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Pinching results for minimal submanifolds and a characterization of spherical caps

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. In this note, we discuss some curvature and energy gap results for free boundary minimal submanifolds in the Euclidean ball. We present a pinching condition on the length of the second fundamental form and profile function of minimal submanifolds that restricts the topology to be either a k-disk or a k-solid tube; we also include a discussion regarding the expected geometric rigidity in lower dimensions. Finally, we show a pinched curvature result towards graphical uniqueness of spherical caps among free boundary CMC hypersurfaces in the Euclidean ball B^n.

Keywords: minimal submanifolds, free boundary, mean curvature.

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1 Introduction

In this note, we study the area and extrinsic curvatures of a k-dimensional submanifolds with constant mean curvature in the unit ball B^n in the

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Euclidean space that meet ∂B^n orthogonally. These submanifolds are critical points of the area functional (possibly wih volume constraints) in the space of k-dimensional submanifolds with boundary in ∂B^n. They are commonly known as free boundary submanifolds. See [25] and [12] for more details. The equatorial disk D^k and the critical catenoid are the simplest examples in the unit Euclidean ball. Besides these two standard examples, we also have spherical caps and pieces of a Delaunay surfaces which are free boundary examples with constant mean curvature (CMC). Several results about the existence and construction of certain types of free boundary CMC submanifolds in B^n have been considered recently (see [14, 11, 15, 21, 20]).

In addition to the question of existence mentioned above, something very important is the question of uniqueness. Related to this question, the most classic result was proved by Nitsche [24] who showed that equatorial disks and right angle spherical caps are in fact the only immersed free boundary CMC surfaces in B^3 that are homeomorphic to a disk (see also Fraser-Schoen [13] for the generalization to minimal surfaces with high co-dimensional case). For the non-trivial topology case, it has been proved Fraser-Schoen [13] that the critical catenoid maximizes the normalized Steklov eigenvalue in the class of annulus surfaces. Building on the result, Tran [26] (see also [10]) proved that If Σ^2 is an embedded free-boundary minimal annulus in B^3 with Morse index equal to four, then Σ^2 is the critical catenoid. In another direction, Brendle [8] proved a rigidity result considering the area of the free-boundary minimal submanifolds in B^n. More precisely, he proved that if Σ^k is a k-dimensional free boundary minimal submanifold in B^n then $|\Sigma^k| \geq |D^k|$. Moreover, the equality holds if, and only if, Σ^k is contained in a k-dimensional hyperplane in $\mathbb{R}^n$. This result greatly generalizes previous result in dimension two proved by Fraser-Schoen [12] (see also Ros-Vergasta [25]).

We are interested on topological and geometric uniqueness results like the ones mentioned above assuming certain gap/pinching conditions involving the volume, Willmore energy, and extrinsic curvatures. The organization of this note is as follows. In Section 2 we review some $O(2)\times O(2)$-invariant minimal hypersurface in B^4 homeomorphic to the solid tube $\mathbb{D}^2\times\mathbb{S}^1$ which are the model rigid candidate for the curvature gap inequality discussed in the following sections. In Section 3, we address a characterization of the disk and critical catenoid proved by Ambrozio-Nunes [4] and discuss generalizations to higher dimensions; we show a gap/pinching condition of the second fundamental form and profile function that characterizes topologically the k-disk and the k-solid tube, Theorem 3.7. Moreover, we also discuss the expected rigidity result in lower dimensions. In Section 4, we briefly recall some rigidity and gap results involving the area and present a gap result for the Willmore energy of 2-dimensional constant mean curvature surfaces in B^3, Theorem 4.4. Finally, we prove in Section 5 two partial results towards graphical uniqueness of spherical caps in the class of graphical free boundary CMC hypersurfaces in B^n, Proposition 5.1 and Corollary 5.2.

2 Equivariant minimal hypersurfaces

In this section we look at minimal hypersurfaces in $\mathbb{R}^{m+2}$ that are $SO(2)\times SO(m)$ invariant. They are natural candidates to satisfy the curvature gap inequality (3.5) addressed in Section 3. These surfaces were studied in [2] and [3]. Their constructions were later extended to the free boundary case in [15]. Following [2] and [15] we begin by recalling how these surfaces are constructed.

If Σ is a minimal hypersurface in $\mathbb{R}^{m+2}$ and invariant by $SO(2)\times SO(m)$, then Σ can be parametrized as

$$X : I\times\mathbb{S}^1\times\mathbb{S}^{m-1}\to\mathbb{R}^{m+2}; \quad X(t,x,y)=(a(t)x,b(t)y).$$

The curve $\gamma(t)=(a(t),b(t))$ satisfies the following ODE:

Lemma 2.1.

$$\frac{b''a'-a''b'}{a'^2+b'^2}+\frac{b'}{a}-(m-1)\frac{a'}{b}=0.$$

Proof. A simple computation gives that $X_t = (a'x, b'y)$, $X_x = (a(t)\partial_x, 0)$, and $X_y = (0, b(t)\partial_y)$. Hence, $g_{tt} = a'^2 + b'^2$, $g_{xx} = a^2$, $g_{yy} = b^2$, and $g_{tx} = g_{ty} = g_{xy} = 0$. Moreover, an unit normal vector N is given by

$$N = \frac{1}{\sqrt{a'^2 + b'^2}}\Big(b'(t)x, -a'(t)y\Big).$$

It follows that $\langle N_t, X_t\rangle = \frac{b''a' - a''b'}{\sqrt{a'^2+b'^2}}$, $\langle N_x, X_x\rangle = \frac{1}{\sqrt{a'^2+b'^2}}ab'$, and $\langle N_y, X_y\rangle = -\frac{1}{\sqrt{a'^2+b'^2}}a'b$. A standard computation gives

$$0 = (m+1)\,H = \frac{1}{\sqrt{a'^2 + b'^2}}\left(\frac{b''a' - a''b'}{a'^2 + b'^2} + \frac{b'a}{a^2} - (m-1)\frac{a'b}{b^2}\right).$$

From this the lemma follows. □

Let $\gamma(s)$ be the curve $\gamma(s) = (a(s), b(s))$ in $\mathbb{R}^2$ with respect to a arc-length parametrization and $\varphi(s)$ and $\theta(s)$ be the functions defined by

1. $\gamma(s) = \sqrt{a^2 + b^2}(\cos(\varphi(s)), \sin(\varphi(s)))$.
2. $\gamma'(s) = (\cos(\theta(s)), \sin(\theta(s)))$.

Using that $\varphi(s) = \arctan(b(s)/a(s))$, we obtain

$$\varphi' = \frac{\sin(\theta - \varphi)}{\sqrt{a^2 + b^2}}.$$

To compute θ' we use Lemma 2.1

$$\theta' = \frac{2}{\sqrt{a^2 + b^2}}\left(\frac{(m-1)\cos(\theta)\cos(\varphi) - \sin(\varphi)\sin(\theta)}{\sin(2\varphi)}\right).$$

The behavior of the surface Σ is understood in terms of the qualitative information given by the integral curves of the vector field (φ', θ') in the plane. This is obtained by studying the integral curves of the following vector field:

$$V(\varphi, \theta) = \Big(\sin(2\varphi)\sin(\theta - \varphi)\,,\, 2((m-1)\cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi))\Big).$$

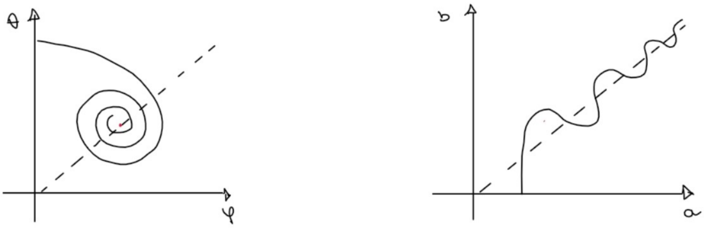

Figure 2.1: An integral curve of $V(\varphi,\theta)$ and the curve $\gamma(s)$.

A careful analysis of the zeros of V in the region $(0,\frac{\pi}{2})\times(-\pi,\pi)$ is given in [2] (see also [15]). When $m\leq 5$ there exists an integral curve

$$\{(\varphi(t),\theta(t):t\in\mathbb{R}\}$$

starting at the saddle point $(0,\frac{\pi}{2})$ and spiraling toward the focal point (v_0,v_0) where $v_0=\arctan(\sqrt{m-1})$, see Figure 2.1. As indicated in [3] when $m\geq 6$, the integral curves of $V(\varphi,\theta)$ does not intersect the diagonal $\{\varphi=\theta\}$ when the curve $\gamma(s)$ is embedded. These integral curves generates, by Lemma 2.1, a properly embedded minimal hypersurface $\Sigma\subset\mathbb{R}^{m+2}$ diffeomorphic to $\mathbb{S}^1\times\mathbb{R}^m$ and asymptotic to the minimal cone

$$\{t\,(x,\sqrt{m}\,y):\,t\geq 0,\,x\in\mathbb{S}^1,\,y\in\mathbb{S}^{m-1}\}.$$

The hypersurface Σ intersects the cone infinitely often when $m\leq 5$ and is disjoint from the cone when $m\geq 6$.

As mentioned, the curve $(\varphi(t),\theta(t))$, $t\in(-\infty,\infty)$, starting at $(0,\frac{\pi}{2})$ intersects the line $\varphi=\theta$ infinitely often when $m\leq 5$. Take t_0 to be the first t for which $\varphi(t)=\theta(t)$, i.e., the curve $\gamma(s(t))$ intersects the circle centered at the origin orthogonally when $s_0=s(t_0)$. It is proved in [15] that $\gamma(s)$, $s\in[0,s_0]$, is contained inside the ball of radius $|\gamma(s_0)|$. Hence, the minimal surface obtained from $\gamma:[0,s_0]\to\mathbb{R}^2$ is a free boundary minimal surface in the ball $B(0,|\gamma(s_0)|)$. The argument also holds for the other values of t solutions of $\varphi(t)=\theta(t)$. After scaling, we obtained an infinite family of $O(2) \times O(m)$ invariant free boundary minimal hypersurfaces in B^n and they converge away from the origin to a minimal cone, see [15].

It is pointed in [15] that since the embedded curves $\gamma(s)$ do not intersect the line $(s, \sqrt{m}\, s)$ for $m \geq 6$, there exist no $SO(2) \times SO(m)$ invariant free boundary minimal surfaces in B^{m+2} in these dimensions.

3 Curvature gap results

In this section we study geometric properties that in a way characterize the equatorial disk as well as others interesting symmetric hypersurfaces. In this direction, we mention the recent work of Ambrozio and Nunes [4] on a geometric characterization of the equatorial disk and the critical catenoid in terms of the curvature and the support function:

Theorem 3.1 (Ambrozio-Nunes [4]). *Let Σ^2 be a compact free boundary minimal surface in B^3. Assume that for all points $x \in \Sigma$,*

$$\langle x, N(x) \rangle^2 |A(x)|^2 \leq 2$$

where $N(x)$ denotes a unit normal vector at the point $x \in \Sigma$ and A denotes the second fundamental form of Σ. Then

1. *$\langle x, N(x) \rangle^2 |A(x)|^2 \equiv 0$ and Σ is a flat equatorial disk;*

2. *$\langle x_0, N(x_0) \rangle^2 |A(x_0)|^2 = 2$ at some point $x_0 \in \Sigma$ and Σ is a critical catenoid.*

The authors in [4] raise the question if the above theorem can be generalized to higher dimension and co-dimension. In [6] we extended their result to 2-dimensional surfaces of any codimension in B^n. In this work we answer their question at the level of topology and show some local second order rigidity properties, see Corollary 3.8 below. Based on our findings, see Remark 3.12, we propose the following conjecture:

Conjecture 3.2. *Let Σ^3 be a capillary minimal hypersurface in the unit ball $B^4 \subset \mathbb{R}^4$. If*

$$|x^\perp|^2\,|A(x)|^2 \leq \frac{3}{2} \tag{3.1}$$

for every $x \in \Sigma^3$, then one of the following is true:

1. *Σ^3 is congruent to the equatorial disk $\mathbb{D}^3$.*
2. *Σ^3 is congruent to one of the $O(2) \times O(2)$-invariant free boundary minimal hypersurface in B^4; these are homeomorphic to the solid tube $\mathbb{D}^2 \times \mathbb{S}^1$.*

Remark 3.3. There exist non-equatorial minimal cones in B^n homeomorphic to $\mathbb{D}^{n-1}$ for every $n \geq 5$, see [19]. In particular, these examples are singular free boundary minimal disks satisfying $|x^\perp|^2|A|^2 = 0$.

3.1 Curvature gap and topological rigidity

The connection between the curvature inequality (3.1) with the topology and geometry of Σ comes from the following computation:

Lemma 3.4. *Let Σ^k be a free boundary minimal submanifold in B^{n+1} and $f : \Sigma^k \to \mathbb{R}$ the function defined by*

$$f(x) = \frac{|x|^2}{2},\ x \in \Sigma^k.$$

Then $\nabla^\Sigma f = x^\top$ for every $x \in \Sigma$ and

$$Hess_\Sigma\, f(x)(X, Y) = \langle X, Y\rangle - \langle A(X, Y), \vec{x}\rangle. \tag{3.2}$$

Proof. Given $X \in \mathcal{X}(\Sigma)$, then

$$X(f) = \frac{1}{2}X\langle \vec{x}, \vec{x}\rangle = \langle X, \vec{x}\rangle = \langle X, x^\top\rangle.$$

This proves that $\nabla^\Sigma f(x) = x^\top$. In particular, $\nabla^\Sigma f(x) = x$ for every $x \in \partial\Sigma$ since Σ is a free boundary minimal surface. Given X and Y vector fields in $\mathcal{X}(\Sigma)$ the $\mathrm{Hess}_\Sigma f\,(X,Y)$ is given by

$$\begin{aligned} Hess_\Sigma f\,(X,Y) &= \langle \nabla_X \nabla f, Y\rangle = \langle \overline{\nabla}_X \nabla f, Y\rangle = \langle \overline{\nabla}_X (x - x^\perp), Y\rangle \\ &= \langle X, Y\rangle - \langle \overline{\nabla}_X x^\perp, Y\rangle = \langle X, Y\rangle + \langle x^\perp, \overline{\nabla}_X Y\rangle \\ &= \langle X, Y\rangle + \langle A(X,Y), \vec{x}\rangle, \end{aligned}$$

where X and Y are vector fields in $\mathcal{X}(\Sigma)$. □

In order to state and prove the main result in this section we need to introduce the definition of the set $C(\Sigma)$ and briefly recall an algebraic technical lemma.

Definition 3.5. Given a free boundary minimal submanifold Σ^k in B^{n+1} we define $m_0 = \inf_\Sigma \frac{|x|^2}{2}$ and

$$C(\Sigma) = \{x \in \Sigma : \frac{|x|^2}{2} = m_0\}. \tag{3.3}$$

Lemma 3.6. *Let $a_1, \ldots, a_n, b$ be a finite sequence of real numbers. If*

$$\sum_{i=1}^n a_i^2 \leq \frac{(\sum_{i=1}^n a_i)^2}{n-1} - \frac{b}{n-1},$$

then $2\,a_i\,a_j \geq \frac{b}{n-1}$ for every $i, j \in \{1, \ldots, n\}$.

Proof. See Lemma 4.1 in Chapter 2 of Chen [9]. □

Theorem 3.7. *Let Σ^k be a free boundary minimal submanifold in a convex domain $\Omega \subset \mathbb{R}^{n+1}$ and such that*

$$|A|^2(x)\,|x^\perp|^2 \leq \frac{k}{k-1} \tag{3.4}$$

for every $x \in \Sigma^k$, then one of the following is true:

1. *$C(\Sigma^k) = \{x_0\}$ and Σ^k is diffeomorphic to the disk $\mathbb{D}^k$.*
2. *$C(\Sigma^k)$ is a closed geodesic and Σ^k is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{k-1}$. Moreover, (3.4) becomes equality when $x \in C(\Sigma^k)$ and*

$$Hess_\Sigma f \Big|_{C(\Sigma)^\perp} = \frac{k}{k-1}\, Id$$

In particular, if $|A|^2(x)|x^\perp|^2 < \frac{k}{k-1}$, then Σ is diffeomorphic to $\mathbb{D}^k$.

Proof. We first show that (3.4) implies that $\mathrm{Hess}_\Sigma f \geq 0$. Let $\{e_1, \dots, e_k\}$ be an orthonormal basis of eigenvectors of $\mathrm{Hess}_\Sigma f$ at $x \in \Sigma$ with respective eigenvalues $\overline{\lambda}_1, \dots, \overline{\lambda}_k$. We want to show that $\lambda_i \geq 0$ for every i. By Lemma 3.4 we have that $\overline{\lambda}_i = 1 + \langle A(e_i, e_i), \overrightarrow{x}\rangle$ which implies

$$\sum_{i=1}^k \overline{\lambda}_i^2 = k + 2\sum_{i=1}^k \langle A(e_i, e_i), \overrightarrow{x}\rangle + \sum_{i=1}^k \langle A(e_i, e_i), \overrightarrow{x}\rangle^2$$

$$= k + \sum_{i=1}^k \langle A(e_i, e_i), \overrightarrow{x}\rangle^2 \leq k + |x^\perp|^2 \sum_{i=1}^k |A(e_i, e_i)|^2 \leq k + |x^\perp|^2 |A|^2.$$

On the other hand, we have $\left(\sum_{i=1}^k \overline{\lambda}_i\right)^2 = k^2$ since Σ^k is minimal. Hence,

$$k + |x^\perp|^2|A|^2 \leq \frac{k^2}{k-1} \Rightarrow \sum_{i=1}^k \overline{\lambda}_i^2 \leq \frac{(\sum_{i=1}^k \overline{\lambda}_i)^2}{k-1}.$$

Therefore, equation (3.4) with Lemma 3.6, where $\overline{\lambda}_i = a_i$ and $b = 0$, imply that $2\overline{\lambda}_i\overline{\lambda}_j \geq 0$. Consequently, the eigenvalues $\overline{\lambda}_i$, $i = 1, \dots, k$, have all the same sign and since $\sum_{i=1}^k \overline{\lambda}_i = k$ we conclude that $\overline{\lambda}_i \geq 0$ for every i.

Next we show that the set of critical points of $f : \Sigma \to \mathbb{R}$ coincides with the $C(\Sigma^k)$. Indeed, let $\gamma(t)$ be a geodesic in Σ joining critical points x_0 and x_1 of f where x_0 is in C, such geodesic exists as the geodesic curvature of $\partial\Sigma$ is positive by the free boundary condition and since Ω is convex. It follows that $(f \circ \gamma)''(t) \geq 0$ which implies that $(f \circ \gamma)'$ is non-decreasing. But since $(f \circ \gamma)'(0) = (f \circ \gamma)'(1) = 0$, we conclude that $f \circ \gamma = \text{const}$, this implies that $f(x_1) = m_0$ and $x_1 \in C(\Sigma^k)$. In particular, every geodesic segment with extremities points at $C(\Sigma^k)$ is contained in $C(\Sigma^k)$, i.e. $C(\Sigma^k)$ is a totally convex set of Σ^k.

If $C(\Sigma^k) = \{x_0\}$, then f has only one critical point $x_0 \in \Sigma$. By standard Morse theory, Milnor [23], we conclude that Σ^k is diffeomorphic to a disk $\mathbb{D}^k$. More generally, if $|A|^2(x)|x^\perp|^2 < \frac{k}{k-1}$, then we choose b from Lemma 3.6 to be

$$\frac{b}{n-1} = \frac{1}{2}\inf_{\Sigma}\left(\frac{k}{k-1} - |A|^2(x)|x^\perp|^2\right).$$

This implies $\overline{\lambda}_i\overline{\lambda}_j > 0$ for every i, j. Hence, $C(\Sigma) = \{x_0\}$ and Σ is diffeomorphic to $\mathbb{D}^k$ be the discussion above.

Let's now study the case where $C(\Sigma^k)$ contains more than one point. We claim that $C(\Sigma^k)$ is a smooth simple closed geodesic. If not, then $C(\Sigma)$ must contain a smooth 2-dimensional surface as a submanifold since $C(\Sigma^k)$ is a totally convex set. It follows that $\dim(\mathrm{Ker}(\mathrm{Hess}_\Sigma f)) \geq 2$ at points in that surface. Let e_1 and e_2 be two null eigenvectors of $\mathrm{Hess}_\Sigma f$, then by Lemma 3.4 we have $\langle A(e_1, e_1), \overrightarrow{x}\rangle = -1$ and $\langle A(e_2, e_2), \overrightarrow{x}\rangle = -1$. By the Cauchy-Schwarz inequality we obtain $|A(e_i, e_i)|^2|x^\perp|^2 \leq 1$ for $i = 1, 2$. Consequently,

$$2 > \frac{k}{k-1} \geq |A|^2|x^\perp|^2 \geq \Big(|A(e_1, e_1)|^2 + |A(e_2, e_2)|^2\Big)|x^\perp|^2 \geq 2.$$

We reach a contradiction and the claim is proved. Since there are no other critical points besides $C(\Sigma)$, Morse theory as in Milnor [23] guarantees a diffeomorphic retraction of Σ onto C, this forces Σ to be diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{k-1}$.

At $C(\Sigma^k)$ we have that $N = \frac{\overrightarrow{x}}{|x|}$ is an unit normal vector for Σ^k, this implies that eigenvectors of $\mathrm{Hess}_\Sigma f$ are the eigenvectors of $A_N := -\overline{\nabla} N$. As $\overline{\lambda}_1 = 0$, we obtain that

$$\left|\nabla^2_\Sigma f|_{C(\Sigma)^\perp}\right|^2 = \sum_{i=2}^{k} \overline{\lambda}_i = k + |x^\perp|^2\,|A_N|^2 \leq \frac{k^2}{k-1} = \frac{(\mathrm{trace}\nabla^2_\Sigma f|_{C(\Sigma)^\perp})^2}{k-1}.$$

Since $\dim C(\Sigma^k) = k-1$, we conclude that $\mathrm{Hess}_\Sigma f|_{C(\Sigma)^\perp} = \lambda\,\mathrm{Id}$. Moreover, as Σ^k is minimal we obtain $\mathrm{trace}\,\mathrm{Hess}_\Sigma f = k$ which implies that $\lambda = \frac{k}{k-1}$ and the Theorem is proved. □

Corollary 3.8. *If Σ^n is a free boundary minimal hypersurface in B^{n+1} such that*

$$|A|^2(x)\,|x^\perp|^2 \leq \frac{n}{n-1} \tag{3.5}$$

for every $x \in \Sigma^n$, then one of the following is true

1. *$C(\Sigma) = \{x_0\}$ and Σ is diffeomorphic to a disk $\mathbb{D}^n$.*

2. $C(\Sigma)$ *is an equatorial circle in* $\mathbb{S}^n(2m_0)$ *and* Σ *is diffeomorphic to* $\mathbb{S}^1 \times \mathbb{D}^{n-1}$. *Moreover, (3.5) becomes equality when* $x \in C(\Sigma^n)$ *and* $B(x)$ *is constant along* $C(\Sigma)$ *with only two principal curvatures:*

$$\frac{-1}{2\,m_0} \quad \textit{and} \quad \frac{1}{2(n-1)m_0}.$$

Proof. The only part to prove is that $C(\Sigma^n)$ is an equator in $\mathbb{S}^n(2\,m_0)$. Since $C(\Sigma^n)$ is a geodesic in Σ^n and since Σ^n is tangent to $\mathbb{S}^n(2\,m_0)$ along $C(\Sigma)$ we conclude that

$$\nabla^{\mathbb{R}^3}_{\gamma'(t)}\gamma'(t) \in T\Sigma^{\perp} = T\mathbb{S}^n(2\,m_0)^{\perp},$$

where $\gamma(t)$ is a parametrization of $C(\Sigma^n)$. This shows that $C(\Sigma^n)$ is also a geodesic in $\mathbb{S}^n(2\,m_0)$, hence, an equator. From (3.2) we have that $\overline{\lambda}_i = 1 + \langle x, N\rangle\lambda_i$, where λ_i are the principal curvatures of Σ^n. As $\overline{\lambda}_1 = 0$ and $\overline{\lambda}_i = \frac{n}{n-1}$ for $i > 1$, the item (2) follows. □

Remark 3.9. Theorem 3.7 e Corollary 3.8 also hold for non-compact properly embedded minimal submanifolds in $\mathbb{R}^{n+1}$. In this setting the pinching curvature implies Σ is either diffeomorphic to $\mathbb{R}^k$ or to $\mathbb{S}^1 \times \mathbb{R}^{k-1}$.

Example 3.10. Let's compute the quantity $|z^{\perp}|^2\,|A(z)|^2$ for the minimal surfaces $\Sigma^{m+1} \subset \mathbb{R}^{m+2}$ constructed in Section 2.

$$\begin{aligned} |z^{\perp}|^2 &= \langle z, N\rangle^2 = (-b'a + a'b)^2 \\ &= (a^2+b^2)(-\sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi))^2 \\ &= (a^2+b^2)\sin^2(\theta-\varphi). \end{aligned}$$

Recall that $|A|^2 = |dN|^2$ and

$$|dN|^2 = \frac{\langle dN(X_s), X_s\rangle^2}{g_{ss}} + \frac{\langle dN(X_x), X_x\rangle^2}{g_{xx}} + \sum_{i=1}^{m-1}\frac{\langle dN(X_{y_i}), X_{y_i}\rangle^2}{g_{y_iy_i}}.$$

Using the expressions for $\langle dN(X_s), X_s\rangle$, $\langle dN(X_x), X_x\rangle$, and $\langle dN(X_{y_i}), X_{y_i}\rangle$

we obtain

$$\begin{aligned}
|A|^2 &= (b''a'-a''b')^2+(\frac{b'}{a})^2+(m-1)(\frac{-a'}{b})^2\\
&= \left((m-1)\frac{a'}{b}-\frac{b'}{a}\right)^2+(\frac{b'}{a})^2+(m-1)(\frac{a'}{b})^2\\
&= (m^2-m)(\frac{a'}{b})^2+2(\frac{b'}{a})^2-2(m-1)\frac{a'b'}{ab}\\
&= \frac{1}{a^2+b^2}\left((m^2-m)\frac{\cos^2(\theta)}{\sin^2(\varphi)}+2\frac{\sin^2(\theta)}{\cos^2(\varphi)}-2(m-1)\frac{\cos(\theta)\sin(\theta)}{\cos(\varphi)\sin(\varphi)}\right)\\
&= \frac{1}{a^2+b^2}\left(\frac{(m^2-m)\cos^2(\theta)\cos^2(\varphi)+2\sin^2(\theta)\sin^2(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right.\\
&\qquad \left. - \quad 2(m-1)\frac{\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|z^\perp|^2|A(z)|^2 &= \sin^2(\theta-\varphi)\left(\frac{(m^2-m)\cos^2(\theta)\cos^2(\varphi)+2\sin^2(\theta)\sin^2(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right.\\
&\qquad \left. - \quad 2(m-1)\frac{\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right).
\end{aligned}$$

If we restrict to the case $m=2$, which corresponds to a 3-dimensional minimal surface $\Sigma^3\subset\mathbb{R}^4$, then we get

$$\begin{aligned}
|z^\perp|^2|A(z)|^2 = 2\sin^2(\theta-\varphi)&\left(\frac{\cos^2(\theta)\cos^2(\varphi)+\sin^2(\theta)\sin^2(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right.\\
&\left. - \quad \frac{\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right)
\end{aligned}$$

$$\begin{aligned}
|z^\perp|^2|A(z)|^2 &= 2\sin^2(\theta-\varphi)\left(4\,\frac{\cos^2(\theta+\varphi)}{\sin^2(2\varphi)}+\frac{\sin(2\theta)\sin(2\varphi)}{\sin^2(2\varphi)}\right)\\
&= 2\frac{4\sin^2(\theta-\varphi)\cos^2(\theta+\varphi)}{\sin^2(2\varphi)}+2\sin^2(\theta-\varphi)\left(\frac{\sin(2\theta)}{\sin(2\varphi)}\right)\\
&= 2\left(\left(\frac{\sin(2\theta)}{\sin(2\varphi)}-1\right)^2+\sin^2(\theta-\varphi)\frac{\sin(2\theta)}{\sin(2\varphi)}\right). \qquad (3.6)
\end{aligned}$$

Let us show that $|z^\perp|^2|A(z)|^2 = \frac{3}{2}$ for every point $z \in C(\Sigma^3)$. First, note that

$$(a(0)x, b(0)y) = \lim_{t\to-\infty} \big(a(s(t))x, b(s(t))y\big) \in C(\Sigma^3).$$

Let λ_1 be a eigenvalue of dN associated to the principal direction $C(\Sigma^3)$ and $z_0 \in C(\Sigma)$. It follows that $\langle z_0, N\rangle\lambda_1 = 1$ by (3.2). Using that $\gamma'(0) = |\gamma(0)|(1,0)$, $\gamma'(0) = (0,1)$ and applying the L'Hôpital's rule we obtain that the other two eigenvalues of dN are equal to $-a''(0)$ at $C(\Sigma^3)$. Since Σ^3 is minimal, we obtain that $2\,a''(0) = \lambda_1$. Therefore,

$$|z_0^\perp|^2|A(z_0)|^2 = \langle z, N\rangle^2(\lambda_1^2 + 2\,(a'')^2) = 1 + \frac{1}{2} = \frac{3}{2}. \tag{3.7}$$

In particular, if one can show that the right hand side of (3.6) is a monotone decreasing function on $[0, s_0]$, then Σ^3 will satisfy $|z^\perp|^2|A(z)|^2 \le \frac{3}{2}$. Let's show that this is true for every $s \in [0,\delta)$ where δ is very small. By (3.7) and L'hôpital's rule we have that

$$\lim_{s\to0}(\varphi'(s), \theta'(s)) = \frac{1}{\sqrt{a^2+b^2}}(1, -\frac{1}{2}).$$

Similarly, using the formulas for $\theta'(s)$ and $\varphi'(s)$ computed earlier we obtain that $\lim_{s\to0}(\varphi''(s), \theta''(s)) = (0,0)$. Let h be the function $h(s) = \frac{\sin(2\theta(s))}{\sin(2\varphi(s))}$, then

$$h'(s) = 2\,\frac{\cos(2\theta)}{\sin(2\varphi)}\,\theta'(s) + \sin(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\varphi'(s).$$

L'hôpital's rule once more gives that $h'(0) = 0$. The second derivative is

$$h''(s) = \left(-4\frac{\sin(2\theta)}{\sin(2\varphi)}\,\theta' + 2\cos(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\varphi'\right)\theta' + 2\,\frac{\cos(2\theta)}{\sin(2\varphi)}\,\theta'' +$$

$$\left(2\cos(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\theta' + \sin(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)''\varphi'\right)\varphi' + \sin(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\varphi''.$$

All quantities above have a limit as $s \to 0$ by L'hôpital's rule. Since $(\varphi'(0), \theta'(0)) = \frac{1}{\sqrt{a^2+b^2}}(1, -\frac{1}{2})$ and $(\varphi''(0), \theta''(0)) = (0,0)$ we obtain

$$h''(0) = \left(\frac{\sin(2(\frac{\pi}{2} - \frac{1}{2}\frac{t}{\sqrt{a^2+b^2}}))}{\sin(2\frac{t}{\sqrt{a^2+b^2}})}\right)''(0) = \frac{1}{2(a^2+b^2)}.$$

As $(|z^\perp|^2|A(z)|^2)'(s) = 4(h-1)h' + 2\sin(2\theta-2\varphi)(\theta-\varphi)'h + 2\sin^2(\theta-\varphi)h'$, we first obtain that $(|z^\perp|^2|A(z)|^2)'(0) = 0$. Similarly,

$$(|z^\perp|^2|A(z)|^2)''(s) = 4h'^2 + 4(h-1)h'' +$$

$$4\cos(2\theta-2\varphi)(\theta-\varphi)'^2h + 2\sin(2\theta-2\varphi)(\theta-\varphi)''h + \sin(2\theta-2\varphi)(\theta-\varphi)'h'$$

$$+ 2\sin(2\theta-2\varphi)(\theta-\varphi)'h + 2\sin^2(\theta-\varphi)h''.$$

$$\left(|z^\perp|^2|A(z)|^2\right)''(0) = -\frac{1}{a^2+b^2} - \frac{9}{2(a^2+b^2)} + \frac{1}{a^2+b^2} = -\frac{9}{2(a^2+b^2)} < 0.$$

Hence, $(|z^\perp|^2|A(z)|^2)(s)$ is a decreasing function on $[0,\delta)$ for some $\delta > 0$ very small. Therefore, Σ^3 satisfies t $|z^\perp|^2|A(z)|^2 \leq \frac{3}{2}$ in a neighborhood of $C(\Sigma)$.

Remark 3.11. The function $|z^\perp|^2|A(z)|^2(s)$ is not a decreasing function on the whole interval $[0,\infty)$ since $|z^\perp|^2(s)$ vanishes infinitely often. Regardless of that, we still expect Σ^3 to satisfy $|z^\perp|^2|A(z)|^2 \leq \frac{3}{2}$. The argument above showed that a capillary piece of Σ^3 in the unit ball B^4 satisfies the curvature inequality $|z^\perp|^2|A(z)|^2 \leq \frac{3}{2}$.

Remark 3.12. The computations in Example 3.10 imply that a minimal hypersurface $\Sigma^3 \subset B^4$ satisfying item (2) in Corollary 3.8 must be tangential to second order with the unique (up to scaling) $O(2)\times O(2)$ invariant minimal hypersurface along the equatorial circle $C(\Sigma)$. We expect that these tangential contact points to have infinite order. If this is true, then the second rigidity statement in Conjecture 3.2 would follow from the analiticity of the minimal surface equation.

3.2 An interesting example

Example 3.13. The n-dimensional catenoid, denoted by Σ_c, is the minimal surface $SO(n)$-invariant in $\mathbb{R}^{n+1}$. By Lemma 2.1, Σ_c can be parametrized by $\{(r, b(r)y) : r \in I,\ y \in \mathbb{S}^{n-1}\}$ where $b(s)$ satisfies

$$\frac{1}{1+b'^2}b'' - \frac{n-1}{b} = 0, \quad b(0) > 0, \quad b'(0) = 0.$$

Note that $\gamma(r) = (r, b(r))$ satisfies $\langle \gamma, \gamma'' \rangle \geq 0$. A simple computation will give the following:

$$\mathrm{Hess}_{\Sigma_c} f\,(\partial_s, \partial_s) = 1 + a\,a'' + b\,b'',$$

where $f(x) = \frac{1}{2}|x|^2$ and $\gamma(s) = (a(s), b(s)$ is a arc length parametrization of $\gamma(r) = (r, b(r))$. Since $\langle \gamma, \gamma'' \rangle \geq 0$, we conclude that $\nabla^2_{\Sigma_c} f \geq 0$.

Proposition 3.14. *Let Σ^n be a minimal hypersurface in $\mathbb{R}^{n+1}$ which satisfies Hess$_\Sigma f \geq 0$. If dim $C(\Sigma) = n-1$, then Σ^n is isometric to the n-dimensional catenoid.*

Proof. Since $\mathrm{Hess}_\Sigma f \geq 0$ and $\dim(C(\Sigma)) = n-1$, we have that $C(\Sigma) \subset \mathbb{S}^{n-1}$ up to scaling. Without loss of generality, let us assume that $C(\Sigma) = \mathbb{S}^{n-1}$. Let $J_\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be an one parameter family of isometries of $\mathbb{S}^{n-1}$, i.e., J_θ is a curve on $SO(n)$. Noting that J_θ is also a curve on $SO(n+1)$ also. Consider the function ϕ defined on Σ given by $\phi(x) = \langle \frac{dJ_\theta}{d\theta}(x), N(x) \rangle$. Since Σ is a minimal hypersurface, we have

$$\Delta\phi + |A|^2\,\phi = 0.$$

Note also that $\phi \equiv 0$ on $C(\Sigma)$. Let us now look at the gradient of ϕ on $C(\Sigma)$. If $v \in C(\Sigma) = \mathbb{S}^{n-1}$, then $v(\phi) = 0$ since $J_\theta \in SO(n)$. Assuming now that $v \in T_x\Sigma$ and is orthogonal to $C(\Sigma)$, then

$$v(\phi) = \langle \overline{\nabla}_v \frac{dJ_\theta}{d\theta}, N \rangle + \langle \frac{dJ_\theta}{d\theta}, dN(v) \rangle = 0 + 0 = 0,$$

since v is a principal direction of Σ by Lemma 3.2. By Aronsajn [5], the function ϕ can only vanish to finite order on $C(\Sigma)$. On the other hand, since $\phi = \nabla\phi = 0$ on $C(\Sigma)$, we obtain by Lemma 1.9 in [18] that $\{\phi = 0\} \cap \{\nabla\phi = 0\}$ has Hausdorff dimension at most $n-2$. As this contradicts that $\dim(C(\Sigma)) = n-1$, we have that $\phi \equiv 0$. Hence, Σ is $SO(n)$ invariant, i.e. Σ is isometric to a n-dimensional catenoid. □

Remark 3.15. If Σ^n is a free boundary minimal hypersurface in B^{n+1} satisfying $\mathrm{Hess}_\Sigma f \geq 0$, then one can easily check that

$$|\mathrm{Hess}_\Sigma f|^2 \leq (\mathrm{trace}(\mathrm{Hess}_\Sigma f))^2.$$

Since $\mathrm{Hess}_\Sigma f(X,Y) = \langle X,Y\rangle + \langle A(X,Y), \overrightarrow{x}\rangle$ by Lemma 3.4, we obtain that $|\mathrm{Hess}_\Sigma f|^2 = n + |x^\perp|^2|A|^2$. On the other hand, since $\mathrm{trace}(\mathrm{Hess}_\Sigma f) = n$ by the minimality of Σ, we conclude that

$$|x^\perp|^2\,|A(x)|^2 \leq n(n-1).$$

In contrast with our previous pinching curvature assumption, this inequality is not strong enough to imply that $Hess_\Sigma f \geq 0$

4 Willmore Energy

In this section, we will prove a ε-regularity for free boundary cmc surfaces which can be interpreted as a natural generalization of the case of minimal surfaces in B^3 proved by Ketover in [21, Proposition 2.1]. The quantity to consider in this case is the Willmore energy instead of area. At the end we state a higher dimension and co-dimension generalization regarding the area of minimal submanifolds in B^n obtained by these authors in [7].

Definition 4.1. Let Σ^2 be a surface with boundary in a space form $\mathbb{M}^{n+1}(c)$. The Willmore energy $\mathcal{W}(\Sigma)$ of Σ is defined as follows

$$\mathcal{W}(\Sigma) = \int_\Sigma (H^2 + c)\, d_\Sigma + \int_{\partial\Sigma} k_g\, d\sigma\,,$$

where k_g is the geodesic curvature of $\partial\Sigma$.

The following result is a sharp inequality for the Willmore energy.

Theorem 4.2 (Ros-Vergasta [25], Fraser-Schoen [14], Volkmann [27])**.** *Let Σ^2 be a free-boundary surface in a ball B^{n+1}. Then*

$$\mathcal{W}(\Sigma) \geq 2\pi\,.$$

Equality holds if, and only if, Σ is umbilical: either an equatorial disk or a spherical cap.

In the case of free boundary minimal submanifolds in B^n we have similar rigidity result due to Brendle:

Theorem 4.3 (Brendle [8])**.** *Let Σ^k be a k-dimensional free boundary minimal surface in B^n. Then*

$$|\Sigma^k| \geq |D^k|$$

Moreover, the equality holds if, and only if, Σ^k is contained in a k-dimensional plane in $\mathbb{R}^n$.

We observe that J. Zhu [29] obtained Brendle's sharp area lower bound in the context of space forms for codimension one free boundary minimal hypersurfaces. For the higher codimensional case, Freidin-McGrath ([16], [17]) obtained some partial results.

The main result in this section is the existence of a gap result for the Willmore energy of constant mean curvature surfaces.

Theorem 4.4. *There exists $\varepsilon > 0$ so that whenever Σ is a free boundary surface with constant mean curvature in a unit ball B^3 satisfying*

$$2\pi \,\leq\, \mathcal{W}(\Sigma) \,<\, 2\pi + \varepsilon,$$

then Σ is either an equatorial disk or a spherical cap. The constant ε is independent of the value of the mean curvature.

Remark 4.5. We note that, as the Wilmore energy is conformal invariant in dimension two. In particular, the result above is also true for space forms.

The key in proving Theorem 4.4 is the following excess inequality proved by Vokmann [27]:

$$\mathcal{W}(\Sigma) - 2\pi \geq \int_\Sigma \left| \overrightarrow{H} - \frac{(x-x_0)^\perp}{|x-x_0|^2} \right|^2 d_\Sigma. \tag{4.1}$$

The equality $\mathcal{W}(\Sigma) = 2\pi$ if, and only if, Σ is a spherical cap or a flat disk.

Proof of Theorem 4.4. Assume for every i that Σ_i is a free boundary surface with constant mean curvature H_i and

$$\mathcal{W}(\Sigma_i) \to 2\pi. \tag{4.2}$$

We start showing that (4.2) implies that $|\Sigma_i| \leq C_0$ for some $C_0 > 0$. Indeed, the free boundary condition and the 1Âª Minkowski formula gives that

$$2|\Sigma_i| = |\partial\Sigma_i| - \int_{\Sigma_i} \langle H_i, x\rangle d_{\Sigma_i} \leq |\partial\Sigma_i| + \left(\int_{\Sigma_i} H_i^2 d_{\Sigma_i}\right)^{\frac{1}{2}} |\Sigma_i|^{\frac{1}{2}}. \tag{4.3}$$

By our assumption $|\partial\Sigma_i| \leq 3\pi$ and $\int_{\Sigma_i} H_i^2\, d\Sigma \leq 3\pi$. Hence, $|\Sigma_i| \leq C$ after dividing both sided by $|\Sigma_i|^{\frac{1}{2}}$. Following [21], we show that (4.2) and (4.1) imply curvature estimates for Σ_i.

Lemma 4.6. *Either Σ_i is totally umbilical or there is $C > 0$ such that*

$$\sup_{\Sigma_i} |A_{\Sigma_i}| \leq C. \tag{4.4}$$

Assuming Lemma 4.6, one can finish the proof of the theorem. Indeed, by Lemma 4.6 and Theorem 6.1 in [22], the sequence Σ_i converges to a weakly embedded free boundary surface Σ_∞ in B^3 with constant mean curvature and satisfying $\mathcal{W}(\Sigma_\infty) = 2\pi$. Here, weakly means that Σ_∞ might have some tangential self-intersection. Theorem 4.1 in [27] deals with immersed surfaces and it follows that Σ_∞ is either a spherical cap or a flat disk. Therefore, Σ_i is topologically a disk for i large enough and the result follows from Nitsche's Theorem. □

Proof of Lemma 4.6. Let $\lambda_i = \sup_{x\in\Sigma_n} |A_i|^2(x)$ and assume that λ_i diverges to ∞. For each i choose $x_i \in \Sigma_i$ with the property that $\sup_{\Sigma_i} |A_i|^2 = |A_i|^2(x_i)$ and consider the new surface $\hat{\Sigma}_i = \lambda_i(\Sigma_i - x_i)$ which satisfies

$$\sup_{x\in\hat{\Sigma}_i} |A|(x) \leq 1 \quad \text{and} \quad |A_{\hat{\Sigma}_i}|(0) = 1. \tag{4.5}$$

The surface $\hat{\Sigma}_i$ is a free boundary surface with constant mean curvature $\hat{H}_i$ in the region $\lambda_i(B_1^3(0) - x_i)$. Property (4.5) implies by Theorem 6.1

in [22] that $\hat{\Sigma}_i \to \Sigma_\infty$ graphically in the interior. Σ_∞ is either weakly embedded without boundary or it is weakly embedded in a half space and with boundary contained in a plane. By regularity results for free boundary surfaces, see [1], we also have smooth convergence up to the boundary. It follows from (4.5) that

$$|A_{\Sigma_\infty}|(0) = 1 \quad \text{and} \quad \lim_{i\to\infty} \hat{H}_i = H_\infty.$$

By the excess formula (4.1), we conclude that

$$\begin{aligned}
&\int_{\Sigma_\infty} \left| \overrightarrow{H}_\infty - \frac{x^\perp}{|x|^2} \right|^2 d_{\Sigma_\infty} \\
&\leq \liminf_{i\to\infty} \int_{\hat{\Sigma}_i} \left| \overrightarrow{\hat{H}}_i - \frac{x^\perp}{|x|^2} \right|^2 d_{\Sigma_\infty} = \liminf_{i\to\infty} \int_{\Sigma_i} \left| \overrightarrow{H}_i - \frac{(x-x_i)^\perp}{|x-x_i|^2} \right|^2 \\
&\leq \liminf_{i\to\infty} \mathcal{W}(\Sigma_i) - 2\pi = 0.
\end{aligned}$$

Hence, $\overrightarrow{H}_\infty = \frac{x^\perp}{|x|^2}$. If Σ_∞ is non-compact, then $\overrightarrow{H}_\infty = 0$ and this implies that Σ_∞ is a plane, a contradiction since $A_{\Sigma_\infty}(0) = 1$. Hence, Σ_∞ is compact and $\overrightarrow{H}_\infty = \frac{x^\perp}{|x|^2}$. Applying Proposition 4.1 in [27], we conclude that Σ_∞ is a spherical cap or an equatorial disk. The strong convergence implies that Σ_i is a topologically a disk and, hence, a spherical cap by Nitsche's Theorem. □

We briefly mention that we have similar rigidity estimate in every dimension and codimension if we restrict to the area functional. The proof builds on the methods developed in Brendle [8] and the key point is to deal with the lack of similar excess inequality such as (4.1) in this other setting.

Theorem 4.7 (Barbosa-Viana [7]). *There exists $\varepsilon(k,n) > 0$ such that whenever Σ^k is a k-dimensional free boundary minimal surface in B^n satisfying*

$$|\Sigma^k| < |D^k| + \varepsilon(k,n),$$

then Σ^k is, up to ambient isometries, the equatorial disk D^k.

5 Towards graphical uniqueness for spherical caps

Proposition 5.1. *Let Σ^n be a free boundary constant mean curvature hypersurface in B^{n+1} with $H > 0$. If Σ is a graph , then*

$$\frac{\langle x, N\rangle}{1-|x|^2} \geq \frac{H}{2n}.$$

The inequality is strict unless Σ is a spherical cap. In particular, Σ is strictly star-shaped.

We remark that Σ does not need to be a disk in Proposition 5.1 since the result is independent of the number of boundary components of Σ. The proof of Proposition 5.1 is inspired by previous work of G. Wheeler and V.-M Wheeler [28] which proved that a smooth free boundary minimal graph in B^n is an equatorial disk.

Under a stronger star-shaped property, we obtain:

Corollary 5.2. *Let Σ^n be a free boundary constant mean curvature hypersurface in B^{n+1}. If Σ is a graph and satisfies*

$$\frac{\langle x, N\rangle}{1-|x|^2} \geq \frac{|A|^2}{2H}, \tag{5.1}$$

for every $x \notin \partial\Sigma$, then (5.1) becomes equality and Σ is a spherical cap.

Proof of Proposition 5.1. Let Φ be the function defined as

$$\Phi = \frac{1}{2}(|x|^2-1)H + n\langle x, N\rangle.$$

It is well known that $\Delta\langle x, N\rangle = -|A|^2\langle x, N\rangle - H$ and $\Delta\frac{|x|^2}{2} = n + H\langle x, N\rangle$. Hence,

$$\Delta\Phi = -(n|A|^2 - H^2)\langle x, N\rangle. \tag{5.2}$$

The free boundary condition implies that Φ restricted to $\partial\Sigma$ is identically zero. Arguing by contradiction, let us suppose that $\Phi < 0$ in some open set $U \subset \Sigma$. Hence, $\Phi = 0$ at ∂U. From (5.2), we immediately obtain

$$\Delta\Phi^2 = -2\Phi(n|A|^2 - H^2)\langle x, N\rangle + 2|\nabla\Phi|^2.$$

Since Σ is a graph, let us consider the function $\varphi^2 = \frac{1}{\langle N,a\rangle^2}$. Here we are assuming that $\langle N, a\rangle > 0$ also on the boundary. One can easily check that $\Delta\langle N, a\rangle + |A|^2\langle N, a\rangle = 0$. Hence,

$$\Delta\varphi^2 = 2|A|^2\varphi^2 + 6|\nabla\varphi|^2.$$

Now we define the function $Q = \Phi^2\,\varphi^2$ and verify that

$$\begin{aligned}\Delta Q &= \Phi^2\Big(2\,|A|^2\,\varphi^2 + 6\,|\nabla\varphi|^2\Big) + 2\langle\nabla\Phi^2, \nabla\varphi^2\rangle \\ &\qquad + \Big(-2\,\Phi\,(n|A|^2 - H^2)\langle x, N\rangle + 2|\nabla\Phi|^2\Big)\varphi^2.\end{aligned}$$

$$\begin{aligned}\Delta Q &= -2\varphi^2\big(n|A|^2 - H^2\big)n\Phi\left(\frac{\Phi}{n} - \frac{|x|^2-1}{2}\frac{H}{n}\right) + 2\Phi^2|A|^2\varphi^2 \\ &+ \Phi^2|\nabla\varphi|^2 + 2\varphi^2|\nabla\Phi|^2 + \left\langle\frac{\nabla Q}{\varphi^2} - \frac{\Phi^2}{\varphi^2}\nabla\varphi^2, \nabla\varphi^2\right\rangle + \langle\nabla\Phi^2, \nabla\varphi^2\rangle \\ &= \varphi^2\Phi H\big(|x|^2-1\big)\left(|A|^2 - \frac{H^2}{n}\right) + \frac{2\varphi^2H^2\Phi^2}{n} \\ &+ 4\varphi\Phi\langle\nabla\Phi, \nabla\varphi\rangle + 2\varphi^2|\nabla\Phi|^2 + 2\Phi^2|\nabla\varphi|^2 + \left\langle\nabla Q, \frac{\nabla\varphi^2}{\varphi^2}\right\rangle \\ &= \varphi^2\Phi H(|x|^2-1)\left(|A|^2 - \frac{H^2}{n}\right) + \frac{2\varphi^2H^2\Phi^2}{n} \\ &\qquad +2|\varphi\nabla\Phi + \Phi\nabla\varphi|^2 + \left\langle\nabla Q, \frac{\nabla\varphi^2}{\varphi^2}\right\rangle \\ &\geq \left\langle\nabla Q, \frac{\nabla\varphi^2}{\varphi^2}\right\rangle.\end{aligned} \tag{5.3}$$

The Maximum Principle applied to (5.3) implies that Q does not have an interior maximum in U. Since $Q \geq 0$ over Σ and $Q|_{\partial\Sigma} = 0$, we obtain that $Q \equiv 0$. Hence, $\Phi \equiv 0$ and by (5.2) we obtain that either $n|A|^2 - H^2 \equiv 0$ or $\langle x, N\rangle \equiv 0$ (minimal cones). Therefore, Σ is a spherical cap on U, a contradiction since $\Phi \equiv 0$ on spherical caps. The second statement follows from the Maximum Principle applied to (5.2) □

Proof of Corollary 5.2. It follows from Proposition 5.1 that

$$\Phi = \frac{1}{2}(|x|^2 - 1)H + n\langle x, N\rangle \geq 0.$$

Recall the function $Q = \Phi^2\varphi^2$. The free boundary condition implies that Q restricted to $\partial\Sigma$ is identically zero. A straightforward rearrangement in equation (5.3) using definition of Φ gives

$$\begin{aligned} \Delta Q &= \varphi^2 \Phi H\Big((|x|^2-1)|A|^2 + 2H\langle x, N\rangle\Big) \\ &+ 2|\varphi\nabla\Phi + \Phi\nabla\varphi|^2 + \left\langle \nabla Q, \frac{\nabla\varphi^2}{\varphi^2}\right\rangle \geq \left\langle \nabla Q, \frac{\nabla\varphi^2}{\varphi^2}\right\rangle. \end{aligned}$$

By the Maximum Principle, Q has an interior maximum only if it is constant. Since $Q \geq 0$ over Σ and $Q|_{\partial\Sigma} = 0$, we obtain that $Q \equiv 0$ and $\Phi \equiv 0$. By (5.2), we obtain that $n|A|^2 - H^2 \equiv 0$; hence, Σ is a spherical cap. □

Remark 5.3. The proofs of Proposition 5.1 and Corollary 5.2 straightforwardly extend to the setting of capillary constant mean curvature hypersurfaces in the ball B^{n+1}, i.e., surfaces that meet the boundary ∂B^{n+1} at a constant angle $\theta \in (0, \frac{\pi}{2}]$. The key point is to consider the function $\Phi = \frac{1}{2}(|x|^2 - 1)H + n\big(\langle x, N\rangle - \cos(\theta)\big)$ which still obeys the equation (5.2) and satisfy the crucial Dirichlet boundary condition needed in the proof.

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