Matemática
Contemporânea
Vol. 49, 236–250
http://doi.org/10.21711/231766362022/rmc499

Minimal real Kaehler submanifolds

S. Chion[1] and **M. Dajczer**[2]

[1]Universidad de Murcia, Departamento de Matemáticas, E-30100 Espinardo, Murcia, Spain
[2]IMPA, Estrada Dona Castorina 110, Rio de Janeiro, Brazil

Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. We show that generic rank conditions on the second fundamental form of an isometric immersion $f\colon M^{2n}\to\mathbb{R}^{2n+p}$ of a Kaehler manifold of complex dimension $n\geq 2$ into Euclidean space with low codimension p imply that the submanifold has to be minimal. If M^{2n} if simply connected, this amounts to the existence of a one-parameter associated family of isometric minimal immersions unless f is holomorphic.

Keywords: Kaehler submanifolds, Minimal immersions, complex s-nullities.

2020 Mathematics Subject Classification: 53B25, 53B35.

This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia. Sergio Chion was partially supported by Fundación Séneca project 19901/GERM/15, Spain. e-mail: sjchiona@gmail.com. Marcos Dajczer was partially supported by MICINN/FEDER project PGC2018-097046-B-I00 and Fundación Séneca project 19901/GERM/15, Spain. e-mail: marcos@impa.br.

1 Introduction

An isometric immersion $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ is called a *real Kaehler submanifold* if (M^{2n}, J) is a Kaehler manifold of complex dimension $n \geq 2$ immersed into Euclidean space with codimension p. We are interested in the case when f is minimal but not holomorphic. By the latter condition we mean that p is even and f is holomorphic with respect to a constant complex structure in $\mathbb{R}^{2n+p}$.

Minimal real Kaehler submanifolds have been intensively studied since in [3] it was shown that they enjoy several of the basic properties of Euclidean minimal surfaces. For instance, if simply connected a minimal real Kaehler submanifold $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ is either holomorphic or has a nontrivial one-parameter associated family of minimal isometric immersions, all of them carrying the same oriented Gauss map. Moreover, f can be realized as the "real part" of its holomorphic representative $\sqrt{2}F = (f, \bar{f})\colon M^{2n} \to \mathbb{C}^{2n+p}$ where $\bar{f}\colon M^{2n} \to \mathbb{R}^{2n+p}$ is the conjugate immersion to f in the associated family.

Real Kaehler hypersurfaces $f\colon M^{2n} \to \mathbb{R}^{2n+1}$, in particular the minimal ones, have been parametrically classified by Dajczer and Gromoll [3] in terms of pseudoholomorphic surfaces in spheres by means of the so called Gauss parametrization. A parametric classification of the complete minimal Real Kaehler submanifold in codimension two was obtained by Dajczer and Gromoll in [4]. A local Weierstrass type representation for the minimal real Kaehler submanifolds of any possible codimension was given by Arezzo, Pirola and Solci in [1]; see the Appendix of Chapter 15 in [6].

We have to recall some definitions. Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kaehler submanifold and let $\alpha\colon TM \times TM \to N_fM$ denote its second fundamental form taking values in the normal bundle of f. The *first normal space* $N_1^\kappa(x) \subset N_fM(x)$ of f at $x \in M^{2n}$, $\kappa \leq p$, is defined as

$$N_1^\kappa(x) = \operatorname{span}\{\alpha(X, Y)\colon X, Y \in T_xM\}.$$

If $U^s \subset N_1^\kappa(x)$ is an s-dimensional vector subspace we denote $\alpha_{U^s} = \pi_{U^s} \circ \alpha$

where $\pi_{U^s}\colon N_1^\kappa(x) \to U^s$ is the projection. Then let $\mathcal{N}_c(\alpha_{U^s}) \subset T_xM$ be the complex vector tangent subspace given by

$$\mathcal{N}_c(\alpha_{U^s}) = \{Y \in T_xM\colon \alpha_{U^s}(X,Y) = \alpha_{U^s}(X,JY) = 0 \ \text{ for all } \ X \in T_xM\}$$

and $\nu^c(\alpha_{U^s}) = \dim \mathcal{N}_c(\alpha_{U^s})$. The *complex s-nullity* $\nu_s^c(x)$ of f at $x \in M^{2n}$, $1 \le s \le \kappa$, is defined by

$$\nu_s^c(x) = \max_{U^s \subset N_1(x)} \nu^c(\alpha_{U^s}).$$

Recall that $\nu_\kappa^c(x)$ is called the index of complex relative nullity of f.

Theorem 1.1. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$, $p \le n$, be a real Kaehler submanifold. Assume that at each point $x \in U$ of an open dense subset U of M^{2n} we have that either*

(i) $\nu_1^c(x) < 2n-2$ and $\nu_{2s}^c(x) < 2n-4s$ for $s \ge 1$, or

(ii) $\nu_{2s+1}^c(x) < 2n-4s-2$ for $s \ge 0$.

Then f is a minimal submanifold.

We observe that Renato Tribuzy to whom this paper is dedicated has given many valuable contributions to the subject of isometric immersions of Kaehler manifolds, for instance, see [2], [7], [8], [9] and [10].

2 Flat bilinear forms

Let $\varphi\colon U \times V \to W$ be a bilinear form between finite dimensional real vector spaces. We denote by

$$\mathcal{S}(\varphi) = \operatorname{span}\{\varphi(X,Y)\colon X \in U,\ Y \in V\},$$

the vector subspace of W generated by the *image* of φ. The (right) *kernel* of φ is the vector subspace of V defined by

$$\mathcal{N}(\varphi) = \{Y \in V\colon \varphi(X,Y) = 0 \text{ for all } X \in U\}$$

and the *nullity* of φ is $\nu(\varphi) = \dim \mathcal{N}(\varphi)$.

A vector $X \in U$ is called a (left) *regular element* of φ if $\dim \varphi_X(V) = r_o$ where $\varphi_X \colon V \to W$ is the linear map defined by $\varphi_X Y = \varphi(X,Y)$ and

$$r_o = \max\{\dim \varphi_Z(V) \colon Z \in U\}. \tag{2.1}$$

The set $RE(\varphi)$ of regular elements of φ is easily seen to be an open dense subset of U; cf. Proposition 4.4 in [6].

Let $\varphi \colon V \times V \to W$ be a bilinear form where W is endowed with an inner product $\langle\, ,\, \rangle$ of any signature. Then φ is called *flat* if

$$\langle \varphi(X,Y), \varphi(Z,T)\rangle - \langle \varphi(X,T), \varphi(Z,Y)\rangle = 0$$

for any $X, Y, Z, T \in V$.

Lemma 2.1. *Let $\varphi \colon V \times V \to W$ be a flat bilinear form. If $X \in RE(\varphi)$ then*

$$\mathcal{S}(\varphi|_{V \times \ker \varphi_X}) \subset \varphi_X(V) \cap \varphi_X(V)^{\perp}. \tag{2.2}$$

Proof: See equations (8) and (9) in [12] or Proposition 4.6 in [6]. ∎

Proposition 2.2. *Let V be a real vector space endowed with a complex structure, that is, there is $J \in End(V)$ such that $J^2 = -I$. If $X_1, JX_1, \ldots, X_{k-1}, JX_{k-1}, X_k \in V$ are linearly independent vectors then also $X_1, JX_1, \ldots, X_k, JX_k$ are linearly independent. In particular, we have that V has even dimension.*

Proof: If $JX_k = \sum_{i=1}^{k} a_i X_i + \sum_{j=1}^{k-1} b_j JX_j$ for $0 \neq (a_1, \ldots, a_k, b_1, \ldots, b_{k-1})$ then

$$(1 + a_k^2) X_k + \sum_{j=1}^{k-1} ((a_k a_j - b_j) X_j + (a_k b_j + a_j) JX_j) = 0,$$

and this is a contradiction. ∎

In the sequel U^p denotes a p-dimensional vector space endowed with a positive definite inner product. Then $W^{p,p} = U^p \oplus U^p$ is endowed with the inner product of signature (p,p) given by

$$\langle\!\langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle\!\rangle = \langle \xi_1, \eta_1 \rangle_{U^p} - \langle \xi_2, \eta_2 \rangle_{U^p}.$$

Proposition 2.3. *Let $\rho\colon V^{2n}\times V^{2n}\to U^p$ be a symmetric bilinear form and $J\in End(V)$ a complex structure. Let $\sigma\colon V^{2n}\times V^{2n}\to W^{p,p}$ be the associated bilinear form given by*

$$\sigma(X,Y)=(\rho(X,Y),\rho(X,JY))$$

and let $V_1,V_2\subset V^{2n}$ be vector subspaces where V_2 is J-invariant. Then the bilinear form $\sigma_0=\sigma|_{V_1\times V_2}\colon V_1\times V_2\to\mathcal{S}(\sigma_0)\subset W^{p,p}$ satisfies:

(i) The vector subspace $\mathcal{N}(\sigma_0)$ of V_2 is J-invariant.

(ii) There exists a complex structure $T\in End(\mathcal{S}(\sigma_0))$.

(iii) The vector subspace $\mathcal{S}(\sigma_0)\subset W^{p,p}$ has even dimension.

Proof: We prove part (ii). If $(\xi,\eta)\in\mathcal{S}(\sigma_0)$ let $X_i\in V_1$, $Y_i\in V_2$, $1\le i\le\ell$, be such that

$$(\xi,\eta)=\sum_{i=1}^{\ell}\sigma_0(X_i,Y_i)=\sum_{i=1}^{\ell}(\xi_i,\eta_i).$$

Then

$$\sum_{i=1}^{\ell}\sigma_0(X_i,JY_i)=\sum_{i=1}^{\ell}(\eta_i,-\xi_i)=(\eta,-\xi)\in\mathcal{S}(\sigma_0),$$

and hence $T\in\mathrm{End}(\mathcal{S}(\sigma_0))$ defined by $T(\xi,\eta)=(\eta,-\xi)$ satisfies $T^2=-I$. Now part (iii) follows from Proposition 2.2. ∎

Let $\alpha\colon V^{2n}\times V^{2n}\to U^p$ be a bilinear form and let $J\in\mathrm{End}(V)$ be a complex structure. In the sequel $\beta\colon V^{2n}\times V^{2n}\to W^{p,p}$ is the associated bilinear form given by

$$\beta(X,Y)=(\alpha(X,Y)+\alpha(JX,JY),\alpha(X,JY)-\alpha(JX,Y)). \tag{2.3}$$

Notice that Proposition 2.3 applies to β. It follows from Proposition 2.2 that the subspace

$$\mathcal{N}(\beta)=\{X\in V^{2n}:\alpha(X,JY)=\alpha(JX,Y)\ \text{ for all }\ Y\in V^{2n}\}$$

is even dimensional. We also have that β verifies

$$\beta(X,X) = (\xi, 0), \;\; \beta(X, JX) = -(0, \xi), \tag{2.4}$$

$$\beta(X,Y) = (\xi,\eta) \;\text{ if and only if }\; \beta(Y,X) = (\xi,-\eta) \tag{2.5}$$

and

$$\beta(X,Y) = (\xi,\eta) \;\text{ if and only if }\; \beta(X,JY) = (\eta,-\xi). \tag{2.6}$$

From (2.6) we obtain that

$$\beta(X,Y) = \beta(S,T) \;\text{ if and only if }\; \beta(X,JY) = \beta(S,JT) \tag{2.7}$$

whereas from (2.5) and (2.6) that

$$\beta(X,Y) = \beta(JX,JY). \tag{2.8}$$

Proposition 2.4. *Assume that $\beta\colon V^{2n} \times V^{2n} \to W^{p,p}$ is flat. If a vector subspace $L \subset V^{2n}$ satisfies $\beta|_{L\times L} = 0$ then $L \subset \mathcal{N}(\beta)$.*

Proof: If $\beta(X,Y) = (\xi,\eta)$, we obtain from (2.5) that

$$\langle\!\langle \beta(X,Y), \beta(Y,X)\rangle\!\rangle = \|\xi\|^2_{U^p} + \|\eta\|^2_{U^p}.$$

Thus

$$\beta(X,Y) = 0 \;\text{ if and only if }\; \langle\!\langle \beta(X,Y), \beta(Y,X)\rangle\!\rangle = 0. \tag{2.9}$$

Hence, since

$$\langle\!\langle \beta(X,Z), \beta(Z,X)\rangle\!\rangle = \langle\!\langle \beta(Z,Z), \beta(X,X)\rangle\!\rangle = 0$$

if $Z \in L$ and $X \in V^{2p}$, then $\beta(X,Z) = 0$. ∎

Proposition 2.5. *If $\beta\colon V^{2n}\times V^{2n} \to W^{p,p}$, $p \le n$, is flat then $\nu(\beta) = 2n - r_o$ where r_o is given by (2.1). In particular, we have $\nu(\beta) \ge 2n - \dim \mathcal{S}(\beta)$.*

Proof: Let $L = \ker B_X$ where $X \in RE(\beta)$ and $B_X\colon V^{2n} \to W^{p,p}$ is the linear map given by $B_X Y = \beta(X,Y)$. Notice that $\mathcal{N}(\beta) \subset L$. By (2.2) we have

$$\langle\!\langle \beta(X,Y), \beta(Y,X)\rangle\!\rangle = 0$$

for any $X, Y \in L$. Then (2.9) gives $\beta|_{L\times L} = 0$ and therefore $L = \mathcal{N}(\beta)$ by Proposition 2.4. Then $\nu(\beta) = \dim L = 2n - r_o$. ∎

The following result gives an alternative presentation and proof of Lemma 7 in [11].

Proposition 2.6. *Assume that $\beta\colon V^{2n} \times V^{2n} \to W^{p,p}$, $p \leq n$, is flat. If $\nu(\beta) = 2(n-p)$ there exists a basis $\{X_i, JX_i\}_{1\leq i\leq n}$ of V^{2n} such that*

(i) $\mathcal{N}(\beta) = span\{X_j, JX_j,\ p+1 \leq j \leq n\}$.

(ii) $\beta(Y_i, Y_j) = 0$ if $Y_k \in span\{X_k, JX_k\}$ where $i \neq j$ and $k = i, j$.

(iii) $\{\beta(X_j, X_j), \beta(X_j, JX_j), 1 \leq j \leq p\}$ is an orthonormal basis of $W^{p,p}$.

Proof: Assume that the result holds for $p = n$. By part (i) of Proposition 2.3 the vector subspace $\mathcal{N}(\beta)$ is J-invariant. By Proposition 2.2 there is a decomposition $V^{2n} = V_0^{2p} \oplus \mathcal{N}(\beta)$ where V_0^{2p} is J-invariant. The bilinear form $\beta_0 = \beta|_{V_0\times V_0}\colon V_0^{2p} \times V_0^{2p} \to W^{p,p}$ is flat and $\mathcal{N}(\beta_0) = 0$. In fact, if $Z \in \mathcal{N}(\beta_0)$ decompose $X \in V^{2n}$ as $X = X_1 + X_2$ with $X_1 \in V_0^{2p}$ and $X_2 \in \mathcal{N}(\beta)$. Since $\beta(Z, X_2) = 0$ we have from (2.5) that $\beta(X_2, Z) = 0$. Then $Z \in \mathcal{N}(\beta)$ since $\beta(X, Z) = \beta(X_1, Z) + \beta(X_2, Z) = 0$, and thus $Z = 0$.

By the initial assumption there exists a basis $\{X_j, JX_j\}_{1\leq j\leq p}$ of V_0^{2p} such that parts (ii) and (iii) hold. Then, by Proposition 2.2 we can complete the basis of V_0^{2p} to a basis $\{X_j, JX_j\}_{1\leq j\leq n}$ of V^{2n} such that also part (i) holds.

By the above, it remains to argue for the case $p = n$, that is, when $\nu(\beta) = 0$.

Fact 1. If $p \geq 2$ there exist non-zero vectors $X, Y \in V^{2p}$ such that $\beta(X, Y) = 0$.

If $X \in \mathrm{RE}(\beta)$ and since $\nu(\beta) = 0$, then from Proposition 2.5 the map $B_X\colon V^{2p} \to W^{p,p}$ is an isomorphism. Since $\mathrm{RE}(\beta)$ is open and dense in V^{2p} there is a basis $Z_1, \ldots, Z_{2p}$ of V^{2p} such that $Z_2 \notin \mathrm{span}\{Z_1, JZ_1\}$ and $\{\beta(Z_k, Z_j)\}_{1\leq j\leq 2p}$ is for $k = 1$ as well as for $k = 2$ a basis of $W^{p,p}$. Let

$A = (a_{ij})$ be the $2p \times 2p$ matrix given by

$$\beta(Z_2, Z_j) = \sum_{r=1}^{2p} a_{rj} \beta(Z_1, Z_r).$$

Let $\lambda \in \mathbb{C}$ be a eigenvalue of A where $(v^1, \ldots, v^{2p}) \in \mathbb{C}^{2p}$ is the corresponding eigenvector. Extending β linearly from $V^{2p} \otimes \mathbb{C}$ to $W^{p,p} \otimes \mathbb{C}$, we have

$$\sum_{j=1}^{2p} v^j \beta(Z_2, Z_j) = \lambda \sum_{j=1}^{2p} v^j \beta(Z_1, Z_j).$$

Hence $\beta(S, T) = 0$ where $S = Z_2 - \lambda Z_1$ and $T = \sum_{j=1}^{2p} v^j Z_j$. Then

$$\beta(S_1, T_1) = \beta(S_2, T_2) \text{ and } \beta(S_1, T_2) + \beta(S_2, T_1) = 0 \tag{2.10}$$

where $S = S_1 + iS_2$ and $T = T_1 + iT_2$. If $X = S_1 - JS_2$ and $Y = T_1 + JT_2$, we obtain using (2.7), (2.8) and (2.10) that

$$\begin{aligned}\beta(X, Y) &= \beta(S_1, T_1) + \beta(S_1, JT_2) - \beta(JS_2, T_1) - \beta(JS_2, JT_2) \\ &= \beta(S_1, T_1) + \beta(S_1, JT_2) + \beta(S_2, JT_1) - \beta(S_2, T_2) = 0.\end{aligned}$$

Similarly, for $X' = S_1 + JS_2$ and $Y' = T_1 - JT_2$ we obtain $\beta(X', Y') = 0$. The vectors X and X' are both non-zero. For instance, if $X = 0$ then

$$JS_2 + iS_2 = S = Z_2 - \lambda Z_1.$$

Thus $JS_2 = Z_2 - \mathrm{Re}(\lambda) Z_1$ and $S_2 = -\mathrm{Im}(\lambda) Z_1$. Then $Z_2 \in \mathrm{span}\,\{Z_1, JZ_1\}$, and this is a contradiction. Finally, if $Y = Y' = 0$ then $T = 0$, and this is a contradiction.

Fact 2. There exists $Z_0 \in V^{2p}$ such that $\dim \ker B_{Z_0} = 2(p-1)$.

Fact 2 holds for $p = 1$. In fact, given $0 \neq X \in V^2$ we have from Proposition 2.4 that $\beta(X, X) \neq 0$. From (2.4) the vectors $\beta(X, X), \beta(X, JX)$ are linearly independent, and thus $\ker B_X = 0$.

For $p \geq 2$ we argue by induction. Assume that Fact 2 is true for any $q \leq p - 1$. By Fact 1 there are nonzero vectors $X, Y \in V^{2p}$ such that

$\beta(X,Y)=0$. By part (iii) of Proposition 2.3 the dimension of $B_X(V)$ is even. If $\dim B_X(V)=2r$ then $r<p$ since $B_XY=0$. Moreover, we have that $N_X=\ker B_X\neq V^{2p}$. If otherwise, we would have from (2.5) that $X\in\mathcal{N}(\beta)$ and hence $X=0$. Thus $\dim N_X=2p-2r$, $1\leq r\leq p-1$.

Let $U_1^s=\pi_1(B_X(V))$ where $\pi_1\colon W^{p,p}\to U^p$ is the projection onto the first component. We claim that $s=r$ and that

$$B_X(V)=U_1^r\oplus U_1^r=\{\beta(Z,X)\colon Z\in V^{2p}\}. \tag{2.11}$$

To prove the claim, we first show that

$$B_X(V)+\{\beta(Z,X)\colon Z\in V^{2p}\}=U_1^s\oplus U_1^s,\ \ s\geq r. \tag{2.12}$$

If $\beta(X,Z)=(\xi,\eta)$ then (2.6) gives $\eta\in U_1^s$, and hence $(\xi,\eta)\in U_1^s\oplus U_1^s$. From (2.5) if $\beta(Z,X)=(\zeta,\eta)$ then $\zeta\in U_1^s$. Moreover, since $\beta(X,JZ)=-(\eta,\zeta)$ from (2.5) and (2.6), then $\eta\in U_1^s$, and thus $(\zeta,\eta)\in U_1^s\oplus U_1^s$. For the other inclusion, let $(\xi_1,\xi_2)\in U_1^s\oplus U_1^s$. Then there are $Z_1,Z_2\in V^{2p}$ such that $\beta(X,Z_i)=(\xi_i,\eta_i)$, $i=1,2$. Then using (2.5) and (2.6) we obtain that

$$(\xi_1,\xi_2)=\frac{1}{2}(\beta(X,Z_1-JZ_2)+\beta(Z_1+JZ_2,X)),$$

and (2.12) has been proved. If $U^p=U_1^s\oplus U_2^{p-s}$ is an orthogonal splitting, we show that

$$\mathcal{S}(\beta|_{N_X\times N_X})\subset U_2^{p-s}\oplus U_2^{p-s}. \tag{2.13}$$

The flatness of β gives

$$\langle\!\langle\beta(X,Z),\beta(S,T)\rangle\!\rangle=\langle\!\langle\beta(X,T),\beta(S,Z)\rangle\!\rangle=0$$

for any $S,T\in N_X$ and $Z\in V^{2p}$. Moreover, from (2.5) we have $\beta(S,X)=0$, and thus

$$\langle\!\langle\beta(Z,X),\beta(S,T)\rangle\!\rangle=\langle\!\langle\beta(Z,T),\beta(S,X)\rangle\!\rangle=0,$$

and then (2.12) gives (2.13).

N_X is J-invariant by Proposition 2.3 and $\beta|_{N_X\times N_X}\colon N_X^{2p-2r}\times N_X^{2p-2r}\to U_2^{p-s}\oplus U_2^{p-s}$ is flat. Then Proposition 2.5 gives that $L=\mathcal{N}(\beta|_{N_X\times N_X})$ satisfies $\dim L\geq 2s-2r\geq 0$. On the other hand, since $\beta|_{L\times L}=0$ it follows from Proposition 2.4 that $L\subset\mathcal{N}(\beta)=0$, and thus $s=r$. Then (2.11) holds since the first equality follows from (2.12) and the second equality by (2.5).

The assumption of induction applies to $\beta|_{N_X\times N_X}\colon N_X\times N_X\to U_2^{p-r}\oplus U_2^{p-r}$ since $r\geq 1$ and $\mathcal{N}(\beta|_{N_X\times N_X})$=0. Therefore there exists $Z_0\in N_X$ such that

$$\dim\ker B_{Z_0}|_{N_X}=2(p-r-1). \tag{2.14}$$

We have that

$$\langle\langle\beta(S,X),\beta(Z_0,T)\rangle\rangle=\langle\langle\beta(S,T),\beta(Z_0,X)\rangle\rangle=0$$

for any $S,T\in V^{2p}$. It follows from (2.11) that $B_{Z_0}(V)\subset U_2^{p-r}\oplus U_2^{p-r}$. Proposition 2.4 gives $B_{Z_0}Z_0\neq 0$. Hence, in view of (2.14) there is a basis $X_1=Z_0, X_2=JZ_0, X_3,\ldots,X_{2(p-r)}$ of N_X such that $\beta(Z_0,X_j)=0$, $3\leq j\leq 2(p-r)$. Since $\mathcal{N}(\beta|_{N_X\times N_X})=0$, we have from Proposition 2.5 that

$$\operatorname{span}\{\beta(X_i,X_j)\ 1\leq i,j\leq 2(p-r)\}=U_2^{p-r}\oplus U_2^{p-r}. \tag{2.15}$$

By (2.4) we may set $\beta(Z_0,Z_0)=(\xi,0)$. Then (2.5), (2.6) and (2.8) give

$$\beta(Z_0,JZ_0)=(0,-\xi),\ \ \beta(JZ_0,Z_0)=(0,\xi)\ \text{ and }\ \beta(JZ_0,JZ_0)=(\xi,0). \tag{2.16}$$

Flatness yields

$$\langle\langle\beta(X_s,X_t),\beta(Z_0,W)\rangle\rangle=\langle\langle\beta(X_s,W),\beta(Z_0,X_t)\rangle\rangle=0 \tag{2.17}$$

for $3\leq s,t\leq 2(p-r)$ and any $W\in V^{2p}$. From (2.15), (2.16) and (2.17) we obtain

$$\begin{aligned}&\operatorname{span}\{\beta(X_s,X_t)\colon 3\leq s,t\leq 2(p-r)\}\\&\qquad=(\operatorname{span}\{\xi\})^{\perp}\cap U_2^{p-r}\oplus(\operatorname{span}\{\xi\})^{\perp}\cap U_2^{p-r}.\end{aligned}$$

It follows from (2.15) and (2.17) that $B_{Z_0}(V) = \text{span}\,\{\xi\} \oplus \text{span}\,\{\xi\}$, and this gives the proof of Fact 2.

We conclude the proof by means of a recursive construction. Notice that it suffices to construct an orthogonal basis of $W^{p,p}$ since by (2.4) it can be replaced by an orthonormal one. By Fact 2 there is $X_1 \in V^{2p}$ such that $N_{X_1} = \ker B_{X_1}$ satisfies $\dim N_{X_1} = 2p-2$ and by Proposition 2.3 the vector subspace N_{X_1} is J-invariant. Proposition 2.4 gives $\beta(X_1, X_1) = (\xi_1, 0) \neq 0$ and (2.6) that $\beta(X_1, JX_1) = (0, -\xi_1)$. If $p = 1$, then X_1, JX_1 is the desired basis. If $p \geq 2$ we have by flatness that

$$\langle\!\langle \beta(X_1, X_1), \beta(N_{X_1}, N_{X_1})\rangle\!\rangle = 0 = \langle\!\langle \beta(X_1, JX_1), \beta(N_{X_1}, N_{X_1})\rangle\!\rangle.$$

Since $\mathcal{N}(\beta) = 0$ we have from Proposition 2.4 that the bilinear form

$$\hat{\beta} = \beta|_{N_{X_1} \times N_{X_1}} \colon N_{X_1} \times N_{X_1} \to (\text{span}\,\{\xi_1\})^\perp \oplus (\text{span}\,\{\xi_1\})^\perp$$

satisfies $\mathcal{N}(\hat{\beta}) = 0$. By Fact 2 there is $X_2 \in N_{X_1}$ such that $\dim \ker \hat{B}_{X_2} = 2p-4$. As above, we have that $\beta(X_2, X_2) = (\xi_2, 0) \neq 0$ and $\beta(X_2, JX_2) = (0, -\xi_2)$, where ξ_2 is perpendicular to ξ_1. Since N_{X_1} is J-invariant, then

$$\beta(X_1, X_2) = 0 = \beta(X_1, JX_2).$$

If $p = 2$ then X_1, JX_1, X_2, JX_2 is the desired basis. If $p \geq 3$ we just reiterate the construction. ∎

In the sequel, let $\gamma \colon V^{2n} \times V^{2n} \to W^{p,p}$ be the bilinear form associated to the symmetric bilinear form $\alpha \colon V^{2n} \times V^{2n} \to U^p$ given by

$$\gamma(X, Y) = (\alpha(X, Y), \alpha(X, JY)). \tag{2.18}$$

Proposition 2.7. *Let the bilinear forms $\gamma, \beta \colon V^{2n} \times V^{2n} \to W^{p,p}$ be flat and satisfy that*

$$\langle\!\langle \beta(X, Y), \gamma(Z, T)\rangle\!\rangle = \langle\!\langle \beta(X, T), \gamma(Z, Y)\rangle\!\rangle \tag{2.19}$$

for any $X, Y, Z, T \in V^{2n}$. If $V_1^s = \pi_1(\mathcal{S}(\beta))$ and α_{V_1} denotes taking the V_1^s-component of α then $\mathcal{N}(\beta) = \mathcal{N}_c(\alpha_{V_1})$.

Proof: We first show that

$$\mathcal{S}(\beta) = V_1^s \oplus V_1^s. \tag{2.20}$$

If $(\xi,\eta) \in \mathcal{S}(\beta)$ then $(\xi,0),(0,\xi),(\eta,0) \in \mathcal{S}(\beta)$. In fact, if

$$(\xi,\eta) = \sum_k \beta(X_k,Y_k) = \sum_k (\xi_k,\eta_k),$$

we obtain from (2.5) and (2.6) that

$$\sum_k \beta(Y_k,X_k) = (\xi,-\eta), \;\; \sum_k \beta(X_k,JY_k) = (\eta,-\xi),$$

$$\sum_k \beta(JY_k,X_k) = (\eta,\xi).$$

It follows that $\mathcal{S}(\beta) \subset V_1 \oplus V_1$. For the other inclusion, let $(\xi,\eta) \in V_1 \oplus V_1$. Then there are $\delta,\bar\delta \in U^p$ such that $(\xi,\delta),(\eta,\bar\delta) \in \mathcal{S}(\beta)$, and by the above $(\xi,\eta) \in \mathcal{S}(\beta)$.

From (2.19) and (2.20) we have $\mathcal{S}(\gamma|_{V\times\mathcal{N}(\beta)}) \subset V_1^\perp \oplus V_1^\perp$. Thus

$$\langle \alpha(X,Y),\xi\rangle = \langle\!\langle \gamma(X,Y),(\xi,0)\rangle\!\rangle = 0$$

and

$$\langle \alpha(X,JY),\xi\rangle = -\langle\!\langle \gamma(X,Y),(0,\xi)\rangle\!\rangle = 0$$

for any $X \in V^{2p}$, $Y \in \mathcal{N}(\beta)$ and $\xi \in V_1$. Thus $\mathcal{N}(\beta) \subset \mathcal{N}_c(\alpha_{V_1})$. The remaining inclusion follows from (2.3) and (2.20). ∎

3 The proof

Proposition 3.1. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kaehler submanifold. At $x \in M^{2n}$ the bilinear forms $\beta,\gamma\colon T_xM \times T_xM \to W^{p,p} = N_fM(x) \oplus N_fM(x)$ defined by (2.3) and (2.18) are flat and the condition (2.19) is satisfied.*

Proof: The curvature tensor of a Kaehler manifold M^{2n} satisfies

$$R(X,Y) = R(JX,JY) \ \text{ and } \ R(X,Y)JZ = JR(X,Y)Z$$

for any $X,Y,Z \in T_xM$; cf. Proposition 15.1 in [6]. Then straightforward computations using the Gauss equation give the result. ∎

Proof of Theorem 1.1: We claim that $V_1^s(x) = \pi_1(\mathcal{S}(\beta))$ satisfies $s = 0$. Suppose otherwise that $s > 0$. From Proposition 2.5, Proposition 2.7 and (2.20) we obtain

$$\nu_s^c(x) \geq \nu^c(\alpha_{V_1^s}(x)) \geq \nu(\beta) \geq 2(n-s).$$

In particular, $s \neq 1$ since otherwise we have a contradiction with the assumptions of the theorem.

Suppose that $s \geq 2$. If we have $\nu(\beta) > 2(n-s)$ then again from Proposition 2.5, Proposition 2.7 and (2.20) we obtain

$$\nu_{s-1}^c(x) \geq \nu_s^c(x) \geq \nu^c(\alpha_{V_1^s}(x)) \geq \nu(\beta(x)) \geq 2(n-s+1)$$

and this is contradiction with both parts of Theorem 1.1. Hence $\nu(\beta) = 2(n-s)$.

Let $\{X_i, JX_i\}_{1\leq i\leq n}$ be the basis of T_xM given by Proposition 2.6 and $\xi_j = \pi_1(\beta(X_j,X_j))$ for $1 \leq j \leq s$. If $i \neq j$ we obtain from (2.4) and (2.19) that

$$0 = \langle\langle \gamma(X,X_j), \beta(X_j,X_i)\rangle\rangle = \langle\langle \gamma(X,X_i), \beta(X_j,X_j)\rangle\rangle = \langle A_{\xi_j}X_i, X\rangle$$

and

$$0 = \langle\langle \gamma(X,JX_j), \beta(X_j,X_i)\rangle\rangle = \langle A_{\xi_j}JX_i, X\rangle$$

for any $X \in T_xM$. Then

$$\text{span}\,\{X_i, JX_i,\ 1 \leq i \leq n \text{ and } i \neq j\} \subset \ker A_{\xi_j} \cap \ker A_{\xi_j}J$$

and thus

$$\dim(\ker A_{\xi_j} \cap \ker A_{\xi_j}J) = 2n-2,\ 1 \leq j \leq s.$$

Then $\nu^c_{s_0}(x) \geq 2(n - s_0)$ for any $1 \leq s_0 \leq s$. In particular, we have $\nu^c_{s-1}(x) \geq 2(n - s + 1)$ and this has been seen to be a contradiction. Thus $s = 0$, that is,

$$\alpha(JX, Y) = \alpha(X, JY) \tag{3.1}$$

for any $X, Y \in \mathfrak{X}(M)$. In particular, the submanifold is minimal. ∎

Remark 3.2. *Theorem* 1.2 *in [5] or Theorem* 15.7 *in [6] give that the condition* (3.1) *of f being pluriharmonic is equivalent to minimality.*

Remark 3.3. *We observe that Theorem 1.1 does not apply for $p = 1$ since in this case the nonflat examples have $\nu^c_1 = 2n - 2$.*

References

[1] C. Arezzo, G. Pirola and M. Solci, The Weierstrass representation for pluriminimal submanifolds, *Hokkaido Math. J.* **33** (2004), 357–367.

[2] F. Burstall, J.-H. Eschenburg, M. Ferreira and R. Tribuzy, Kähler submanifolds with parallel pluri-mean curvature, Differential Geom. Appl. **20** (2004), 47–66.

[3] M. Dajczer and D. Gromoll, Real Kaehler submanifolds and uniqueness of the Gauss map, *J. Differential Geom.* **22** (1985), 13–28.

[4] M. Dajczer and D. Gromoll, The Weierstrass representation for complete minimal real Kaehler submanifolds of codimension two, *Invent. Math.* **119** (1995), 235–242.

[5] M. Dajczer and L. Rodríguez, Rigidity of real Kaehler submanifolds, *Duke Math. J.* **53** (1986), 211–220.

[6] M. Dajczer and R. Tojeiro, *"Submanifold theory beyond an introduction"*, Series: Universitext. Springer, 2019.

[7] J.-H. Eschenburg, M. Ferreira and R. Tribuzy, Isotropic ppmc immersions, *Differential Geom. Appl.* **25** (2007), 351–355.

[8] J.-H. Eschenburg, A. Kollross and R. Tribuzy, Codimension of immersions with parallel pluri-mean curvature, *Differential Geom. Appl.* **27** (2009), 691–695.

[9] M. Ferreira and R. Tribuzy, Codimension two Kähler submanifolds of space forms, *Arch. Math.* **79** (2002), 520–528.

[10] M. Ferreira and R. Tribuzy, On the nullity of isometric immersions from Kähler manifolds, *Rend. Semin. Mat. Univ. Politec. Torino* **65** (2007), 345–352.

[11] L. Florit, W. Hui and F. Zheng, On real Kaehler Euclidean submanifolds with non-negative Ricci curvature, *J. Eur. Math. Soc.* **7** (2005), 1–11.

[12] J. Moore, Submanifolds of constant positive curvature I, *Duke Math. J.* **44** (1977), 449–489.