Matemática
Contemporânea
Vol. 50, 4–12
http://doi.org/10.21711/231766362022/rmc501

On the Cylinder Theorem in $M^2 \times \mathbb{R}$

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. Consider a surface M with Gaussian curvature either < 0 or > 0. We prove that in $M^2 \times \mathbb{R}$ cylinders are characterized as the surfaces with both the extrinsic and intrinsic curvatures equal to zero.

Keyword: Cylinder.

2020 Mathematics Subject Classification: 53C42.

1 Introduction

Except for a note by A.V. Pogorelof in the Doklady in 1956 (see details in the Dover edition of Differential Geometry of Curves and Surfaces p. 415), the first time that the theorem of the cylinder in $\mathbb{R}^3$ was in print was in a paper by P. Hartman and L. Nirenberg [2] which treats a more general situation but mentions explicitly the case of surfaces, namely, if the Gauss curvature vanishes everywhere, the surface is a cylinder. Direct proofs of this particular case were published, almost simultaneously, by Massey [3] and Stoker [6].

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In 2020 Barbosa and do Carmo publish a paper [1] generalizing this result to $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ is the 2-dimension hyperbolic space. This result was generalized by J. Park [4] to include the case when the ambient space is the sphere times $\mathbb{R}$.

In the present paper we are extending that result to $M^2 \times \mathbb{R}$, where M^2 is a surface with Gaussian curvature either < 0 or > 0. In this ambient space we define a *cylinder* as the surface given by $\alpha \times I$ where α is a regular curve in M^2 and I is an open interval of $\mathbb{R}$. Observe that a cylinder is complete if and only if α is a complete curve in M and $I = \mathbb{R}$.

We know that any surface in a 3-dimension space has two curvatures: the Gaussian Curvature and the Extrinsic Curvature, that is given by the product of the two principal curvatures, say $k_1 k_2$. In $\mathbb{R}^3$ these two curvatures coincide. In $M^2 \times \mathbb{R}$ they are in general different and give different information about the surface.

The goal of this paper is to prove that:

Let Σ be a complete and connected surface in $M^2 \times \mathbb{R}$, where M^2 is a surface with Gaussian curvature either < 0 or > 0. Then Σ is a cylinder if and only if both its Gaussian and its extrinsic curvatures vanish.

Of course to prove this statement we have to prove two propositions:

Theorem 1.1. *Let Σ be a connected surface in $M^2 \times \mathbb{R}$, where M^2 is a surface with Gaussian curvature either < 0 or > 0. If Σ is the cylinder $\alpha \times I$ then it has the Gaussian and extrinsic curvatures equal to zero.*

Theorem 1.2. *Let Σ be a connected surface in $M^2 \times \mathbb{R}$, where M^2 is a surface with Gaussian curvature either < 0 or > 0. If Σ has both the intrinsic and the extrinsic curvatures equal to zero then Σ is a cylinder.*

2 Proof of theorem 1.1

Along this work, we are going to use moving frames having as reference the book [5].

In this section, we are going to compute the Gaussian and the extrinsic curvatures of a cylinder.

Let Σ be the cylinder $\gamma \times \mathbb{R}$, where γ is a curve in M^2

To study the geometry of this cylinder we proceed as follows. First observe that the metric in $M^2 \times \mathbb{R}$ is given by

$$d\sigma^2 = d\zeta^2 + dt^2$$

where $d\zeta$ is the metric in M^2 and dt is the standard metric in $\mathbb{R}$. The covariant differential $\bar{D}$ in $M^2 \times \mathbb{R}$ decomposes as

$$\bar{D} = D + d$$

where D is the covariant differential in M^2 and d is the standard differentiation in $\mathbb{R}$. If we represent by $\partial/\partial t$ the unit vector field tangent to the lines $p \times \mathbb{R}$, $p \in M^2$, then we obtain

$$\bar{D}(\partial/\partial t) = 0\,.$$

We assume that α is parameterized by the arc length s. Take an adapted frame field in a neighborhood of the cylinder: $e_1 = \partial/\partial t$, $e_2 = \alpha'$, and e_3 normal to α and tangent to M^2.

The corresponding dual forms are $\omega_1 = dt$, $\omega_2 = ds$. The metric in Σ is then given by

$$dt^2 + ds^2\,.$$

Hence Σ is isometric to the plane and so it is flat and, consequently, its Gaussian curvature is zero.

Given a point $p \in \Sigma$, extend the above mentioned frame field to a neighborhood of p in the ambient space.

Represent by ω_{ij} the connection forms given by

$$\bar{D}e_i = \sum \omega_{ij} \wedge e_j\,.$$

We then have

$$0 = \bar{D}(\partial/\partial t) = \bar{D}e_1 = \omega_{12}e_2 + \omega_{13}e_3\,.$$

Then $\omega_{12} = 0$ and $\omega_{13} = 0$.

Therefore

$$\bar{D}e_2 = \omega_{21}e_1 + \omega_{23}e_3 = \omega_{23}e_3 \,.$$

Since in Σ, $e_2 = \alpha'$ that is tangent to M^2 then $\bar{D}e_2$ along Σ can not have a component in the direction of ω_1. Hence, in Σ, $\omega_{23} = -\lambda\omega_2$.

Summarizing: at the point p, $\omega_{31} = 0$ and $\omega_{32} = \lambda\omega_2$. Hence e_1 and e_2 are principal vectors and the respective principal curvatures are 0 and λ. Thus the extrinsic curvature at p is zero. Since the point p can be any point of Σ, the theorem is proved.

3 Proof of theorem 1.2

Let p be a point of Σ. Take a frame field e_1, e_2 and e_3 in a neighborhood of p in the ambient space, such that this frame is adapted to Σ, that is, restricted to points of Σ, e_1 and e_2 are principal vectors tangent to Σ and e_3 is normal. Let ω_i be its dual forms and ω_{ij} be the connection forms associated to such frame.

We then have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} - \bar{\Omega}_{12}$$

where $\bar{\Omega}$ is the curvature form of the ambient space. When we restrict this equation to Σ, we have

$$d\omega_{12} = -K\omega_1 \wedge \omega_2$$

where K is the Gaussian curvature of Σ.

We then have

$$K\omega_1 \wedge \omega_2 = \bar{\Omega}_{12}|_{\Sigma} - \omega_{13} \wedge \omega_{32} \,.$$

Observe that $\omega_{13} \wedge \omega_{32}(e_1, e_2)$ is precisely the product of the principal curvatures. Since the Gaussian curvature of Σ is zero and the product of the principal curvatures is also zero then we conclude that, at the point p

$$\bar{\Omega}_{12}(e_1, e_2) = 0\,.$$

Observe that the value of $\bar{\Omega}_{12}(e_1, e_2)$ is the same for any orthonormal basis of the plane expanded by e_1, e_2.

Lemma 3.1. *If $\bar{\Omega}_{12}(e_1, e_2) = 0$ then the vector $\partial/\partial t$ belongs to the subspace expanded by e_1, e_2.*

Assume this Lemma is proved.

As we have seen, for any point $p \in \Sigma$ we have that the curvature $\bar{\Omega}$ of $T_p\Sigma$ is zero. Therefore, $T_p\Sigma$ contains the vector $\partial/\partial t$.

Hence, $\partial/\partial t$ is a vector field well defined over Σ. Consequently, its integral curves are curves in Σ. But such curves are intervals. Thus, Σ is foliated by intervals. Hence, it is a product $\alpha \times I$ where α is a curve in M^2.

Therefore Σ is a cylinder. This proves Theorem (1.2).

Proof of the Lemma (3.1): Choose a frame field E_1, E_2 and $E_3 = \partial/\partial t$. Let ω_1, ω_2 and $\omega_3 = dt$ be its dual forms and ω_{ij} be the corresponding connection forms defined by

$$\bar{D}E_i = \sum \omega_{ij} E_j$$

By the choice of E_3 we have that E_1 and E_2 are tangent to $M^2 \times \{t\}$ and

$$\bar{D}E_3 = 0.$$

Then we have $\omega_{31} = 0$ and $\omega_{32} = 0$. It then follows that

$$d\omega_{ij} = -\bar{\Omega}_{ij}, \quad \text{where ,} \quad 1 \leq i, j \leq 3.$$

It also implies that

$$\bar{\Omega}_{13} = \bar{\Omega}_{23} = 0\,. \tag{3.1}$$

Observe that the plane generated by E_1 and E_2 is tangent to $M^2 \times \{t\}$. Set $c(p) = \bar{\Omega}_{12}(E_1, E_2)$ at the point p . Then

$$\bar{\Omega}_{12} = c(p)\,\omega_1 \wedge \omega_2$$

We are interested in computing the curvature of any plane Π tangent to $M \times \mathbb{R}$ at some point p. If such plane is vertical, that is, if it contains the vector $\partial/\partial t$ at the point p, then its curvature will be zero. This was proved along the proof of Theorem (1.1). If the plane has $\partial/\partial t$ as its normal vector, then it will be tangent to the totally geodesic surface $M \times \{t\}$ and so, its curvature will be the curvature c of $M \times \{t\}$ at the point p.

Then, consider a non vertical plane Π which is also not tangent to $M^2 \times \{t\}$. We may choose the frame field E_1, E_2 and E_3, as before, having the care to choose $E_1(p)$ to be the common vector to $T_p(M^2 \times \{t\})$ and Π. We now take a frame field e_1, e_2 and e_3, where $e_1 = E_1$, e_2 is tangent to Π and e_3 is normal to Π given by

$$\begin{aligned} e_1 &= E_1 \\ e_2 &= aE_2 + bE_3 \\ e_3 &= -bE_2 + aE_3 \end{aligned}$$

where a and b are constant real numbers such that $a^2 + b^2 = 1$. It follows that

$$\begin{aligned} E_1 &= e_1 \\ E_2 &= ae_2 - be_3 \\ E_3 &= be_2 + ae_3 \, . \end{aligned}$$

Let θ_1, θ_2 and θ_3 be the dual forms of the frame e_1, e_2, e_3 and ω_1, ω_2 and ω_3 be the dual forms of the frame E_1, E_2 and E_3. We then have

$$\begin{aligned} \theta_1 &= \omega_1 \\ \theta_2 &= a\omega_2 + b\omega_3 \\ \theta_3 &= -b\omega_2 + a\omega_3 \, . \end{aligned}$$

Conversely we have

$$\begin{aligned}\omega_1 &= \theta_1\\ \omega_2 &= a\theta_2 - b\theta_3\\ \omega_3 &= b\theta_2 + a\theta_3\,.\end{aligned}$$

We then have

$$\bar{D}e_1 = \bar{D}E_1 = \omega_{12}E_2 = \omega_{12}(ae_2 - be_3) = \theta_{12}e_2 + \theta_{13}e_3$$

$$\begin{aligned}\bar{D}e_2 &= \bar{D}(aE_2 + bE_3) = a\bar{D}E_2 + b\bar{D}E_3 = a\bar{D}E_2\\ &= a\omega_{21}E_1 + a\omega_{23}E_3 = a\omega_{21}E_1 = a\omega_{21}e_1\\ &= \theta_{21}e_1 + \theta_{23}e_3\end{aligned}$$

$$\begin{aligned}\bar{D}e_3 &= \bar{D}(-bE_2 + aE_3) = -b\bar{D}E_2 + a\bar{D}E_3 = -b\bar{D}E_2\\ &= -b\omega_{21}E_1 - b\omega_{23}E_3 = -b\omega_{21}E_1 = -b\omega_{21}e_1\\ &= \theta_{31}e_1 + \theta_{32}e_2\,.\end{aligned}$$

where the last equality in the last three equations is just the definition of θ_{ij}. It follows from these equations that

$$\theta_{12} = a\omega_{12} \qquad \theta_{13} = -b\omega_{12} \qquad \theta_{23} = 0.$$

We will represent by R the curvature tensor of $M^2 \times \mathbb{H}$, so that

$$R(e_i, e_j, e_m, e_k) = \bar{\Omega}_{ij}(e_m, e_k).$$

We now compute the value of the curvature at the plane Π. Using the 4-linearity of R.

$$\begin{aligned}R(e_1, e_2, e_1, e_2) &= R(E_1, aE_2 + bE_3, E_1, aE_2 + bE_3)\\ &= a^2R(E_1, E_2, e_1, e_2) + abR(E_1.E_2.E_1.E_3) +\\ &+ baR(E_1, E_3, E_1, E_2) + b^2R(E_1, E_3, E_1, E_3)\end{aligned}$$

Let c be the value of the curvature $M^2 \times \{t\}$ at the point p. Observe that

$$\begin{aligned} R(E_1, E_2, E_1, E_2) &= \bar{\Omega}_{12}(E_1, E_2) = c, \\ R(E_1, E_2, E_1, E_3) &= \bar{\Omega}_{12}(E_1, E_3) = 0, \\ R(E_1, E_3, E_1, E_2) &= R(E_1, E_2, E_1, E_3) = 0 \end{aligned}$$

and

$$R(E_1, E_3, E_1, E_3) = \bar{\Omega}_{13}(E_1, E_3) = 0$$

In the last equality we have used (3.1). Thus

$$R(e_1, e_2, e_1, e_2) = a^2 c$$

By assumption $c \neq 0$. Thus $R(e_1, e_2, e_1, e_2) = 0$ if and only if $a = 0$. But this is equivalent to

$$\begin{aligned} e_1 &= E_1 \\ e_2 &= E_3 = \partial/\partial t \\ e_3 &= -E_2 \end{aligned}$$

what is equivalent to the fact that the vector $\partial/\partial t$ belongs to Π. This concludes de proof of the Lemma.

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