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Travelling wave solutions for an internal wave model

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Abstract. In this work we consider a reduced system that models internal waves at the interface of two immiscible, inviscid, incompressible and irrotational fluids limited by a rigid lid at the top and a flat bottom. The system was previously derived by asymptotic expansions from the Euler equations for each layer and the complementary conditions that coupled them and consists of two integro-differential evolution equations for the interface wave profile and the upper layer averaged horizontal velocity. The aforementioned system is further reduced here to a single integro-differential evolution equation (the Regularized Benjamin equation) for the interface wave profile, in order to prove the existence of smooth periodic travelling wave solutions which capture the behaviour of internal waves observed in the ocean layers.

Keywords: internal waves, Regularized Benjamin equation, travelling waves, asymptotic methods, fixed point index methods.

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1 Introduction

Internal waves are gravity waves that move within stratified fluid layers and can travel for long distances maintaining well defined shapes. Along the vertical direction, oceans exhibit abrupt density variations due mainly to differences in temperature and salinity, which generate stratification in nearly horizontal layers separated by relatively thin interfaces along which internal waves evolve. Internal waves are essential in natural processes like transportation and mixing of nutrients and substances in oceans and are important for marine and submarine human activity. For example, vessels operations can be affected by the so called dead water phenomenon when they travel over a thin layer of fresh water which remains immiscible with the denser lower layer, as reported by Nansen in [27]. Internal waves also appear in lakes, reservoirs and in different layers of the atmosphere. For a broad discussion about internal waves, see [33].

Inspired by the surface waves models for a single fluid layer that arised as reductions of the Euler equations in two spatial variables, there are several internal waves models that follow a similar approach, see for example [24, 15, 16, 28, 5, 32, 35, 19, 2]. The classical reduced surface wave model is the Boussinesq partial differential equations system for shallow waters [12, 34]. It is obtained by averaging the velocity vector components along the vertical variable in the Euler equations and then performing asymptotic expansions. In this way, the Boussinesq system involves only two unknown scalar functions (the surface perturbation and the vertically averaged horizontal velocity), both depending merely on the horizontal spatial variable and the temporal variable. This reduces the study of the evolution of the surface and the velocity vector in the two-dimensional domain occupied by the fluid to a one-dimensional system that preserves the main features of the surface wave. The Boussinesq system can be further reduced by restricting the propagation of the wave to one direction, leading to the KdV equation [34], which is of special interest in the study of travelling and solitary waves. Returning to the case of internal waves, at least two layers need to be considered and the same reduction by averaging and asymptotic expansions can be done in each layer or in one of them [16]. The connection between layers occurs through the continuity of the pressure condition.

A systematic and unifying approach for deriving a wide class of asymptotic models for internal waves at the interface between two layers can be found in [11], where the authors obtained a general reduced system which involves two nonlocal operators and then, performing asymptotic expansions of those nonlocal operators, different asymptotic models are obtained depending on the scaling regime, recovering the models obtained by Hamiltonian formalism in [17], and also introducing new ones. Both works intended to capture in their own uniform and general approaches the largest class of internal wave models as possible and are excellent references to summarize and compare the extensive amount of reduced models available in the literature. Notably, the systems considered in [15, 16] are important guides in these two general works; from them, again restricting the propagation of the wave to one direction, equations with travelling wave solutions like KdV type equations, the ILW equation ([22, 24]) and the Benjamin-Ono (BO) equation ([8, 18]) were obtained for shallow, deep (referred here as intermediate) and infinitely deep lower layer configuration, respectively. Travelling wave solutions are able to capture the behaviour of well-identified disturbances that move with practically constant speed for long periods of time as described in [20, 4] and are also of great theoretical interest.

In this work we consider a simplified two dimensional model consisting of a two-layer fluid configuration limited by a rigid lid at the top and a flat bottom as illustrated in Figure 1.1. Fluids of different constant densities $\rho_1 < \rho_2$ occupy each layer, the denser one is located below. The two fluids are immiscible, inviscid, incompressible and irrotational. The coordinate system is set at the undisturbed interface between the layers. The thickness $h_1 > 0$ of the upper layer is much smaller than the characteristic wavelength $L > 0$ at the interface, this is precisely the shallow water regime.

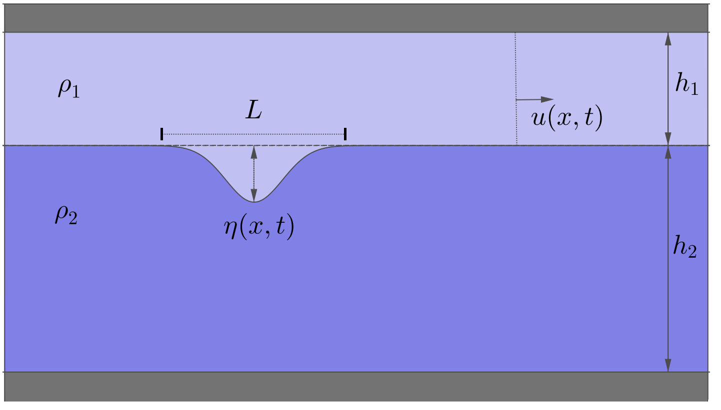

Figure 1.1: Two fluids configuration.

The thickness $h_2 > 0$ of the lower layer is comparable with L, which corresponds to an intermediate water regime. An internal wave travels at the interface between the layers and the whole system evolves according to the Euler equations together with appropriate boundary conditions as described in [16], where the authors proposed a dispersive strongly nonlinear reduced model which was improved in [31, 32] to a higher order asymptotic expansion model that includes the more general situation of an arbitrary irregular bottom. In the flat bottom case and with normalized shallow water velocity, the aforementioned higher order system reads

$$\begin{cases} \eta_t = \left[(1-\eta)u\right]_x, \\ u_t + u\,u_x - \eta_x = \sqrt{\beta}\,\frac{\rho_2}{\rho_1}\mathcal{T}_\delta\left[(1-\eta)u\right]_{xt} + \frac{\beta\rho_2}{2\rho_1}\left(\mathcal{T}_\delta\left[\left((1-\eta)u\right)_x\right]^2\right)_x \\ +\frac{\beta}{3(1-\eta)}\left((1-\eta)^3\left(u_{xt} + u\,u_{xx} - u_x\,u_x\right)\right)_x + \beta\,\frac{\rho_2}{\rho_1}\mathcal{T}_\delta\left[\eta\,\mathcal{T}_\delta\left[(1-\eta)u\right]_x\right]_{xt} \\ +\beta\,\frac{\rho_2}{\rho_1}\left(\eta\left((1-\eta)u\right)_{xt} + \frac{1}{2}\left((1-\eta)u\right)_x^2\right)_x + \mathcal{O}\left(\beta^{3/2}\right). \end{cases} \tag{1.1}$$

System (1.1) is already written in nondimensional variables, x and t

represent the spatial and temporal variables, $\eta(x,t)$ describes the interface between the fluids and $u(x,t)$ accounts for the upper layer vertically averaged horizontal velocity. Subscripts x and t stand for partial derivatives. The nondimensional dispersion parameter $\beta = (h_1/L)^2$, which is small because of the relative sizes of h_1 and L, plays a fundamental role in the asymptotic expansion which yields terms with successive powers of $\sqrt{\beta}$, including terms of order β discarded in the system proposed in [16] and discarding powers of order $\beta^{3/2}$. Nevertheless, the first equation is exact, this means that it is a direct consequence of the Euler equations and no approximation from the asymptotic expansion was made [31, 32]. There are also terms involving the integral operator $\mathcal{T}_\delta$ known as Hilbert transform on the strip which is very important because it carries the information from the lower layer to the upper layer averaged equations up to the order required in the asymptotic expansion. This is achieved by means of the continuity of the pressure condition and in such a way that there is no need for equations from the lower layer in the reduced system (1.1). Specifically, $\mathcal{T}_\delta$ appears in the asymptotic reduction process of the lower layer Euler equations when transforming the normal derivative data of the Neumann problem involving the Laplace equation for the lower layer velocity potential into the tangential derivative along the interface at rest. In order to obtain said tangential derivative, the aforementioned Neumann problem is treated in the frequency domain and the symbol of the nonlocal operator $\mathcal{T}_\delta$, in the $2l$-periodic case, $l > 0$, results to be

$$\widehat{\mathcal{T}_\delta[f]}(k) = \boldsymbol{i}\coth(\delta k\pi/l)\widehat{f}(k), \quad k \in \mathbb{Z}^*,$$

where $\delta = h_2/L$ and $\boldsymbol{i}$ denotes the imaginary unit. The integral operator $\mathcal{T}_\delta$ applied to the normal derivative data returns the corresponding tangential derivative along the interface at rest which allows to substitute the pressure term in the upper layer averaged equations with an expression originated from the lower layer Euler equations and involving only the unknown η, up to the order required in the asymptotic expansion.

The explicit expression of $\mathcal{T}_\delta$ in the spatial domain, though it is not used in the present work, was obtained in [1] and consists of the following principal value integral

$$\mathcal{T}[f](x) = \frac{1}{2l}\, PV\!\int_{-l}^{l} \tilde{T}(x-\xi;\delta,l) f(\xi) d\xi$$

describing a convolution with kernel $\widetilde{T}$ given by

$$\widetilde{T}(x;\delta,l) = -\frac{2K}{\pi}\left[\mathrm{Z}\left(\frac{Kx}{l}\right) + \mathrm{dn}\left(\frac{Kx}{l}\right)\mathrm{cs}\left(\frac{Kx}{l}\right)\right],$$

where the Jacobi Zeta function Z and the Jacobian elliptic functions dn and cs all have modulus k, determined by the condition $K'(k)/K(k) = \delta/l$, that involves the associated elliptic integral of the first kind $K'(k)$ and the complete elliptic integral of the first kind $K = K(k)$, that is,

$$K = K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}} d\theta,\ K'(k) = K\left(\sqrt{1-k^2}\right).$$

For more details about the operator $\mathcal{T}_\delta$ and its properties, see [31, 3, 13, 1, 7, 6].

System (1.1) is a strongly nonlinear reduced model because no restriction is imposed on the wave amplitude. Introducing the nonlinearity parameter α, which is the ratio between the typical absolute wave amplitude value and the thickness h_1, different weakly nonlinear reduced models can be obtained depending on the relation between α and β. Despite the restrictive amplitude, weakly nonlinear models capture relevant wave behaviours and are more treatable theoretically and numerically. The nonlinearity parameter is also nondimensional and assuming a weakly nonlinear wave propagation regime with small absolute wave amplitude if compared with h_1, in such a way that α is of the same order of β, ($\alpha = \mathcal{O}(\beta)$), additional asymptotic expansions of the variables involved in system (1.1) can be performed. That is, by scaling $\eta = \alpha\eta^*$, $u = \alpha u^*$, and omitting the asterisks, the following dispersive, weakly nonlinear reduced model was derived from the strongly nonlinear system (1.1) in [13] and also in [25, 26], after discarding terms of order $\alpha\sqrt{\beta}, \alpha\beta$ and so on:

$$\begin{cases} \eta_t - \big[(1-\alpha\eta)u\big]_x = 0, \\ u_t + \alpha u\, u_x - \eta_x = \sqrt{\beta}\,\dfrac{\rho_2}{\rho_1}\mathcal{T}_\delta\left[u_{xt}\right] + \dfrac{\beta}{3}u_{xxt} + \mathcal{O}\left(\alpha\sqrt{\beta}, \beta^{3/2}\right). \end{cases} \tag{1.2}$$

System (1.2) provides an approximation for the original Euler problem of order $\mathcal{O}\left(\alpha\sqrt{\beta}, \beta^{3/2}\right)$, according to the expansion in the second equation. As pointed out in [2], where system (1.2) is used as a starting point to derive other asymptotic models, it is also a particular case of the system of equations (5.40) and (5.41) derived in [15] by a similar but less direct manner due to the generality of the upper top considered there.

The purpose of this work is to further reduce system (1.2) to the single unidirectional equation

$$\eta_t + \eta_x - \frac{3\alpha}{2}\eta\eta_x - \frac{\sqrt{\beta}}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_\delta[\eta_{xt}] - \frac{\beta}{6}\eta_{xxt} = 0, \tag{1.3}$$

to which we refer as a Regularized Benjamin equation (RB) and to prove the existence of travelling wave solutions for it. The proof relies on the fixed point index theory in cones for positive operators on Banach spaces introduced by Krasnosel'skii in [23]. This theory has been successfully applied to prove the existence of travelling wave solutions for different models like the Benjamin equation in [9], a regularized Benjamin-Ono system in [30] and a general kind of dispersive equations in [14]. This technique is predominantly used with the problem formulated in the frequency domain, but it can also be used considering the physical domain as in [10], where FrÃ©chet spaces are also used instead of Banach spaces.

The rest of this article is organized as follows: in section 2 the unidirectional reduction is performed and in section 3 the existence of periodic travelling wave solutions is proven using a fixed point index theory in cones for positive operators on Banach spaces. Final remarks and future developments are considered in the conclusions.

2 Unidirectional reduction to the RB equation

First, let us present the derivation of the regularized equation from system (1.2) using asymptotic expansions. Following [34], let us consider the first order approximation of system (1.2):

$$\begin{cases} \eta_t - u_x = \mathcal{O}(\alpha), \\ u_t - \eta_x = \mathcal{O}(\sqrt{\beta}, \beta, \alpha). \end{cases} \tag{2.1}$$

Since the linear system

$$\begin{cases} \eta_t - u_x = 0, \\ u_t - \eta_x = 0, \end{cases} \tag{2.2}$$

supports travelling wave solutions of the form $u = -\eta$, (among other possibilities), let us propose the following higher-order correction for approximate travelling waves of the nonlinear system (1.2):

$$u = -\eta + \alpha P + \beta Q + \sqrt{\beta} R + \mathcal{O}(\alpha\sqrt{\beta}, \beta^{3/2}), \tag{2.3}$$

where P, Q and R are functions to be determined of η, its derivatives and $\mathcal{T}_\delta[\eta]$. Differentiating (2.3) with respect to t and x we have:

$$u_t = -\eta_t + \alpha P_t + \beta Q_t + \sqrt{\beta} R_t + \mathcal{O}(\alpha\sqrt{\beta}, \beta^{3/2}) \tag{2.4}$$

and

$$u_x = -\eta_x + \alpha P_x + \beta Q_x + \sqrt{\beta} R_x + \mathcal{O}(\alpha\sqrt{\beta}, \beta^{3/2}). \tag{2.5}$$

Substituting expression (2.3) in the first equation of system (1.2), up to the desired order of the expansion we obtain:

$$\eta_t + \eta_x - \alpha(P_x + (\eta^2)_x) - \beta Q_x - \sqrt{\beta} R_x + \mathcal{O}(\alpha\sqrt{\beta}, \beta^{3/2}) = 0. \tag{2.6}$$

Analogously, substituting the expressions (2.3), (2.4) and (2.5) in the second equation of system (1.2), after some simplifications and up to the desired order of the expansion we get:

$$\begin{aligned} -\eta_t - \eta_x + \alpha(P_t + \eta\eta_x) + \beta\left(Q_t + \frac{1}{3}\eta_{xxt}\right) + \\ + \sqrt{\beta}\left(R_t + \frac{\rho_2}{\rho_1}\mathcal{T}_\delta[\eta_{xt}]\right) = \mathcal{O}(\alpha\sqrt{\beta}, \beta^{3/2}). \end{aligned} \tag{2.7}$$

Furthermore, substituting equation (2.5) in the first equation of system (2.1), we see that $\eta_t = -\eta_x + \mathcal{O}(\sqrt{\beta}, \beta, \alpha)$. Since P, Q and R are functions of η, its derivatives and $\mathcal{T}_\delta[\eta]$, they also satisfy that $P_t = -P_x + \mathcal{O}(\sqrt{\beta}, \beta, \alpha)$, $Q_t = -Q_x + \mathcal{O}(\sqrt{\beta}, \beta, \alpha)$ and $R_t = -R_x + \mathcal{O}(\sqrt{\beta}, \beta, \alpha)$, respectively. Therefore, P_t, Q_t and R_t in equation (2.7) can be replaced by minus the corresponding spatial derivative, maintaining the desired order of approximation. Equations (2.6) and (2.7) are then consistent if and only if

$$\begin{cases} P_x = -\dfrac{(\eta^2)_x}{4}, \\ Q_x = \dfrac{1}{6}\eta_{xxt}, \\ R_x = \dfrac{\rho_2}{2\rho_1}\mathcal{T}_\delta[\eta_{xt}]. \end{cases} \tag{2.8}$$

Substituting the equations in (2.8) in equation (2.6) we obtain that

$$\eta_t + \eta_x - \alpha\left(-\frac{1}{2}\eta\eta_x + (\eta^2)_x\right) - \beta\frac{1}{6}\eta_{xxt} - \sqrt{\beta}\frac{\rho_2}{2\rho_1}\mathcal{T}_\delta[\eta_{xt}] + \mathcal{O}(\alpha\sqrt{\beta}, \beta^{3/2}) = 0,$$

which leads us to the Regularized Benjamin equation (1.3), repeated here for convenience:

$$\eta_t + \eta_x - \frac{3\alpha}{2}\eta\eta_x - \frac{\sqrt{\beta}}{2}\frac{\rho_2}{\rho_1}\mathcal{T}_\delta[\eta_{xt}] - \frac{\beta}{6}\eta_{xxt} = 0$$

and also written with general positive parameters as

$$\eta_t + \eta_x - c_1\eta\eta_x - c_2\mathcal{T}_\delta[\eta_{xt}] - c_3\eta_{xxt} = 0. \tag{2.9}$$

3 Existence of periodic travelling wave solutions for the RB equation

In this section we will prove the existence of periodic travelling wave solutions for equation (1.3). We will use the fixed point index theory in cones for positive operators on Banach spaces used to prove the existence of travelling wave solutions for the Benjamin equation in [9], adapting some results and notations from [30].

To begin with, let us reduce the number of parameters in equation (2.9) by performing the change of variables $\eta^* = -\frac{c_1}{2}\eta$ in it to obtain

$$\eta_t + \eta_x + 2\eta\eta_x - c_2\mathcal{T}_\delta[\eta_{xt}] - c_3\eta_{xxt} = 0, \tag{3.1}$$

after dropping the asterisks. Supposing that equation (3.1) admits a travelling wave solution $\eta(x,t) = \varphi(x - ct)$ let us substitute this expression in it. Integrating the resulting equation once and considering the integration constant to be equal to zero yields

$$-c\varphi + \varphi + \varphi^2 + c_2c\mathcal{T}_\delta[\varphi'] + c_3c\varphi'' = 0. \tag{3.2}$$

In fact, the integration constant needs not be zero since it can be removed by a change of variables similar to what was done in [9].

Introducing the change of variables $\varphi = c\varphi^*$, and dropping the asterisks we get

$$B\varphi - c_2\mathcal{T}_\delta[\varphi'] - c_3\varphi'' = \varphi^2, \tag{3.3}$$

where $B = (c-1)/c$. In the following computations it will be necessary that $B > 0$, thus we suppose that $c > 1$. However, if we consider the change of variables $\varphi = c\varphi^* + c - 1$ in equation (3.2), we will obtain equation (3.3), but with a value $B = (1-c)/c$ and we must impose $0 < c < 1$ in order to have $B > 0$. Therefore, the following results can be extended to the case $0 < c < 1$.

Equation (3.3) has two trivial solutions $\varphi \equiv 0$ and $\varphi \equiv B$. Additionally, we will prove that it admits a $2l$-periodic and even non-trivial solution represented by the Fourier series

$$\sum_{k\in\mathbb{Z}} a_k \cos(k\pi x/l), \tag{3.4}$$

where

$$a_k = \frac{1}{l}\int_0^l \varphi(x)\cos(k\pi x/l)dx = a_{-k}.$$

The Fourier coefficients of φ^2 are given by the convolution with itself of the sequence of Fourier coefficients of φ that returns $(a * a)_k = \sum_{m\in\mathbb{Z}} a_{k+m}a_m$, as given in [9]. Substituting expression (3.4) in equation (3.3) we can see that coefficients a_k should satisfy that

$$a_k = \frac{1}{w_k}(a * a)_k,\ k \in \mathbb{Z}, \tag{3.5}$$

where $w_k = B + \frac{c_2}{\delta}\phi(\delta k\pi/l) + c_3(k\pi/l)^2$ and ϕ is the symbol in the frequency domain for the composition of one spatial derivative with operator $\mathcal{T}_\delta$, that is,

$$\phi(k) = \begin{cases} 1, & k = 0, \\ k\coth k, & k \neq 0. \end{cases} \tag{3.6}$$

The sequences of Fourier coefficients corresponding to $\varphi \equiv 0$ and $\varphi \equiv B$ are solutions of equation (3.5). Looking for a non-trivial sequence which will correspond to a non-trivial solution of equation (3.3), we can write equation (3.5) in the operator form $a = \mathcal{A}a$, where $(\mathcal{A}a)_k = w_k^{-1}(a * \ a)_k$. We remark that non-trivial fixed points of operator $\mathcal{A}$ occur in pairs as in [9]. That is, if a is a fixed point of $\mathcal{A}$, so is $a^* = \{a_k^*\}_{k\in\mathbb{Z}}$, where

$$\begin{cases} a_k^* = a_k & \text{if } |k| = 0, 2, 4, \ldots \\ a_k^* = -a_k & \text{if } |k| = 1, 3, 5, \ldots \end{cases}$$

In this way, our original problem of finding travelling waves solutions for equation (3.1) was reformulated as the search of non-trivial fixed points of operator $\mathcal{A}$ (which acts in sequences corresponding to Fourier coefficients) and then we will focus on its properties.

Operator $\mathcal{A} : l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$, where $l_2(\mathbb{Z})$ is the space of all square summable complex valued sequences, is well defined because by the Cauchy-Bunyakovsky-Schwarz inequality, $|(a * a)_k| \leq \|a\|_2^2 = \sum_{k\in\mathbb{Z}} |a_k|^2$, for all $k \in \mathbb{Z}$ and $w^{-1} = \{w_k^{-1}\}_{k\in\mathbb{Z}} \in l_2(\mathbb{Z})$. Thus, we aim to prove the existence of a real sequence a which is a non-trivial fixed point of $\mathcal{A}$. To that end, let us state the main results of the fixed point index theory to be used.

Definition 3.1. A subset K in a Banach space is a cone if the following three statements are valid:

1. $0 \in K$.

2. If $x \in K$, then $\lambda x \in K$ for all $\lambda \geqslant 0$.

3. If $-x$ and x belong to K, then $x = 0$.

Let K be a closed convex cone in a Banach space and $A : K \to K$ a continuous and compact operator. If $\Omega \subset K$ is open in the relative topology of K and there are no fixed points of A on $\partial\Omega$, then we denote by $i(A, K, \Omega)$ the fixed point index of A in Ω. The definition of the integer number $i(A, K, \Omega)$ is quite technical but can be found in detail in [29]. For our purposes, it is enough to summarize here that index i has the following properties:

1. Let $\Omega \subset K$ be an open subset in the relative topology of K. If $Ax = x_0$, $\forall x \in \Omega$ for some $x_0 \in \Omega$, then $i(A, K, \Omega) = 1$.

2. Let $\Omega_1, \Omega_2 \subset K$ be two open and disjoint subsets in the relative topology of K. If $\overline{\Omega}_1 \cap \overline{\Omega}_2$ is an empty set then

$$i(A, K, \Omega_1 \cup \Omega_2) = i(A, K, \Omega_1) + i(A, K, \Omega_2).$$

3. Let $\Omega \subset K$ be an open subset in the relative topology of K. If $i(A, K, \Omega) \neq 0$, then A has a fixed point in Ω.

For $0 < r < R$, let us define $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$ and $K_r^R = \{x \in K : r < \|x\| < R\}$. The following lemma is adapted from [30].

Lemma 3.2. *Let K be a closed convex cone in a Banach space and $A : K \to K$ a continuous and compact operator, then the following statements are valid:*

1. *Let $r > 0$ be a constant. If $x - tAx \neq 0$, $\forall x \in \partial K_r$ and $t \in [0, 1]$, then $i(A, K, K_r) = 1$.*

2. *Let $r > 0$ be a constant. If there exists $\zeta \in K$ with $\zeta \neq 0$ such that $x - Ax \neq s\zeta$, $\forall x \in \partial K_r$ and $s \geq 0$, then $i(A, K, K_r) = 0$.*

3. *Let $\Omega \subset K$ be an open subset in the relative topology of K such that A has no fixed points in $\partial\Omega$. If there exists $\zeta \in K$ with $\zeta \neq 0$ such that $x - Ax \neq s\zeta$, $\forall x \in \partial\Omega$ and $s \geq 0$, then $i(A, K, \Omega) = 0$.*

The strategy to apply the previous results for proving the existence of a non-trivial fixed point for operator $\mathcal{A}$ is the following: determine a suitable cone $\mathcal{K}$ such that operator $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ satisfies the hypotheses of lemma 3.2 and then construct a subset Ω of $\mathcal{K}$ such that $i(A, K, \Omega) \neq 0$.

Denoting $\lambda = \|w^{-1}\|_2$, we define the cone $\mathcal{K}$ as

$$\mathcal{K} = \{a \in l_2(\mathbb{Z}) : \|a\|_2 \leq \lambda w_0 a_0 \text{ and } a_k = a_{-k} \geq 0, \forall k \in \mathbb{Z}\}.$$

Note that $\mathcal{K}$ is a closed subset that admits non-trivial sequences since $\lambda w_0 > 1$ and it is convex by the triangular inequality. For $a \in \mathcal{K}$, we have that $b = \mathcal{A}a \in l_2(\mathbb{Z})$. In addition, $b_k = b_{-k} \geq 0$, since $w_k = w_{-k} \geq 0$. On the other hand, we obtain that $b_k \leq \frac{1}{w_k}\|a\|_2^2$, and in particular $b_0 = \frac{1}{w_0}\|a\|_2^2$. Therefore, $\|b\|_2 \leq \lambda\|a\|_2^2 = \lambda w_0 b_0$, that is $b \in \mathcal{K}$, and $\mathcal{A}$ maps $\mathcal{K}$ into itself. The following lemma proves that operator $\mathcal{A}$ satisfies the hypotheses of lemma 3.2.

Lemma 3.3. *Operator $\mathcal{A} : \mathcal{K} \to \mathcal{K}$ is continuous and compact.*

Proof. To prove the continuity of $\mathcal{A}$, consider $a, b \in \mathcal{K}$, then,

$$\begin{aligned} |(a * a)_k - (b * b)_k| &= \left| \sum_{m\in\mathbb{Z}} a_{k+m}a_m - b_{k+m}b_m \right| \\ &= \left| \sum_{m\in\mathbb{Z}} a_{k+m}(a_m - b_m) + b_m(b_{k+m} - a_{k+m}) \right| \\ &\leq \left| \sum_{m\in\mathbb{Z}} a_{k+m}(a_m - b_m) \right| + \left| \sum_{m\in\mathbb{Z}} b_m(b_{k+m} - a_{k+m}) \right|. \end{aligned}$$

By the Cauchy-Bunyakovsky-Schwarz inequality we have that

$$\left| \sum_{m\in\mathbb{Z}} a_{k+m}(a_m - b_m) \right| \leq \|a\|_2\|a - b\|_2$$

and

$$\left| \sum_{m\in\mathbb{Z}} b_m(b_{k+m} - a_{m+k}) \right| \le \|b\|_2 \|a-b\|_2.$$

Therefore, $|(a*a)_k - (b*b)_k| \le (\|a\|_2 + \|b\|_2)\|a-b\|_2$, which implies that

$$\|\mathcal{A}a - \mathcal{A}b\|_2 \le \sqrt{\sum_{k\in\mathbb{Z}} w_k^{-2} |(a*a)_k - (b*b)_k|^2} \le \lambda(\|a\|_2 + \|b\|_2)\|a-b\|_2.$$

Therefore, $\mathcal{A}$ is continuous.

For the compactness let us consider the family of finite rank operators $\mathcal{A}_N$ which are defined by

$$(\mathcal{A}_N a)_k = \begin{cases} (\mathcal{A}a)_k & \text{for } |k| \le N, \\ 0 & \text{for } |k| > N. \end{cases}$$

Since $\|\mathcal{A}_N a - \mathcal{A}a\|_2 \le \|a\|_2^2 \left(\sum_{|k|>N} w_k^{-2}\right)^{1/2} \le \lambda \|a\|_2^2$, and as $N \to \infty$, $\sum_{|k|>N} w_k^{-2} \to 0$, operator $\mathcal{A}$ is the uniform limit on bounded sets of the finite rank compact operators $\mathcal{A}_N$ and therefore it is compact. □

The subset of $\mathcal{K}$ constructed to contain a non-trivial fixed point of $\mathcal{A}$ has the form $\mathcal{K}_r^R \backslash V_\epsilon(d)$, where $0 < r < R$ and $V_\epsilon(d)$ is a small open disc, contained in $\mathcal{K}_r^R$, to be determined later. The next two lemmas show how to choose the values of r and R.

Lemma 3.4. *Let r be a positive constant. If $r < 1/\lambda$, then $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_r) = 1$.*

Proof. Consider $a \in \partial\mathcal{K}_r$ and take the component $k = 0$ of $a - t\mathcal{A}a$. Since $a_0 = \|a\|_2/(\lambda w_0)$, it follows that

$$a_0 - \frac{t}{w_0}\|a\|_2^2 = \frac{1}{w_0}\left(\frac{1}{\lambda} - t\|a\|_2\right)\|a\|_2.$$

Using that $\|a\|_2 < \frac{1}{\lambda}$ and $t \in [0,1]$ we obtain $(a - t\mathcal{A}a)_0 > 0$. Therefore $a - t\mathcal{A}a \ne 0$, for all $a \in \partial\mathcal{K}_r$ and $t \in [0,1]$. By the item 1 of lemma 3.2 we conclude that $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_r) = 1$. □

Lemma 3.5. *Let R be a positive constant. If $R > w_0$ then $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_R) = 0$.*

Proof. Consider $\zeta \in \mathcal{K}$ such that $\zeta_0 > 0$ and $a \in \partial\mathcal{K}_R$. Take the component $k = 0$ of $a - \mathcal{A}a$, since $\|a\|_2 > w_0$ we obtain that

$$a_0 - \frac{1}{w_0}\|a\|_2^2 < \|a\|_2 - \frac{1}{w_0}\|a\|_2^2 < 0.$$

Therefore, $(a - \mathcal{A}a)_0 \neq s\zeta_0$, $\forall s \geq 0$, and by the item 2 of lemma 3.2, we conclude that $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_R) = 0$. □

With lemmas 3.4 and 3.5 we can conclude that $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_r^R) = -1$. Unfortunately, since $r < w_0 < R$, the sequence $d = \{\{d_k\}_{k\in\mathbb{Z}} : d_0 = w_0 \text{ and } d_k = 0 \text{ for } k \neq 0\}$ corresponding to $\varphi \equiv B$ belongs to $\mathcal{K}_r^R$ and we have no guarantee that there is a non-trivial fixed point of $\mathcal{A}$ in $\mathcal{K}_r^R$. Thus, we must isolate this sequence in a small neighborhood $V_\epsilon(d) = \{a \in \mathcal{K} : \|d - a\|_2 < \epsilon\}$ for $\epsilon > 0$ such that $i(\mathcal{A}, \mathcal{K}, V_\epsilon(d)) = 0$. Thus, using the properties of index i we obtain that $\mathcal{A}$ has a non-trivial fixed point in $\mathcal{K}_r^R \backslash V_\epsilon(d)$ as desired. The following lemma shows under what conditions $i(\mathcal{A}, \mathcal{K}, V_\epsilon(d)) = 0$.

Lemma 3.6. *If $2w_0 > w_1$ and $\epsilon > 0$ is chosen sufficiently small, then $i(\mathcal{A}, \mathcal{K}, V_\epsilon(d)) = 0$.*

Proof. An element $b \in \partial V_\epsilon(d)$ is of the form $b = d + h$, where $\|h\|_2 = \epsilon > 0$, thus

$$(b * b)_k = 2w_0 h_k + (h * h)_k.$$

Consequently,

$$(b - \mathcal{A}b)_k = d_k + \left(1 - \frac{2w_0}{w_k}\right) h_k - \frac{1}{w_k}(h * h)_k.$$

By the definition of b we have that $h_k \geq 0$ for $k \neq 0$, thus

$$-(h * h)_1 < 2|h_0|h_1 < 2\epsilon h_1.$$

Therefore,

$$(b - \mathcal{A}b)_1 = \left(1 - \frac{2w_0}{w_1}\right) h_1 - \frac{1}{w_1}(h * h)_1 < \left(1 - \frac{2w_0}{w_1} + \frac{2\epsilon}{w_1}\right) h_1.$$

If the half-period l is chosen large enough, then the inequality $2w_0 > w_1$ is valid. Additionally, if $\epsilon > 0$ is sufficiently small, then $(b - \mathcal{A}b)_1 < 0$. Thus, choosing $\zeta \in K$ so that $\zeta_1 > 0$, we have $b - \mathcal{A}b \neq s\zeta$, $\forall b \in \partial V_\epsilon(d), s \geq 0$, and finally $i(\mathcal{A}, \mathcal{K}, V_\epsilon(d)) = 0$. □

Now we are able to prove the following theorem:

Theorem 3.7. *If $2w_0 > w_1$, then operator $\mathcal{A}$ has a non-trivial fixed point in $\mathcal{K}_r^R$ for $0 < r < w_0 < R$.*

Proof. From lemmas 3.4 and 3.5 we have that $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_r^R) = -1$. If $2w_0 > w_1$ and $\epsilon > 0$ is chosen small enough, then $i(\mathcal{A}, \mathcal{K}, V_\epsilon(d)) = 0$, by lemma 3.6. Moreover, if $\epsilon > 0$ is such that $V_\epsilon(d) \subset \mathcal{K}_r^R$, then we have that $i(\mathcal{A}, \mathcal{K}, \mathcal{K}_r^R \backslash V_\epsilon(d)) = -1$. Therefore, operator $\mathcal{A}$ has a non-trivial fixed point in $\mathcal{K}_r^R \backslash V_\epsilon(d)$. □

The non-trivial fixed point a of operator $\mathcal{A}$ corresponds to a function $\varphi \in L^2_{\text{per}}$, however we need a smooth function φ. So let us enunciate the theorem which will guarantee the existence of a C^∞ travelling solution.

Theorem 3.8. *If the half-period l is chosen large enough, then there exists a non-trivial $2l$-periodic C^∞ function φ, which is a solution of equation (3.3).*

Proof. Due to the considerations made previously and since the inequality $2w_0 > w_1$ is valid when l is chosen large enough, we just need to prove that the function φ corresponding to the non-trivial fixed point a of $\mathcal{A}$ belongs to the periodic Sobolev spaces $H^s_{\text{per}}([-l, l])$, for all $s \geq 0$, that is, $\varphi \in L^2_{\text{per}}$ is such that

$$\sum_{k \in \mathbb{Z}} \left(1 + |k|^2\right)^s |\widehat{\varphi}(k)|^2$$

is finite.

Since the sequence a is a fixed point of $\mathcal{A}$, it satisfies that $0 \leq w_k a_k \leq \|a\|_2^2$. Therefore,

$$\sum_{k \in \mathbb{Z}} |a_k| \leq \sum_{k \in \mathbb{Z}} \frac{\|a\|_2^2}{w_k} = \|a\|_2^2 \sum_{k \in \mathbb{Z}} \frac{1}{w_k} < \infty,$$

and also

$$\sum_{k\in\mathbb{Z}} w_k|a_k|^2 \leq \sum_{k\in\mathbb{Z}} |a_k|\|a\|_2^2 \leq \|a\|_2^2 \sum_{k\in\mathbb{Z}} |a_k| < \infty.$$

Observe that w_k grows like $1+|k|^2$, thus $\sum_{k\in\mathbb{Z}}(1+|k|^2)|a_k|^2 < \infty$, that is, $\varphi \in H^1_{\mathrm{per}}([-l,l])$. By theorem 3.200 of [21], that establishes that $H^s_{\mathrm{per}}([-l,l])$ is a Banach algebra for all $s > 1/2$, we also have that $\varphi^2 \in H^1_{\mathrm{per}}([-l,l])$ and

$$\sum_{k\in\mathbb{Z}}(1+|k|^2)|(a*a)_k|^2 = \sum_{k\in\mathbb{Z}}(1+|k|^2)w_k^2|a_k|^2 < \infty.$$

Therefore, $\sum_{k\in\mathbb{Z}}(1+|k|^2)^3|a_k|^2 < \infty$, that is $\varphi \in H^3_{\mathrm{per}}([-l,l])$. Repeating this process we conclude that $\varphi \in H^s_{\mathrm{per}}([-l,l])$ for all $s \geq 0$ so φ is a C^∞ function and a solution of equation (3.3). □

Returning to the original variables we have that equation (2.9) admits non-trivial periodic and C^∞ travelling wave solutions.

4 Conclusions

In this work the Regularized Benjamin equation was obtained as a unidirectional reduction of a system of integro-differential equations using an asymptotic expansion method. The existence of C^∞ periodic travelling wave solutions for the RB equation was proven by means of the fixed point index theory in cones for positive operators on Banach spaces. This theory was succesfully applied in [30] for a system similar to system (1.2), only with dispersive terms in both equations. This is not the case of system (1.2), because its first equation is a nonlinear first order differential equation. Because of this, we were not able to prove directly the existence of travelling wave solutions for system (1.2), a challenge we expect to address in future works.

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