Matemática
Contemporânea
Vol. 49, 251–262
http://doi.org/10.21711/231766362022/rmc4910

Equivariant embeddings of symmetric spaces

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. For an equivariant embedding of a compact symmetric space $X = G/K$ into a Euclidean G-space the following statements are equivalent:

(a) The embedding is extrinsic symmetric.

(b) The maximal torus T_X of X is rectangular and the representation of G has lowest possible highest weight.

(c) The maximal torus T_X is embedded as a Clifford torus (an extrinsic product of planar circles).

Keywords: Equivariant embedding, Clifford torus, symmetric spaces.

2020 Mathematics Subject Classification: 53C35.

1 Introduction

In 1854, Bernhard Riemann coined the notions of an abstract manifold of arbitrary dimension and of a Riemannian metric. As Riemann

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indicates, examples are algebraic submanifolds in higher dimensional Euclidean space. Since that time, the two theories – abstract Riemannian manifolds and submanifolds of Euclidean space – coexist, and traditionally, the Brazilian differential geometry group was particularly well known for their contributions to the second area. When I met Renato Tribuzy in Berkeley 40 years ago, I started studying submanifolds for the first time, and since then we wrote many joined papers. One of our common projects was trying to generalize the notion of constant mean curvature surfaces to higher dimensions [1] by what was called parallel pluri-mean curvature (ppmc) submanifolds. However, we were somewhat unhappy since we knew only few examples.

In February 2014 we were visiting our co-author Maria Joao Ferreira at Lisbon. One of the subjects we discussed was a Ph.D. project at the Federal University of Amazonas: the quest for new ppmc submanifolds. The known ones were the extrinsic symmetric embeddings of compact Kähler symmetric spaces. But every compact symmetric space has infinitely many other embeddings respecting its symmetry, so called *equivariant embeddings*. Are some of them ppmc? This became the theme of Kelly Karina Santos' PhD thesis [14]. Sadly for us, she disproved the ppmc property in the most promissing cases, e.g. for all equivariant embeddings of $\mathbb{CP}^n$ which are not extrinsic symmetric. But her investigations opened us the door to new research on this interesting class of symmetric submanifolds, minimally embedded in the sphere. (Joined work with E. Heintze, P. Quast [7], M.S. Tanaka [8].)

2 Equivariant and extrinsic symmetric embeddings

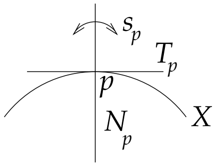
John Nash has shown that every closed Riemannian manifold X is isometric to a submanifold of Euclidean space V. But when X is acted on by a compact group G of isometries, we want more: We look for embeddings such that G is carried over into a group $\hat{G}$ of motions of V preserving the submanifold $X \subset V$. Since $\hat{G}$ is compact, it fixes a point 0, hence

$\hat{G}$ acts linearly on V, that is $\hat{G} \subset GL(V)$. Further, an inner product on V is preserved by $\hat{G}$, thus $\hat{G} \subset O(V)$ (the orthogonal group on V). This is *equivariance*: there is a Lie group homomorphism $\rho : G \to O(V)$ (a *representation*) with

$$\phi(gx) = \rho(g)\phi(x) \text{ for all } g \in G \text{ and } x \in X,$$

where $\phi : X \to V$ denotes the embedding.

We are interested in the case of *symmetic spaces* X, that is, for any $p \in X$ there is an isometry s_p called *symmetry at* p with order 2 ($s_p s_p = \text{id}$) and with p being an isolated fixed point of s_p. Symmetric spaces are particularly important in Riemannian geometry since their curvature tensor R (the quantity which distinguishes between Riemannian and Euclidean geometry) is "constant", $\nabla R = 0$. We consider embeddings $\phi : X \to V$ which are equivariant for the group G generated by all symmetries of X, the *symmetry group.* Then $\hat{s}_p \in O(V)$ is an order-2 element with $\hat{s}_p = -\text{id}$ on the tangent space T_p, thus it is a reflection along a subspace N_p^+ of the normal space N_p. The embedding is called *extrinsic symmetric* if $\hat{s}_p$ is the reflection along the full normal space, that is $N_p^+ = N_p$, as in the case of the round sphere $X = \mathbb{S}^n \subset V = \mathbb{R}^{n+1}$.



Essentially, an equivariant map $\phi : X = G/K \to V$ is given by the representation $\rho = \rho_\phi : G \to O(V)$, $s_p \mapsto \hat{s}_p$, at least when ρ is *irreducible* (that is: it does not allow nontrivial invariant subspaces). In fact, after picking a base point $o = eK$ we put $v_o = \phi(o)$, then $\phi(go) = \rho(g)v_o$, and in particular, $\rho(K)$ fixes v_o. Thus the fixed space V^K of $\rho(K)$ in nonzero, containing v_o. Such representation ρ with $V^K \neq 0$ is called *spherical.* Thus an equivariant embedding is given by a spherical representation ρ of G on V and some $v_o \in V^K$. Élie Cartan [3] has shown that V^K is one-dimensional when ρ is irreducible.[1] Thus v_o is unique up to scalar multiples, and so is the equivariant map $\phi = \phi_\rho$ with $\phi(gK) = \rho(g)v_o$.

The grandmother of all spherical representations is $C^\infty(X)^c$, the space of complex valued smooth functions on X with the G-action by precomposition: $g.f := f \circ g^{-1}$. It is the direct sum of all irreducible spherical representations, and each equivalence class occurs precisely once [3, 15]. In particular, there are infinitely many equivalence classes.

For arbitrary compact symmetric spaces, our principal aim was to distinguish the extrinsic symmetric embeddings among the equivariant ones. There are compact symmetric spaces without any extrinsic symmetric embedding, e.g. SU_n (although the natural inclusion $U_n \subset \mathbb{C}^{n\times n}$ *is* extrinsic symmetric). So two natural questions arise:

(1) Which X allow an extrinsic symmetric embedding?

(2) Which spherical representations ρ are extrinsic symmetric?

Question (1) was answered by Ottmar Loos [12, 13]: It depends on the *maximal torus* T_X of X. A maximal torus is a maximal flat totally geodesic submanifold of X; every geodesic in X is contained in a maximal torus, and any two of them are congruent, therefore we can talk about *the* maximal torus of X. Now Loos' theorem says: A compact symmetric space X has an extrinsic symmetric embedding if and only if T_X is rectangular, that is a Riemannian product of circles. Shortly:

Theorem 2.1. $\exists$ *extr. symm. embedding* $X \hookrightarrow V \iff T_X$ *rectangular.*

Loos' proof [13] used an algebraic structure called Jordan triple systems, and extended computations were needed to verify the defining identities. In [8, 7] we obtained a new proof which is less computational and includes an anwer to Question (2). We will give a sketch in the following two sections.

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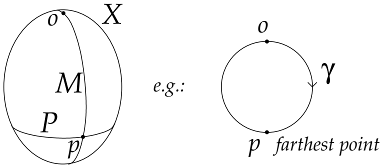
[1] Actually, Cartan considered complex representations. But when G is the full symmetry group, his proof can be easily extended to real representations.

3 Dimension reduction using meridians

"⇒" [8]

Let $X \subset V$ be extrinsic symmetric. The main idea is to reduce $\dim X$ while keeping a maximal torus T_X. More precisely, we replace X by a proper totally geodesic submanifold $M \subset X$, a so called *meridian*, which is still extrinsic symmetric and contains T_X as its maximal torus. To explain this notion we have to consider the fixed set of the symmetry s_o at the base point $o \in X$. Certainly o is a fixed point (an isolated one), but there might be others. E.g. the "farthest point" on any closed geodesic γ through o, the point opposite to o, is fixed by s_o since s_o preserves o and γ (cf. right figure). A positive dimensional connected component of $\mathrm{Fix}(s_o)$ is called a *polar* P. Every polar $P \subset X$ has a sort of orthogonal complement through any point $p \in P$, which is the connected component of $\mathrm{Fix}(s_p s_o)$. This is called a *meridian* M (left figure). It is extrinsic symmetric in the fixed space of $\hat{s}_p \hat{s}_o$.

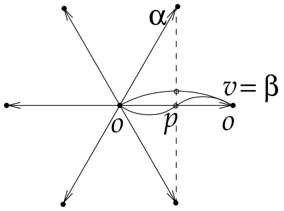


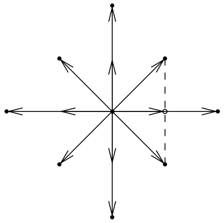
Now let T_X be a maximal torus of X containing both o and p. Then $p \in \mathrm{Fix}(s_o) \cap T_X$. But any fixed point of a symmetry on a torus T_X is isolated (locally it looks like $-\mathrm{id}$). Thus $s_p = s_o$ on T_X, that is $s_p s_o = \mathrm{id}$ on T_X, which means $T_X \subset M$.

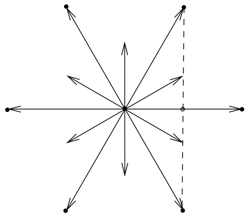
Now we repeat this argument with X replaced by $X_1 := M$, that is, we pass to a meridian X_2 of X_1 containing T_X. The process stops when we arrive at some X_k without polars. But this happens only when X_k is a Riemannian product of spheres (see Theorem below) whose maximal torus is clearly a Riemannian product of great circles, hence it is rectangular. $\square$

Theorem. *Only products of spheres are without polars (and therefore without meridians).*

Sketch of proof. We show first that a compact symmetric space X without polars is a Riemannian product of a simply connected symmetric space and possibly a torus [8, Lemma 8]. Then by [8, Thm. 3] the torus part of X is rectangular (a Riemannian product of circles) once X is extrinsic symmetric. It remains to consider simply connected and indecomposable compact symmetric spaces. The DNA of such a space X is its *root system* [10], [4, Sect. 10], a certain finite set of tangent vectors spanning the tangent space $\mathfrak{a}$ of its maximal torus T_X. The root system of the sphere $\mathbb{S}^n$ is $A_1 = \{\pm e_1\}$. We are going to construct a polar for all other indecomposable root systems. Since every such root system contains a sub-root system of dimension one or two which is different from A_1, we can restrict our attention to those, which are BC_1, A_2, B_2, BC_2, G_2. They are depicted in the following figures (the middle figure of BC_2 contains also BC_1 and B_2).







E.g. let us consider the case A_2 (left figure). Recall that $T_X = \mathbb{R}^k/\Gamma$ for some lattice $\Gamma \subset \mathbb{R}^k$. In our case the lattice is hexagonal, its points are marked black. The lattice points closest to 0 are the roots $\alpha, \beta, \dots$. (marked by an arrow).[2] The inner product between v and α determines the sectional curvature of the plane spanned by v and any nonzero vector v_α in the corresponding root space $\mathfrak{p}_\alpha$.[3] More precisely, $\langle \alpha, v\rangle = \frac{1}{2}\langle \beta, v\rangle$ implies $\sec(v, v_\alpha) = \frac{1}{4}\sec(v, v_\beta)$ for $v_\alpha \in \mathfrak{p}_\alpha$, $v_\beta \in \mathfrak{p}_\beta$. The geodesic $\exp(tv)$ for $0 \leq t \leq 1$ is simply closed since there are no lattice points between

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[2] Roots are 1-forms on $\mathfrak{a}$ which are viewed as vectors in $\mathfrak{a}$. By scaling the metric we can arrange that roots and lattice points behave as in the figures. This can be easily seen by looking at the rank-one subspaces determined by each root [10, p.407].

[3] The root space $\mathfrak{p}_\alpha \subset \mathfrak{p} = T_oX$ is the common eigenspace of the Jacobi operator $R(.,v)v$ for the eigenvalue $\langle \alpha, v\rangle^2$, for all $v \in \mathfrak{a}$.

0 and v. Its midpoint $p = \exp(v/2)$ is the point opposite to o and hence it is fixed by s_o. The same holds for all neighboring geodesics from o to o. However, when the curvature is large, p could be a "node", passed by all neighboring geodesics. This may happen in the (v, v_β)-plane, but not in the (v, v_α)-plane, due to smaller curvature. Hence the midpoints of the neighboring geodesics in the (v, v_α)-plane form a nontrivial curve fixed by s_o, hence contained in a polar. Similar pictures arise in the other cases, see figures in the center and right. □

4 Smallest highest weights and Clifford tori

"$\Leftarrow$" [7]

Let us assume that X is an indecomposable compact symmetric space with a rectangular torus $T_X \cong \mathbb{S}^1_1 \times \ldots \times \mathbb{S}^1_k$. Changing slightly our notation, we let G be the transvection group of X, which is the connected component of both the isometry group and the symmetry group, and $K \subset G$ is the stabilizer of a chosen base point $o \in X$. We have to construct a spherical representation $\rho : G \to O(V)$ such that ϕ_ρ is an extrinsic symmetric embedding.

A representation $\rho : G \to GL(V)$ on a complex vector space V can be restricted to a maximal connected abelian subgroup $T \subset G$ (*maximal torus* of G). As a T-representation, V decomposes into irreducible components: $V = \sum_\mu V_\mu$, where $\mu \in \mathrm{Hom}(T, \mathbb{S}^1)$ denote the irreducible subrepresentations of $\rho|_T$. They are called *weights* and V_μ *weight spaces.* The set of weights is partially ordered (using a "Weyl chamber"). Hermann Weyl has shown in 1926 that irreducible representations ρ have a *highest* weight $\lambda = \lambda_\rho$, and vice versa, any "positive" homomorphism $\lambda : T \to \mathbb{S}^1$ determines an irreducible representation ρ_λ of G which is unique up to equivalence. Sigurdur Helgason [11, 15] has refined this theory for spherical representations where the maximal torus T of G is chosen such that $T.o \subset X$ is a maximal torus T_X of X, in other words, $T_X = T/(T \cap K)$:

Theorem of Helgason. *An irreducible representation ρ of G is spherical if and only if its highest weight λ descends from T to T_X, that is $\lambda(T \cap K) = 1$.*

Now we can choose our representation. Since T_X is rectangular we have distinguished homomorphisms $\epsilon_j : T \to T_X = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1 \to \mathbb{S}^1$, namely the action $t \mapsto t.o$ followed by the projection onto the j-th $\mathbb{S}^1$-factor. These are mutually equivalent, and ϵ_1 is the largest. Let $\rho_1 : G \to GL(V_1)$ be the complex irreducible representation with highest weight $\lambda = \epsilon_1$. Clearly ρ_1 is spherical since ϵ_1 descends to T_X. Moreover ϵ_1 is *minimal*, wrapping the first factor of T_X one-to-one onto $\mathbb{S}^1$:

Minimality of ϵ_1. *All weights μ which descend to T_X (that is $\mu(T \cap K) = 1$) and which are smaller than ϵ_1 are equivalent to ϵ_1 (that means $\mu = \epsilon_j$ or $\bar\epsilon_j$) or zero.*

Now we put $\rho = \text{Re}\ \rho_1$ on $V = \text{Re}\ V_1$ and $\phi = \phi_\rho : g.o \mapsto \rho(g)w_o$ with $w_o = \text{Re}\ v_o$ (this can be chosen nonzero). The concept "Real part" makes sense since we may assume $V_1 \subset C^\infty(X)^c$.

Definition. A *Clifford torus* in a Euclidean vector space V is the image of the standard Clifford torus $\mathbb{S}^1_1 \times \ldots \times \mathbb{S}^1_k \subset \mathbb{C}^k$ under an affine isometric map $\psi : \mathbb{C}^k \to V$ where $\mathbb{S}^1_j = \{z \in \mathbb{C} : |z| = r_j\}$ for some $r_1, \ldots, r_k > 0$. The circles $c_j = \psi(\mathbb{S}^1_j)$ are called *generating circles.* A submanifold $X \subset V$ has *Clifford type* if any two points in X lie in a common Clifford torus $C \subset V$ which is totally geodesic in X.

Claim (1) $\phi(T_X)$ is a *Clifford torus* and $\phi(X)$ is of *Clifford type.*

Proof of Claim (1). We decompose v_o with respect to the weight spaces: $v_o = \sum_\mu v_\mu$ with $0 \neq v_\mu \in V_\mu$. Then $\langle v_\mu, v_o\rangle \neq 0 \overset{!}{\Rightarrow} \mu(T \cap K) = 1$.
In fact $\mu(t) = 1$ for all $t \in T \cap K$ since

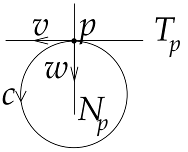
$$\mu(t)\langle v_\mu, v_o\rangle = \langle \rho(t)v_\mu, v_o\rangle = \langle v_\mu, \rho(t^{-1})v_o\rangle = \langle v_\mu, v_o\rangle.$$

By minimality of ϵ_1 we have either $\mu = 0$ or $\mu \in \{\epsilon_j, \bar\epsilon_j\}$ for some j. Let $x = t.o \in T_X$ for some $t \in T$. We can explicitly compute $\phi(x)$:

$\phi(t.o) = \rho(t)w_o = \mathrm{Re}\sum_\mu \rho(t)v_\mu = \mathrm{Re}\sum_\mu \mu(t)v_\mu = \mathrm{Re}\sum_j \epsilon_j(t)v_{\epsilon_j} + w_o^o$ where w_o^o is the component of w in V_0 (= weight space V_μ for $\mu = 0$).
$\Rightarrow \phi(T_X)$ is a Clifford torus (with generating circles $t \mapsto \epsilon_j(t)v_{\epsilon_j}$)
$\Rightarrow \phi(X)$ is of Clifford type by congruence of maximal tori.

Claim (2) $X \subset V$ of Clifford type $\Rightarrow X \subset V$ extrinsic symmetric.

Proof of Claim (2). Let $p \in X$ and s_p the reflection along N_p.
Let $q \in X$ and $C \subset X$ a Clifford torus containing both p and q. Let c be a generating circle through p.



Then $c(0) = p$, $c'(0) = v \in T_p$ and $c''(0) = w$. Then $w \in N_p$ since c is a geodesic in $X \subset V$. Thus s_p preserves c and all generating circles through p, hence it preserves C. In particular, from $q \in C$ we obtain $s_p(q) \in C \subset X$. $\square$

5 Extrinsic symmetric $\overset{(1)}{\iff}$ Clifford type $\overset{(2)}{\iff}$ $\rho = \mathrm{Re}\ \rho_{\epsilon_1}$

- $\phi : X \hookrightarrow V$ of Clifford type $\overset{(2)}{\iff}$ $V = \mathrm{Re}\ V_{\epsilon_1}$.

"$\Leftarrow$": Claim (1) above.
"$\Rightarrow$": Let $c_1, \dots, c_k$ be the generating circles of T_X. Then $\phi \circ c_j$ is a planar circle in V. On the other hand we compute $\phi(c_j(t)) = \rho_\phi(\exp te_j)v_o$ using the weights of ρ_ϕ on $V^c = V \otimes \mathbb{C}$. Comparing the two formulas we obtain a restriction for the weights as claimed.

- $X \subset V$ extrinsic symmetric $\overset{(1)}{\iff}$ $X \subset V$ of Clifford type.

"$\Leftarrow$" Claim (2) above.
"$\Rightarrow$" Let $X = G/K \subset V$ be a full extrinsic symmetric space and $p \in X$.

Let $\alpha : S(T_p) \to N_p$ be the second fundamental form at p where $S(T_p)$ denotes the space of symmetric 2-tensors on T_p. This map α is linear, K-equivariant, onto, and it characterizes the extrinsic symmetric space $X \subset V$ [5]. Hence as a K-space N_p is equivalent to a sum of irreducible components of $S(T_p)$.

> **Example.** $X = \mathbb{S}^n$ with $K = SO_n$. Then $S(T_p) = S_o(T_p) \oplus \mathbb{R} \cdot \mathrm{id}$ where $S_o(T_p) = S(T_p) \cap \{\text{trace } 0\}$. The fixed space $N_p^K \subset N_p$ always contains the radial vector p since X lies in a sphere. Hence either $N_p \cong_K S(T_p)$ or N_p is the fixed space $\mathbb{R}p$. The first case gives $X = \mathbb{RP}^n \subset S_o(\mathbb{R}^{n+1})$ which is not an embedding of $\mathbb{S}^n$ while the second case is the standard embedding $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

By the argument in [8] (see section 2 above) we may assume that X is intrinsically a product of round spheres, $X = S_1 \times \ldots \times S_k$. Hence $T_p = \sum_{j=1}^k T_j$ and $S(T_p) = \sum_j S(T_j) \oplus \sum_{i<j} T_i \otimes T_j$. But there is a large number of possibilities for N_p. To reduce it we need two extra ideas.

(1) Ferus [9, 6] has shown: When $X \subset V$ is full extrinsic symmetric, there is no locally diffeomorphic G-orbit near $X = Gv_o$ in the unit sphere of V (since v_o is corner of a Weyl chamber in $V = \mathfrak{p}$). Hence the only parallel normal fields along X are radial and $N_p^K = \mathbb{R}p$.

(2) For any extrinsic symmetric space $X = G/K \subset V$, the isotropy group K acts on N_p as the *normal holonomy group*, so we can use Olmos' normal holonomy theorem [2, 4.2.1.]:

> **Theorem of Olmos.** *There are decompositions* $N_p = N_p^K \oplus N_1 \oplus \cdots \oplus N_s$ *and* $K = K_1 \times \ldots \times K_s$ *such that each* K_i *acts irreducibly on* N_i *and trivially on* N_j *for* $j \neq i$.

In our case, the possible irreducible components of N_p are equivalent to $S_o(T_i)$ or to $T_i \otimes T_j$. But both are acted on by the factor $SO(T_i)$ of K, hence not both of them can occur in N_p. We may assume that $X \subset V$ is indecomposable. Then the only possible cases are:

$k = 1$ with $N_p = \mathbb{R}p$ or $N_p \cong_K S(T_p)$,
$k = 2$ with $N_p \ominus \mathbb{R}\,p \cong_K T_1 \otimes T_2$.

The corresponding extrinsic symmetric spaces are $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ or $\mathbb{RP}^n$ or $S_1 \otimes S_2 = (S_1 \times S_2)/\pm$, but the latter two cases are not sphere products. ☐

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