollary 1.2

This corollary follows immediately from the general result proved by Barbosa and Bérad in [4], but it also follows from our proof above. In fact, we just need to guarantee that the functions w_i and $\bar{w}_i$ satisfy the zero mean integral condition. In other words, it means that additionally they solve 6 new equations, namely

$$\int_M w_i dv = \int_M \bar{w}_i dv = 0, \quad i \in \{1, 2, 3\}.$$

So we have a non trivial solution if $2g(\Sigma) > 6k$ and the result follows.

□

Acknowledgements

The authors wishe to thank Celso Viana for his interesting comments and helpful discussion during the preparation of this article and the referee for valuation suggestions. The first author was partially supported by Brazilian National Council for Scientific and Technological Development (CNPq Grants 309733/2019-7, 201322/2020-0 and 405468/2021-0) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES-COFECUB 88887.143161/ 2017-0 and CAPES-MATHAMSUD 88887.368700/2019-00).

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