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Cyclic conformally flat hypersurfaces revisited

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. In this article we classify the conformally flat Euclidean hypersurfaces of dimension three with three distinct principal curvatures of $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ with the property that the tangent component of the vector field $\partial/\partial t$ is a principal direction at any point. Here $\partial/\partial t$ stands for either a constant unit vector field in $\mathbb{R}^4$ or the unit vector field tangent to the factor $\mathbb{R}$ in the product spaces $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$, respectively. Then we use this result to give a simple proof of an alternative classification of the cyclic conformally flat hypersurfaces of $\mathbb{R}^4$, that is, the conformally flat hypersurfaces of $\mathbb{R}^4$ with three distinct principal curvatures such that the curvature lines correspondent to one of its principal curvatures are extrinsic circles. We also characterize the cyclic conformally flat hypersurfaces of $\mathbb{R}^4$ as those conformally flat hypersurfaces of dimension three with three distinct principal curvatures for which there exists a conformal Killing vector field of $\mathbb{R}^4$ whose tangent component is an eigenvector field correspondent to one of its principal curvatures.

Keywords: conformally flat hypersurface, linear Weingarten surface, product spaces, conformal Killing vector field.

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1 Introduction

E. Cartan proved in [Ca] that if $f\colon M^n \to \mathbb{R}^{n+1}$ is an isometric immersion of a Riemannian manifold M^n of dimension $n \geq 4$, then M^n is conformally flat if and only if f has a principal curvature of multiplicity at least $n-1$. Recall that a Riemannian manifold M^n is conformally flat if each point of M^n has an open neighborhood that is conformally diffeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. Cartan also proved that any hypersurface $f\colon M^3 \to \mathbb{R}^4$ with a principal curvature of multiplicity greater than one is conformally flat, and realized that the converse is no longer true in this case. Thus, generic conformally flat Euclidean hypersurfaces of dimension $n \geq 4$ are envelopes of one-parameter families of hyperspheres, but in dimension $n = 3$ there appears an interesting further class of conformally flat hypersurfaces which have three distinct principal curvatures.

Cartan's investigations were taken up by Hertrich-Jeromin [H-J], who showed that any conformally flat Euclidean hypersurface of dimension three with three distinct principal curvatures carries local principal coordinates u_1, u_2, u_3 with respect to which the induced metric can be written as

$$ds^2 = \sum_{i=1}^{3} v_i^2 du_i^2,$$

with the Lamé coefficients v_i, $1 \leq i \leq 3$, satisfying the Guichard condition, say, $v_2^2 = v_1^2 + v_3^2$. Then he used the conformal invariance of this condition to associate with each such hypersurface a Guichard net in $\mathbb{R}^3$, that is, a conformally flat metric on an open subset of $\mathbb{R}^3$ satisfying the Guichard condition, which is unique up to a Moebius transformation. He also proved in [H-J] (see also Section 2.4.6 in [H-J$_2$]) that each conformally flat 3-metric satisfying the Guichard condition gives rise to a unique (up to a Moebius transformation) conformally flat hypersurface in $\mathbb{R}^4$ (see also [CT$_1$]).

Improving earlier work by Suyama (see [Su1], [Su2]), Hertrich–Jeromin and Suyama [H-JS] gave a classification of conformally flat hypersurfaces whose associated Guichard nets in $\mathbb{R}^3$ are cyclic, that is, one of their coordinate line families consists of circular arcs. These include the so-called *conformal product* conformally flat hypersurfaces, which are the images under Moebius transformations of $\mathbb{R}^4$ of either cylinders over umbilic-free surfaces of constant Gauss curvature in $\mathbb{R}^3$, cones over umbilic-free surfaces of constant Gauss curvature in $\mathbb{S}^3$ or rotation hypersurfaces over umbilic-free surfaces in a half-space $\mathbb{R}^3_+$ of $\mathbb{R}^3$, regarded as the half-space model of $\mathbb{H}^3$, which have constant Gauss curvature with respect to the metric induced from the hyperbolic metric on $\mathbb{R}^3_+$. Conformal product conformally flat hypersurfaces in $\mathbb{R}^4$ have been characterized in [**DT**$_1$] as those conformally flat hypersurfaces with three distinct principal curvatures in $\mathbb{R}^4$ such that the curvature lines correspondent to one of its principal curvatures are arcs of circles or straight lines in $\mathbb{R}^4$.

A class of noncyclic conformally flat hypersurfaces was subsequently studied in [**H-JS**$_2$], whose associated Guichard systems are of Bianchi-type, that is, its coordinate surfaces have constant Gauss curvature. However, until not very long ago, all the known explicit examples of conformally flat hypersurfaces of $\mathbb{R}^4$ with three distinct principal curvatures belonged to the class of cyclic conformally flat hypersurfaces. More recently, a Ribaucour transformation for the class of conformally flat hypersurfaces of $\mathbb{R}^4$ with three distinct principal curvatures, based on the characterization of such hypersurfaces obtained in [**CT**$_1$], was developed in [**CT**$_2$], which allowed to construct explicit noncyclic examples (see also [**H-JSUY**] and [**ST**]).

One of the goals of this article is to give a simple proof of an alternative description of cyclic conformally flat hypersurfaces of $\mathbb{R}^4$. This will be derived as a consequence of a classification of independent interest of the conformally flat hypersurfaces of dimension three with three distinct principal curvatures of $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ with the property that the tangent component of $\partial/\partial t$ is a principal direction at any point. Here $\partial/\partial t$ stands for either a constant unit vector field in $\mathbb{R}^4$ or the unit vector field tangent to the factor $\mathbb{R}$ in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$.

First recall that there exists a conformal diffeomorphism $\Phi\colon \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^n \setminus \{0\}$ given by $(x,t) \mapsto e^t x$. Similarly, there is a conformal diffeomorphism

$$\Psi\colon \mathbb{H}^n \times \mathbb{S}^1 \subset \mathbb{R}^{n+1}_1 \times \mathbb{R}^2 \to \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1}$$

onto the complement of a subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ given as follows. Choose a pseudo-orthonormal basis $e_0, e_1, \ldots, e_{n-1}, e_n$ of the Lorentzian space $\mathbb{R}_1^{n+1}$ with $\langle e_0, e_0 \rangle = 0 = \langle e_n, e_n \rangle$, $\langle e_0, e_n \rangle = -1/2$ and $\langle e_i, e_j \rangle = \delta_{ij}$ for $1 \leq i \leq n-1$ and $0 \leq j \leq n$. Then

$$\Psi(x_0 e_0 + \ldots + x_n e_n, (y_1, y_2)) = \frac{1}{x_0}(x_1, \ldots, x_{n-1}, y_1, y_2).$$

Composing Ψ with the isometric covering map

$$\pi \colon \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n \times \mathbb{S}^1 : (x, t) \mapsto (x, (\cos t, \sin t))$$

produces a conformal covering map $\Phi \colon \mathbb{H}^n \times \mathbb{R} \to \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1}$ given by

$$\Phi(x_0 e_0 + \ldots + x_n e_n, t) = \frac{1}{x_0}(x_1, \ldots, x_{n-1}, \cos t, \sin t). \tag{1.1}$$

In what follows, $\mathbb{Q}_\epsilon^3 \subset \mathbb{R}_\mu^{3+|\epsilon|}$ denotes $\mathbb{S}^3$ if $\epsilon = 1$, $\mathbb{R}^3$ if $\epsilon = 0$ and $\mathbb{H}^3$ if $\epsilon = -1$, with $\mu = 0$ if $\epsilon = 0$ or $\epsilon = 1$, and $\mu = 1$ if $\epsilon = -1$. Given a surface $h \colon M^2 \to \mathbb{Q}_\epsilon^3$, let $h_s \colon M^2 \to \mathbb{Q}_\epsilon^3 \subset \mathbb{R}_\mu^{3+|\epsilon|}$ be the the family of its parallel surfaces, that is,

$$h_s(x) = C_\epsilon(s) h(x) + S_\epsilon(s) N(x),$$

where N is a unit normal vector field to h and the functions C_ϵ and S_ϵ are given by

$$C_\epsilon(s) = \begin{cases} \cos s, & \text{if } \epsilon = 1 \\ 1, & \text{if } \epsilon = 0 \\ \cosh s, & \text{if } \epsilon = -1 \end{cases} \qquad \text{and} \qquad S_\epsilon(s) = \begin{cases} \sin s, & \text{if } \epsilon = 1 \\ s, & \text{if } \epsilon = 0 \\ \sinh s, & \text{if } \epsilon = -1. \end{cases}$$

The classification of the conformally flat hypersurfaces of dimension three with three distinct principal curvatures of $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ with the property that the tangent component of $\partial/\partial t$ is a principal direction at any point is as follows.

Theorem 1. *Let $h \colon M^2 \to \mathbb{Q}_\epsilon^3$ be an umbilic-free linear Weingarten surface, i.e., the extrinsic curvature K_{ext} and the mean curvature H of h satisfy*

$$PK_{ext} + QH = R \tag{1.2}$$

for some P, Q, $R \in \mathbb{R}$. Set $\overline{P} = P + \epsilon R$, $\overline{Q} = Q$, $\overline{R} = P - \epsilon R + 4$ and $\Lambda = 2(\epsilon^2 - 1)R$, let $I \subset \mathbb{R}$ be an open interval where

$$0 < r(s) := \frac{1}{4}(\overline{P}C_\epsilon(2s) + \overline{Q}S_\epsilon(2s) + \Lambda S_\epsilon^2(s) + \overline{R}) < 1$$

and let $a\colon I \to \mathbb{R}$ be the smooth function on I given by

$$a(s) = \int_{s_0}^{s} \sqrt{\frac{1 - r(s)}{r(s)}}\, ds, \quad s_0 \in I. \tag{1.3}$$

Then the map $f\colon M^2 \times I \to \mathbb{Q}_\epsilon^3 \times \mathbb{R} \subset \mathbb{R}_\mu^{4+|\epsilon|}$ given by

$$f(x, s) = h_s(x) + a(s)\partial/\partial t, \tag{1.4}$$

where $\partial/\partial t$ denotes a unit vector field tangent to $\mathbb{R}$, defines, on the open subset $M^3 \subset M^2 \times I$ of its regular points, a conformally flat hypersurface with three distinct principal curvatures such that the tangent component of $\partial/\partial t$ is a principal direction of f at any point.

Conversely, any conformally flat hypersurface $f\colon M^3 \to \mathbb{Q}_\epsilon^3 \times \mathbb{R} \subset \mathbb{R}_\mu^{4+|\epsilon|}$ with these properties is given locally either in this way or as a vertical cylinder over an umbilic-free surface $h\colon M^2 \to \mathbb{Q}_\epsilon^3$ with constant Gauss curvature $K_0 \neq \epsilon$, in which case M^3 splits as a Riemannian product $M^3 = M^2 \times I$, where $I \subset \mathbb{R}$ is an open interval, and f is given by

$$f(x, s) = h(x) + s\,\partial/\partial t. \tag{1.5}$$

Regarding h as an isometric immersion into $\mathbb{R}_\mu^{4+|\epsilon|}$, its normal space at each point $x \in M^2$ is a vector space whose dimension is 2 if $\epsilon = 0$ and 3 otherwise, and which is either Lorentzian or Riemannian, according to whether $\mu = 1$ or $\mu = 0$, respectively. If $\epsilon \neq 0$, it is spanned by the position vector $h(x)$, the normal vector $N(x)$ to h in $\mathbb{Q}_\epsilon^3$ at x and the constant vector field $\partial/\partial t$. If $\epsilon = 0$, it is spanned by last two vectors. Notice that these give rise to parallel vector fields along h with respect to its normal connection. For a fixed $x \in M^2$, for $\epsilon \neq 0$ (respectively, $\epsilon = 0$), we can regard $s \mapsto f(x, s) = C_\epsilon(s)h(x) + S_\epsilon(s)N(x) + a(s)\partial/\partial t$ (respectively, $s \mapsto f(x, s) = sN(x) + a(s)\partial/\partial t$) as a curve in the normal space of h at x, which is contained in a cylinder $\mathbb{Q}_\epsilon^1 \times \mathbb{R}$ in that normal space when

$\epsilon \neq 0$. Thus $f(M)$ is generated by parallel transporting such curve along h with respect to its normal connection. In case f is a vertical cylinder over h, then $f(M)$ is generated by parallel transporting the straight line $s \mapsto s\,\partial/\partial t$ along h with respect to its normal connection.

We derive from Theorem 1 the following alternative classification of cyclic conformally flat hypersurfaces of $\mathbb{R}^4$. In the next statement, the map Φ denotes either the conformal diffeomorphism $\Phi\colon \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{R}^4 \setminus \{0\}$ if $\epsilon = 1$, the conformal covering map $\Phi\colon \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{R}^4 \setminus \mathbb{R}^2$ if $\epsilon = -1$ or the isometry $\Phi\colon \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{R}^4$ if $\epsilon = 0$.

Theorem 2. *Let $f\colon M^3 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ be either a vertical cylinder over an umbilic-free surface $h\colon M^2 \to \mathbb{Q}^3_\epsilon$ with constant Gauss curvature $K_0 \neq \epsilon$, or a hypersurface as in the direct statement of Theorem 1. Then $F = \mathcal{I} \circ \Phi \circ f\colon M^3 \to \mathbb{R}^4$, where $\mathcal{I}$ is either the identity map or an inversion with respect to a hypersphere in $\mathbb{R}^4$, is a cyclic conformally flat hypersurface.*

Conversely, any cyclic conformally flat hypersurface of $\mathbb{R}^4$ is locally given in one of these ways.

In the classification of cyclic conformally flat hypersurfaces given in [**H-JS**], the authors deal with hypersurfaces in the sphere $\mathbb{S}^4$, taking into account the invariance of the conditions involved under conformal diffeomorphisms between the ambient space forms. Then they use a Moebius geometric technology to show that any such hypersurface can be produced, up to such a conformal diffeomorphism, from a hypersurface in some space form that is given in terms of a family of parallel linear Weingarten surfaces and a solution of a certain pendulum-type ordinary differential equation. Our approach is somewhat more elementary in nature, and the parametrization of cyclic conformally flat hypersurfaces $f\colon M^3 \to \mathbb{R}^4$ provided by Theorems (1) and (2) only requires a single integration.

Theorem 2 also yields the following characterization of cyclic conformally flat hypersurfaces. Let $x_1, \ldots, x_{n+1}$ denote the standard coordinates in $\mathbb{R}^{n+1}$ and let ∂_{x_i} be a unit vector field tangent to the x_i-coordinate curve, $1 \leq i \leq n+1$. It is well known that the Lie algebra of conformal Killing vector fields in $\mathbb{R}^{n+1}$ has dimension $\frac{1}{2}(n+2)(n+3)$ and is generated by the constant vector fields ∂_{x_i}, $1 \leq i \leq n+1$, the Killing vector fields $\mathcal{K}_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$, $1 \leq i \neq j \leq n+1$, generating rotations around the linear subspaces $\mathbb{R}^{n-1}$ of $\mathbb{R}^{n+1}$ given by $x_i = 0 = x_j$, by the radial vector field $\mathcal{R} = \sum_{i=1}^{n+1} x_i \partial_{x_i}$, and by the vector fields

$$\mathcal{C}_i = \frac{1}{2}(x_i^2 - \sum_{j \neq i} x_j^2)\partial_{x_i} + x_i \sum_{j \neq i} x_j \partial_{x_j}, \quad 1 \leq i \leq n+1.$$

Corollary 3. *A conformally flat hypersurface $f\colon M^3 \to \mathbb{R}^4$ with three distinct principal curvatures is cyclic if and only if the tangent component of one of the above conformal Killing vector fields is an eigenvector field correspondent to one of its principal curvatures.*

Remark 4. According to Theorem 2, any cyclic conformally flat hypersurface $f\colon M^3 \to \mathbb{R}^4$ is given, by the construction of Theorem 1, as an evolution of a one-parameter family of surfaces, with respect to the parameter s of the curvature lines correspondent to one of the principal curvatures, issuing from a linear Weingarten surface in $\mathbb{Q}_c^3$ at a fixed value of s. As pointed out to us by one of the anonymous referees, it was shown in [**BH-JS**] and [**Su3**] that any generic conformally flat hypersurface can also be produced in a suitable way as an evolution of surfaces with respect to the parameter s issuing from certain analytic surfaces in $\mathbb{S}^3$.

2 Proof of Theorem 1

It is well known that a three-dimensional Riemannian manifold M^3 is conformally flat if and only if its Schouten tensor $L = T - (3/2)sI$ satisfies the Codazzi equation

$$(\nabla_X L)Y = (\nabla_Y L)X \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M)$ (see, e.g, [**DT$_2$**], p. 545), where $(\nabla_X L)Y = \nabla_X LY - L(\nabla_X Y)$. Here T is the endomorphism associated with the Ricci tensor and s is the scalar curvature.

Assume that M^3 carries local coordinates x_1, x_2, x_3 with respect to which its Riemannian metric can be written as

$$g = v_1^2 dx_1^2 + v_2^2 dx_2^2 + v_3^2 dx_3^2,$$

where v_1, v_2 and v_3 are smooth functions. In the sequel, we denote by ψ_i the partial derivative of a function ψ with respect to x_i and by ψ_{ij} its second order partial derivative $\partial_{x_i}\partial_{x_j}\psi$. Denote

$$\phi^{ij} = \frac{v_{j,i}}{v_i}, \quad 1 \leq i \neq j \leq 3, \tag{2.2}$$

where $v_{j,i}$ denotes the derivative of v_j with respect to x_i. Let $\partial_1, \partial_2, \partial_3$ be the coordinate vector fields and set $X_k = v_k^{-1}\partial_k$, $1 \leq k \leq 3$. Then the curvature tensor R of g satisfies (see [**DT**$_2$], p. 20)

$$R(\partial_i, \partial_j)X_k = \left((\phi^{kj})_i - \phi^{ki}\phi^{ij}\right) X_j - \left((\phi^{ki})_j - \phi^{kj}\phi^{ji}\right) X_i, \tag{2.3}$$

for $1 \leq i \neq j \neq k \neq i \leq 3$, and

$$-v_i v_j K_{ij} = -\langle R(\partial_i, \partial_j)X_j, X_i\rangle = (\phi^{ij})_i + (\phi^{ji})_j + \phi^{ki}\phi^{kj},$$

for $1 \leq i \neq j \neq k \neq i \leq 3$, where K_{ij}, $1 \leq i \neq j \leq 3$, is the sectional curvature along the plane spanned by ∂_i and ∂_j. Now suppose further that

$$v_1 = e^{\alpha}, \quad v_2 = e^{\beta} \quad \text{and} \quad v_3 \equiv 1$$

for some smooth functions α and β satisfying

$$\alpha_{23} + \alpha_2(\alpha - \beta)_3 = 0 \quad \text{and} \quad \beta_{13} - \beta_1(\alpha - \beta)_3 = 0. \tag{2.4}$$

Notice that the preceding equations are equivalent to $(\phi^{ij})_3 = 0$ for $1 \leq i \neq j \leq 2$. Then Eq. (2.3) implies that $R(\partial_3, \partial_i)X_j = 0$ for $1 \leq i \neq j \leq 2$, and it follows that $L\partial_i = \ell_i\partial_i$, $1 \leq i \leq 3$, where

$$2\ell_1 = K_{12}+K_{13}-K_{23}, \quad 2\ell_2 = K_{12}+K_{23}-K_{13} \quad \text{and} \quad 2\ell_3 = K_{13}+K_{23}-K_{12}.$$

Define $\psi_i = \ell_i v_i$, $1 \leq i \leq 3$. Then L satisfies the Codazzi equation (2.1) if and only if

$$\psi_{i,j} = \phi^{ji}\psi_j, \quad 1 \leq i \neq j \leq 3.$$

Using this, a straightforward computation yields the following lemma.

Lemma 5. *The metric*

$$\tilde{g} = e^{2\alpha}dx_1^2 + e^{2\beta}dx_2^2 + dx_3^2, \tag{2.5}$$

with the smooth functions α and β satisfying (2.4), is conformally flat if and only α and β satisfy the partial differential equations

$$\begin{aligned}(e^{-2\beta}(\alpha_{22} + (\alpha_2)^2 - \alpha_2\beta_2))_2 + (e^{-2\alpha}(\beta_{11} + (\beta_1)^2 - \alpha_1\beta_1))_2 \\ -(\alpha_{33} + (\alpha_3)^2 + \beta_{33} + (\beta_3)^2 - \alpha_3\beta_3)_2 = 0,\end{aligned} \tag{2.6}$$

$$\begin{aligned}(e^{-2\beta}(\alpha_{22} + (\alpha_2)^2 - \alpha_2\beta_2))_1 + (e^{-2\alpha}(\beta_{11} + (\beta_1)^2 - \alpha_1\beta_1))_1 \\ -(\alpha_{33} + (\alpha_3)^2 + \beta_{33} + (\beta_3)^2 - \alpha_3\beta_3)_1 = 0\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}e^{-2\beta}(\alpha_{22} + (\alpha_2)^2 - \alpha_2\beta_2)_3 + (e^{-2\alpha}(\beta_{11} + (\beta_1)^2 - \alpha_1\beta_1))_3 \\ +(\alpha_3\beta_3 + \beta_{33} + (\beta_3)^2 - \alpha_{33} - (\alpha_3)^2)_3 \\ = 2(\alpha_{33} + (\alpha_3)^2 - \alpha_3\beta_3)\beta_3 - 2e^{-2\alpha}(\beta_{11} + (\beta_1)^2 - \alpha_1\beta_1)\beta_3.\end{aligned} \tag{2.8}$$

Proof of Theorem 1: First notice that if $f\colon M^2 \times I \to \mathbb{Q}_\epsilon^3 \times \mathbb{R}$ is a vertical cylinder over $h\colon M^2 \to \mathbb{Q}_\epsilon^3$, then the metric induced by f is the Riemannian product metric $d\sigma^2 = g + ds^2$, where g is the metric induced by h. It is well-known that such a metric is conformally flat if and only if g has constant Gauss curvature K_0. Since ∂/∂_s is a principal direction with principal curvature identically zero, f has three distinct principal curvatures if and only if h is a surface with nowhere vanishing extrinsic curvature, i.e., $K_0 \neq \epsilon$.

Now, by Theorem 1 in [**To**$_2$], if $f\colon M^2 \times I \to \mathbb{Q}_\epsilon^3 \times \mathbb{R}$ is given by (1.4) in terms of an arbitrary surface $h\colon M^2 \to \mathbb{Q}_\epsilon^3$ and a smooth function $a\colon I \to \mathbb{R}$ with positive derivative, then the tangent component of $\partial/\partial t$ is a principal direction at any point of the restriction of f to the subset $M^3 \subset M^2 \times I$ of its regular points. Conversely, any hypersurface $f\colon M^3 \to \mathbb{Q}_\epsilon^3 \times \mathbb{R}$ with this property, and such that $\partial/\partial t$ is nowhere tangent to $f(M)$, is given in this way. Therefore, Theorem 1 will be proved once we show that such a hypersurface f has three distinct principal curvatures and the metric it induces on M^3 is conformally flat if and only if h and a are as in the statement.

The metric induced by f on M^3 is

$$d\sigma^2 = b^2(s)ds^2 + g_s,$$

where $b(s) = \sqrt{1 + (a'(s))^2}$ and g_s is the metric induced by h_s.

For some of the computations that follow, it is convenient to have in mind the following relations between the functions C_ϵ and S_ϵ:

$$\begin{aligned}
&C_\epsilon^2(s) + \epsilon S_\epsilon^2(s) = 1,\\
&C_\epsilon(2s) = C_\epsilon^2(s) - \epsilon S_\epsilon^2(s) \text{ and } S_\epsilon(2s) = 2C_\epsilon(s)S_\epsilon(s),\\
&C'_\epsilon(s) = -\epsilon S_\epsilon(s) \text{ and } S'_\epsilon(s) = C_\epsilon(s).
\end{aligned}$$

Let N be a unit normal vector field along h and let N_s be the unit normal vector field along h_s given by $N_s(x) = C_\epsilon N(x) - \epsilon S_\epsilon h(x)$. Then

$$\eta(x,s) = -\frac{a'(s)}{b(s}N_s(x) + \frac{1}{b(s)}\frac{\partial}{\partial t}$$

defines a unit normal vector field along f, and the shape operators A of f at (x,s) and A^s of h_s at x with respect to η and N_s, respectively, are related by

$$AX = -\frac{a'(s)}{b(s)}A^sX, \text{ if } X \in T_xM^2, \text{ and } A\partial_s = \frac{a''(s)}{b^3(s)}\partial_s, \tag{2.9}$$

where ∂_s is a unit vector field tangent to I. Thus f has three distinct principal curvatures at (x,s) if and only if x is not an umbilic point for h_s, and hence for h.

Under the assumption that h has no umbilic points, there exist locally principal coordinates x_1, x_2 on M_2 with respect to which the first and second fundamental forms of h are

$$I = v_1^2dx_1^2 + v_2^2dx_2^2 \quad \text{and} \quad II = V_1v_1dx_1^2 + V_2v_2dx_2^2,$$

respectively. Therefore the first fundamental form of h_s and its second fundamental form with respect to N_s are given, respectively, by

$$I^s = (v_1^s)^2dx_1^2 + (v_2^s)^2dx_2^2 \quad \text{and} \quad II^s = V_1^sv_1^sdx_1^2 + V_2^sv_2^sdx_2^2,$$

where

$$v_i^s = C_\epsilon v_i - S_\epsilon V_i \quad \text{and} \quad V_i^s = \epsilon S_\epsilon v_i + C_\epsilon V_i = -v_{i,3}^s, \;\; 1 \leq i \leq 2.$$

Notice that for all $s \in I$ we have

$$\frac{v_{j,i}^s}{v_i^s} = \frac{v_{j,i}}{v_i}, \;\; 1 \leq i \neq j \leq 2, \tag{2.10}$$

and that $k_i^s = \dfrac{V_i^s}{v_i^s}$, $1 \leq i \leq 2$, are the principal curvatures of h_s. It follows easily that the extrinsic curvature and mean curvature

$$K_{ext}^s = k_1^s k_2^s = \frac{V_1^s V_2^s}{v_1^s v_2^s} \;\; \text{and} \;\; H^s = \frac{1}{2}\left(\frac{V_1^s}{v_1^s} + \frac{V_2^s}{v_2^s}\right) \tag{2.11}$$

of h_s are related to the extrinsic curvature K_{ext} and the mean curvature H of h by

$$\begin{aligned} K_{ext}^s &= \frac{\epsilon^2 S_\epsilon^2(s) + \epsilon S_\epsilon(2s)H + C_\epsilon^2(s)K_{ext}}{C_\epsilon^2(s) - S_\epsilon(2s)H + S_\epsilon^2(s)K_{ext}}, \\ H^s &= \frac{\epsilon S_\epsilon(2s) + 2C_\epsilon(2s)H - S_\epsilon(2s)K_{ext}}{2\left(C_\epsilon^2(s) - S_\epsilon(2s)H + S_\epsilon^2(s)K_{ext}\right)}. \end{aligned}$$

Since ∂_s is a principal direction of f by the second equation in (2.9), then $x_1, x_2, x_3 := s$ are local principal coordinates for f with respect to which its induced metric is given by $b^2\tilde{g}$, where $\tilde{g}$ has the form (2.5) with

$$e^\alpha = \frac{v_1^s}{b} \quad \text{and} \quad e^\beta = \frac{v_2^s}{b}. \tag{2.12}$$

It follows from (2.10) that the functions ϕ^{ij}, associated with the metric $\tilde{g}$ by means of (2.2), satisfy $(\phi^{ij})_3 = 0$ for $1 \leq i \neq j \leq 2$. Thus (2.4) holds for α and β. We now investigate when α and β also satisfy (2.6), (2.7) and (2.8).

In terms of the function ρ defined by

$$\begin{aligned} \rho \;\; := \;\; & e^{-2\beta}(\alpha_{22} + (\alpha_2)^2 - \alpha_2\beta_2) + e^{-2\alpha}(\beta_{11} + (\beta_1)^2 - \alpha_1\beta_1), \\ & - \left(\alpha_{33} + (\alpha_3)^2 + \beta_{33} + (\beta_3)^2 - \alpha_3\beta_3\right), \end{aligned} \tag{2.13}$$

Eqs. (2.6), (2.7) and (2.8) are equivalent to

$$\begin{aligned} \rho_i &= 0, \ 1 \leq i \leq 2, \\ \rho_3 &= -2\beta_3\rho - 2\beta_3(\beta_{33} + \beta_3^2) - 2(\beta_{33} + (\beta_3)^2)_3 \end{aligned} \tag{2.14}$$

Differentiating (2.12) we obtain

$$\alpha_2 = \phi^{21} e^{\beta-\alpha} \quad \text{and} \quad \beta_1 = \phi^{12} e^{\alpha-\beta},$$

hence

$$\beta_{11} + \beta_1(\beta - \alpha)_1 = \phi_1^{12} e^{\alpha-\beta} \quad \text{and} \quad \alpha_{22} + \alpha_2(\alpha - \beta)_2 = \phi_2^{21} e^{\beta-\alpha}. \tag{2.15}$$

Therefore, the sum of the first two terms on the right hand-side of (2.13) is

$$e^{-(\alpha+\beta)}\left(\phi_2^{21} + \phi_1^{12}\right) = -\frac{b^2}{v_1^s v_2^s}(V_1^s V_2^s + \epsilon v_1^s v_2^s) = -b^2(K_{ext}^s + \epsilon),$$

where the first equality follows from the Gauss equation of h_s, bearing in mind that $\phi^{ij} = \phi_s^{ij}$ for $1 \leq i \neq j \leq 2$.

On the other hand, setting $B = \log b$ we have

$$\begin{aligned} \alpha_3 &= -\left(\frac{V_1^s}{v_1^s} + \frac{b'}{b}\right) = -(k_1^s + B'), \\ \alpha_{33} &= -\left(\epsilon + \left(\frac{V_1^s}{v_1^s}\right)^2 + \frac{b''}{b} - \left(\frac{b'}{b}\right)^2\right) = -(\epsilon + (k_1^s)^2 + B''), \\ \beta_3 &= -\left(\frac{V_2^s}{v_2^s} + \frac{b'}{b}\right) = -(k_2^s + B'), \\ \beta_{33} &= -\left(\epsilon + \left(\frac{V_2^s}{v_2^s}\right)^2 + \frac{b''}{b} - \left(\frac{b'}{b}\right)^2\right) = -(\epsilon + (k_2^s)^2 + B''). \end{aligned} \tag{2.16}$$

It follows from (2.16) that

$$\alpha_{33} + (\alpha_3)^2 + \beta_{33} + (\beta_3)^2 - \beta_3\alpha_3 = 2B'H^s - K_{ext}^s - 2\epsilon - 2B'' + (B')^2. \tag{2.17}$$

From (2.13), (2.15) and (2.17) we obtain $\rho = 2B'' - (B')^2 - \epsilon e^{2B} + 2\varepsilon - \varphi$, where

$$\varphi(x_1, x_2, x_3) := \left(e^{2B} - 1\right) K_{ext}^s + 2B'H^s. \tag{2.18}$$

Using the last two equations in (2.16), the equations in (2.14) are equivalent to

$$\begin{aligned} \varphi_i &= 0,\ 1 \leq i \leq 2, \\ \varphi_3 &= 2k_2^s\theta + 2B'(\epsilon + \varphi), \end{aligned} \tag{2.19}$$

where

$$\theta = \epsilon e^{2B} + B'' - \epsilon + \varphi - 2(B')^2. \tag{2.20}$$

We have shown so far that, for a hypersurface $f\colon M^2 \times I \to \mathbb{Q}_\epsilon^3 \times \mathbb{R}$ given by (1.4) in terms of an umbilic-free surface $h\colon M^2 \to \mathbb{Q}_\epsilon^3$ and a smooth function $a\colon I \to \mathbb{R}$, the metric induced by f on the subset $M^3 \subset M^2 \times I$ of its regular points is conformally flat if and only if the preceding equations are satisfied. We now show that this is the case if and only if h and a are as in the statement.

Suppose first that Eqs. (2.19) hold. The first two equations imply that φ depends only on $x_3 = s$. Let us compute the derivative of φ with respect to x_3. Differentiating (2.11) with respect to x_3, and using that $v_{i,3}^s = -V_i^s$ and $V_{i,3}^s = \epsilon v_i$, $1 \leq i \leq 2$, gives

$$(K_{ext}^s)_3 = 2H^s(\epsilon + K_{ext}^s) \quad \text{and} \quad (H^s)_3 = \epsilon + 2(H^s)^2 - K_{ext}^s.$$

The preceding relations yield

$$\varphi_3 = 2H^s\theta + 2B'(\epsilon + \varphi). \tag{2.21}$$

Comparing (2.19) and (2.21) gives $0 = (k_1^s - k_2^s)\theta$. Since $k_1^s \neq k_2^s$ at any point, we conclude that θ is identically zero. Therefore, the functions φ and B satisfy the system of ordinary differential equations

$$\left\{ \begin{aligned} \varphi' - 2B'(\epsilon + \varphi) &= 0, \\ \epsilon + 2(B')^2 - \epsilon e^{2B} - B'' - \varphi &= 0. \end{aligned} \right. \tag{2.22}$$

The first equation of (2.22) implies that

$$\varphi = \lambda e^{2B} - \epsilon \tag{2.23}$$

for some $\lambda \in \mathbb{R}$. Substituting this formula into the second one, it becomes

$$(e^{-2B})'' + 4\epsilon e^{-2B} = 2(\epsilon + \lambda),$$

whose general solution is

$$e^{-2B(x_3)} = c_1 C_\epsilon(2x_3) + c_2 S_\epsilon(2x_3) + (1-\epsilon^2)\lambda S_\epsilon^2(x_3) + \frac{\epsilon(\epsilon+\lambda)}{2} \tag{2.24}$$

for some $c_1, c_2 \in \mathbb{R}$. Since $B = \log b$, the above equation is equivalent to

$$(1+(a')^2)^{-1} = c_1 C_\epsilon(2x_3) + c_2 S_\epsilon(2x_3) + (1-\epsilon^2)\lambda S_\epsilon^2(x_3) + \frac{\epsilon(\epsilon+\lambda)}{2}. \tag{2.25}$$

Evaluating (2.18) at $x_3 = 0$ by using (2.23) and (2.24) implies that

$$[\epsilon(\epsilon+\lambda) + 2c_1 - 2]\, K_{ext} + (4c_2)H = \epsilon^2(\epsilon+\lambda) + 2\epsilon c_1 - 2\lambda.$$

Set

$$P = \epsilon(\epsilon+\lambda) + 2c_1 - 2, \quad Q = 4c_2 \quad \text{and} \quad R = \epsilon^2(\epsilon+\lambda) + 2\epsilon c_1 - 2\lambda.$$

Since c_1, c_2 and $\lambda \in \mathbb{R}$ are arbitrary, we see that also $P, Q, R \in \mathbb{R}$ are arbitrary, and now (2.25) implies that a is given as in the statement.

Conversely, let $h\colon M^2 \to \mathbb{Q}_\epsilon^3$ be an umbilic-free linear Weingarten surface, let $a\colon I \to \mathbb{R}$ be a smooth function on the open interval $I \subset \mathbb{R}$ as in the statement and let f be given by (1.4). We must show that the function φ, defined by (2.18) on the open subset $M^3 \subset M^2 \times I$ of regular points of f, with $B = \log b$ and $b = (1+(a')^2)^{1/2}$, satisfies the three equations in (2.19).

In view of (2.11), Eq. (2.18) is equivalent to

$$P(x)K_{ext} + Q(x)H = R(x), \tag{2.26}$$

where $x = (x_1, x_2, x_3 = s)$ and

$$\begin{aligned} P(x) &= (e^{2B}-1)C_\epsilon^2(s) - B' S_\epsilon(2s) - \varphi(x) S_\epsilon^2(s), \\ Q(x) &= \epsilon(e^{2B}-1)S_\epsilon(2s) + 2B' C_\epsilon(2s) + \varphi(x) S_\epsilon(2s), \\ R(x) &= \varphi(x) C_\epsilon^2(s) - \epsilon^2(e^{2B}-1)S_\epsilon^2(s) - \epsilon B' S_\epsilon(2s). \end{aligned} \tag{2.27}$$

Since

$$\begin{aligned} 4e^{-2B(s)} &= \overline{P} C_\epsilon(2s) + \overline{Q} S_\epsilon(2s) + \Lambda S_\epsilon^2(s) + \overline{R}, \\ 4B'(s)e^{-2B(s)} &= \epsilon \overline{P} S_\epsilon(2s) - \overline{Q} C_\epsilon(2s) - \Lambda S_\epsilon(s) C_\epsilon(s), \end{aligned}$$

with $\overline{P} = P + \epsilon R$, $\overline{Q} = Q$, $\Lambda = 2(\epsilon^2 - 1)R$ and $\overline{R} = P - \epsilon R + 4$, we have

$$4e^{-2B(s)}P(x) = -2P - S_\epsilon^2(s)\left(\epsilon(4e^{-2B(s)} + 4 - 2\overline{R}) - \Lambda + 4e^{-2B(s)}\varphi(x)\right),$$
$$4e^{-2B(s)}Q(x) = -2Q + S_\epsilon(2s)\left(\epsilon(2e^{-4B(s)} + 4 - 2\overline{R}) - \Lambda + 4e^{-2B(s)}\varphi(x)\right),$$
$$4e^{-2B(s)}R(x) = -2R + C_\epsilon^2(s)\left(\epsilon(2e^{-4B(s)} + 4 - 2\overline{R}) - \Lambda + 4e^{-2B(s)}\varphi(x)\right).$$

The above equations, together with Eqs. (1.2) and (2.26), imply that

$$\left(\epsilon(4e^{-2B(s)}+4-2\overline{R})-\Lambda+4e^{-2B(s)}\varphi(x)\right)\left(S_\epsilon^2(s)K_{ext}-S_\epsilon(2s)H+C_\epsilon^2(s)\right) = 0.$$

We claim that the function

$$\epsilon(4e^{-2B(s)} + 4 - 2\overline{R}) - \Lambda + 4e^{-2B(s)}\varphi(x)$$

is identically zero. Let us suppose, by contradiction, that there is a point x_0 where such function is nonzero. Then $S_\epsilon^2(s)K_{ext} - S_\epsilon(2s)H + C_\epsilon^2(s)$ is identically zero in an open neighbourhood Ω of x_0, hence in Ω we have

$$\begin{cases} S_\epsilon^2(s)K_{ext} - S_\epsilon(2s)H &= -C_\epsilon^2(s), \\ PK_{ext} + QH &= R. \end{cases}$$

Since K_{ext} and H depend only on (x_1, x_2), the determinant $QS_\epsilon^2(s) + PS_\epsilon(2s)$ must be identically zero in Ω, which is a contradiction. Therefore

$$\varphi(x) = \frac{\epsilon}{2}\left((\overline{R} - 2)e^{2B(s)} - 2\right) + \frac{\Lambda e^{2B(s)}}{4}$$

for all $x \in M^3$. We conclude that $\varphi_i = 0$, $1 \leq i \leq 2$, and $\varphi_3 = \epsilon(\overline{R} - 2)e^{2B(s)}B'(s)$. On the other hand, the function θ introduced in (2.20) satisfies

$$\begin{aligned} 2e^{-2B}\theta &= 2\epsilon(1 - e^{-2B}) + 2e^{-2B}\varphi + 2e^{-2B}(B'' - 2(B')^2), \\ &= 2\epsilon(1 - e^{-2B}) - \epsilon(2e^{-2B} + 2 - \overline{R}) + \frac{\Lambda}{2} - (e^{-2B})'', \\ &= 2\epsilon(1 - e^{-2B}) - \epsilon(2e^{-2B} + 2 - \overline{R}) + \frac{\Lambda}{2} + \left(4\epsilon e^{-2B} - \epsilon\overline{R} - \frac{\Lambda}{2}\right), \\ &= 0. \end{aligned}$$

Therefore, $\theta = 0$. Since

$$\begin{aligned} 2B'(\epsilon + \varphi) &= B'\left(2\epsilon + \epsilon\left((\overline{R}-2)e^{2B}-2\right) + \frac{\Lambda e^{2B}}{2}\right), \\ &= \epsilon(\overline{R}-2)e^{2B}B' + \frac{\Lambda e^{2B}B'}{2}, \end{aligned}$$

the third equation in (2.19) is satisfied. □

3 Cyclic conformally flat hypersurfaces

Let $F\colon M^3 \to \mathbb{R}^4$ be a conformally flat hypersurface with three distinct principal curvatures $\lambda_1, \lambda_2, \lambda_3$ and corresponding unit principal vector fields e_1, e_2 and e_3, respectively. E. Cartan proved (see [**La**], p. 84) that the conformal flatness of M^3 is equivalent to the relations

$$\langle \nabla_{e_i} e_j, e_k \rangle = 0 \tag{3.1}$$

and

$$(\lambda_j - \lambda_k)e_i(\lambda_i) + (\lambda_i - \lambda_k)e_i(\lambda_j) + (\lambda_j - \lambda_i)e_i(\lambda_k) = 0, \tag{3.2}$$

for all $1 \leq i \neq j \neq k \neq i \leq 3$. It follows from Codazzi's equation and (3.1) that

$$\nabla_{e_i} e_i = \sum_{j\neq i} (\lambda_i - \lambda_j)^{-1} e_j(\lambda_i) e_j. \tag{3.3}$$

Proposition 6. *The following assertions are equivalent:*

(i) The integral curves of e_1 are extrinsic circles;

(ii) The functions $\rho_j = \frac{e_j(\lambda_1)}{\lambda_1 - \lambda_j}$, $2 \leq j \leq 3$, satisfy $e_1(\rho_j) = 0$.

(iii) The relation

$$(\lambda_1 - \lambda_j)e_j e_1(\lambda_1) = 2e_1(\lambda_1)e_j(\lambda_1)$$

holds for $2 \leq j \leq 3$.

(iv) The image by F of each integral curve σ of e_1 is contained in a two-dimensional sphere whose normal spaces in $\mathbb{R}^4$ along $F(\sigma)$ are spanned by (the restrictions to $F(\sigma)$ of) the vector fields F_*e_2 and F_*e_3.

(v) The image by F of each leaf of the distribution spanned by e_2 and e_3 is contained in a hypersphere (or affine hyperplane) of $\mathbb{R}^4$;

Proof. The integral curves of e_1 are extrinsic circles if and only if

$$\langle \nabla_{e_1}\nabla_{e_1}e_1, e_j\rangle = 0, \quad 2 \le j \le 3.$$

Using (3.1) and (3.3) we obtain

$$\begin{aligned}\langle \nabla_{e_1}\nabla_{e_1}e_1, e_j\rangle &= e_1\langle \nabla_{e_1}e_1, e_j\rangle - \langle \nabla_{e_1}e_1, \nabla_{e_1}e_j\rangle \\ &= e_1(\rho_j), \quad 2 \le j \le 3,\end{aligned}$$

hence (i) and (ii) are equivalent. The equation $e_1(\rho_j) = 0$ can be written as

$$e_1(\lambda_1 - \lambda_j)e_j(\lambda_1) = (\lambda_1 - \lambda_j)e_1e_j(\lambda_1), \quad 2 \le j \le 3.$$

We have

$$\begin{aligned}e_je_1(\lambda_1) &= e_1e_j(\lambda_1) + [e_j, e_1](\lambda_1) \\ &= e_1e_j(\lambda_1) + (\nabla_{e_j}e_1)(\lambda_1) - (\nabla_{e_1}e_j)(\lambda_1) \\ &= e_1e_j(\lambda_1) + \langle \nabla_{e_j}e_1, e_j\rangle e_j(\lambda_1) - \langle \nabla_{e_1}e_j, e_1\rangle e_1(\lambda_1) \\ &= e_1e_j(\lambda_1) - \frac{e_1(\lambda_j)}{\lambda_j - \lambda_1}e_j(\lambda_1) + \frac{e_j(\lambda_1)}{\lambda_1 - \lambda_j}e_1(\lambda_1).\end{aligned}$$

Thus, for $2 \le j \le 3$, the equation $e_1(\rho_j) = 0$ reduces to the relation in item (iii).

Now, for $2 \le j \le 3$, using (3.3) we obtain

$$\tilde{\nabla}_{e_1}F_*e_j = F_*\nabla_{e_1}e_j = -\langle \nabla_{e_1}e_1, e_j\rangle F_*e_1 = -\rho_jF_*e_1 = -\langle F_*e_j, \xi\rangle F_*e_1,$$

where $\xi = \rho_2F_*e_2 + \rho_3F_*e_3$. The equivalence between the assertions in items (ii) and (iv) follows.

Finally, we prove the equivalence between the assertions in items (iii) and (v). First notice that the normal spaces of the restriction f_σ of f to a leaf σ of the distribution spanned by e_2 and e_3 are spanned by the restrictions to $f(\sigma)$ of f_*e_1 and the unit normal vector field N to F. Since e_1, e_2 and e_3 are principal directions of F, and in view of (3.1), it follows that F_σ has flat normal bundle, with the restrictions of e_2 and e_3 to σ as an orthonormal diagonalizing tangent frame and corresponding principal normal vector fields

$$\begin{aligned}\eta_j &= \left\langle \nabla_{e_j} e_j, e_1 \right\rangle F_* e_1 + \lambda_j N \\ &= \frac{e_1(\lambda_j)}{\lambda_j - \lambda_1} F_* e_1 + \lambda_j N, \quad 2 \le j \le 3.\end{aligned}$$

Using (3.2) we obtain

$$\begin{aligned}\eta_2 - \eta_3 &= \left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} - \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} \right) F_* e_1 + (\lambda_2 - \lambda_3) N \\ &= (\lambda_2 - \lambda_3)(\mu F_* e_1 + N),\end{aligned}$$

where

$$\mu = \frac{e_1(\lambda_1)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}.$$

Thus

$$\zeta = F_* e_1 - \mu N \tag{3.4}$$

is an umbilical normal vector field to $F|_\sigma$, for it is orthogonal to $\eta_2 - \eta_3$, and the assertion in item (v) is equivalent to ζ being parallel with respect to the normal connection of $F|_\sigma$. The latter is, in turn, equivalent to $e_2(\mu) = 0 = e_3(\mu)$.

Notice that $e_j(\mu) = 0$, for $2 \le j \le 3$, is equivalent to

$$(\lambda_j - \lambda_1)(\lambda_k - \lambda_1) e_j e_1(\lambda_1) = e_1(\lambda_1)(e_j(\lambda_j - \lambda_1)(\lambda_k - \lambda_1) + e_j(\lambda_k - \lambda_1)(\lambda_j - \lambda_1)),$$

for $2 \le k \ne j \le 3$. Using (3.2), the expression between brackets on the right-hand-side is equal to

$$(\lambda_j - \lambda_k) e_j(\lambda_1) - e_j(\lambda_1)(\lambda_k + \lambda_j - 2\lambda_1) = 2 e_j(\lambda_1)(\lambda_1 - \lambda_k).$$

Hence, the equation $e_j(\mu) = 0$, $2 \le j \le 3$, reduces to the relation in item (iii). $\square$

4 Proof of Theorem 2 and Corollary 3

To prove Theorem 2, first notice that if $f\colon M^2 \times I \to \mathbb{Q}^3_\epsilon \times \mathbb{R} \subset \mathbb{R}^{4+|\epsilon|}_\mu$ is a vertical cylinder over a surface $h\colon M^2 \to \mathbb{Q}^3_\epsilon$ with constant Gauss curvature, then $\Phi\circ f\colon M^3\to\mathbb{R}^4$ is either a cylinder, a cone or a rotation hypersurface over h, according to whether $\epsilon=0$, 1 or -1, respectively. In other words, the compositions of $\Phi\circ f$ with Moebius transformations of $\mathbb{R}^4$ are precisely the conformal product hypersurfaces referred to at the third paragraph of the introduction.

Now let $f\colon M^2\times I\to\mathbb{Q}_\epsilon^3\times\mathbb{R}\subset\mathbb{R}_\mu^{4+|\epsilon|}$ be given by (1.4) in terms of an umbilic-free linear Weingarten surface $h\colon M^2\to\mathbb{Q}_\epsilon^3$ and the function $a\colon I\to\mathbb{R}$ given by (1.3). By Theorem 1, the metric induced by f on the subset $M^3\subset M^2\times I$ of its regular points is conformally flat and f is a hypersurface with three distinct principal curvatures such that the tangent component of $\partial/\partial t$ is a principal direction of f at any point.

To complete the proof of the direct statement, it suffices to argue that $\Phi\circ f\colon M^3\to\mathbb{R}^4$ is a cyclic conformally flat hypersurface, for the composition with an inversion in $\mathbb{R}^4$ clearly preserves both properties.

We must thus prove that, for each $x\in M^2$, the curve $\gamma\colon I\to M^3$ given by $\gamma(s)=(x,s)$ is a curvature line of $\Phi\circ f$, as well as an extrinsic circle, or a geodesic, of M^3. First notice that if $\bar\gamma\colon I\to\mathbb{Q}_\epsilon^3\times\mathbb{R}$ is given by $\bar\gamma=f\circ\gamma$, then $\bar\gamma(I)$ is contained in the vertical cylinder $\beta(\mathbb{R})\times\mathbb{R}$ in $\mathbb{Q}_\epsilon^3\times\mathbb{R}$ over the geodesic β of $\mathbb{Q}_\epsilon^3$ normal to g at x, which intersects $f(M)$ orthogonally along $\bar\gamma(I)$.

We argue separately for the cases $\epsilon=1$, $\epsilon=-1$ and $\epsilon=0$. If $\epsilon=1$, then the image of the vertical cylinder $\beta(\mathbb{R})\times\mathbb{R}$ under Φ is a two-dimensional subspace of $\mathbb{R}^4$ that intersects $\Phi(f(M))$ orthogonally along $\Phi(\bar\gamma(I))$, for Φ is conformal. Thus γ is a curvature line of $\Phi\circ f$ and also a geodesic of M^3 (see Proposition 9 of [**To**$_1$]).

If $\epsilon=-1$, then the image of the vertical cylinder $\beta(\mathbb{R})\times\mathbb{R}$ under $\Phi\colon\mathbb{H}^3\times\mathbb{R}\to\mathbb{R}^4\setminus\mathbb{R}^2$ is a two-dimensional sphere centered at the subspace $\mathbb{R}^2\subset\mathbb{R}^4$, which intersects $\Phi(f(M))$ orthogonally along $\Phi(\bar\gamma(I))$. Therefore, in this case the curve γ is a curvature line of $\Phi\circ f$ that is an extrinsic circle of M^3 (see again Proposition 9 of [**To**$_1$]). The case $\epsilon=0$ is similar and easier.

To prove the converse statement, let $F\colon M^3\to\mathbb{R}^4$ be a cyclic conformally flat hypersurface. By Proposition 6, since the integral curves of e_1 are extrinsic circles, one has a family $\mathcal{F}$ of hyperspheres (or affine hyperplanes) that contain the images by F of the leaves of the distribution spanned by e_2 and e_3, and a family $\mathcal{G}$ of two-dimensional spheres (or affine subspaces) that contain the images by F of the integral curves of e_1, with the property that each element of the former is orthogonal to every element of the latter, and conversely. By Lemma 6 of [**To**$_1$], there exists an inversion $\mathcal{I}$ in $\mathbb{R}^4$ that takes the families $\mathcal{F}$ and $\mathcal{G}$, respectively, into families of hyperspheres (or affine hyperplanes) and two-dimensional spheres (or affine subspaces) of one of the following types:

(i) a family of parallel affine hyperplanes and a family of orthogonal affine subspaces;

(ii) a family of concentric hyperspheres and a family of affine subspaces through their common center;

(iii) a family of affine hyperplanes intersecting along a two-dimensional affine subspace and a family of two-dimensional spheres centered at that affine subspace;

(iv) a family of hyperspheres whose centers lie in a straight line and a family of two-dimensional affine subspaces intersecting along that straight line;

(v) a family of affine hyperplanes intersecting along a straight line and a family of two-dimensional spheres centered at that straight line.

Let ζ be the vector field given by (3.4). In case (i), the vector field $\mathcal{I}_*\zeta$ is collinear with the constant unit vector field e_4 normal to the family $\mathcal{I}(\mathcal{F})$ of affine hyperplanes, thus the tangent component of e_4 is collinear with $\mathcal{I}_*F_*e_1$, and hence is a principal direction of $\tilde{f} = \mathcal{I} \circ F$. In terms of the orthogonal decomposition $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, with e_4 spanning the factor $\mathbb{R}$, we can write $\tilde{f} = \Phi \circ f$, and hence $F = \mathcal{I} \circ \Phi \circ f$, where $\Phi \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^4$ is the standard isometry and $f \colon M^3 \to \mathbb{R}^3 \times \mathbb{R}$ is a conformally flat hypersurface with three distinct principal curvatures, having the property that the tangent component of the vector field $\partial/\partial t = e_4$ is a principal direction at any point.

In case (ii), the vector field $\mathcal{I}_*\zeta$ is collinear with the radial vector field $\mathcal{R}$ along $\tilde{f} = \mathcal{I} \circ F$, thus the tangent component of $\mathcal{R}$ along $\tilde{f}$ is collinear with $\mathcal{I}_*f_*e_1$, and hence is a principal direction of $\tilde{f}$. In other words, $F = \mathcal{I} \circ \tilde{f}$, where $\tilde{f}\colon M^3 \to \mathbb{R}^4$ is a conformally flat hypersurface with three distinct principal curvatures having the property that the tangent component of the radial vector field $\mathcal{R}$ along $\tilde{f}$, that is, of the position vector field of $\tilde{f}$, is a principal direction of $\tilde{f}$. It follows that $\tilde{f} = \Phi \circ f$, and hence $F = \mathcal{I} \circ \Phi \circ f$, where $\Phi\colon \mathbb{S}^3 \times \mathbb{R} \to \mathbb{R}^4 \setminus \{0\}$ is the conformal diffeomorphism given by $\Psi(x,t) = e^t x$ and $f\colon M^3 \to \mathbb{S}^3 \times \mathbb{R}$ is a conformally flat hypersurface with three distinct principal curvatures having the property that the tangent component of the unit vector field $\frac{\partial}{\partial t}$ is a principal direction at any point, for the vector fields $\frac{\partial}{\partial t}$ and $\mathcal{R}$ are Φ-related, that is, $\Phi_*(x,t)\frac{\partial}{\partial t} = \mathcal{R}(\Phi(x,t))$.

In case (iii) we assume that the affine subspace in the intersection of all affine hyperplanes of the family $\mathcal{I}(\mathcal{F})$ is, say, the subspace $\{(y_0, y_1, y_2, y_3) : y_0 = 0 = y_1\}$. Then the vector field $\mathcal{I}_*\zeta$ is collinear along $\tilde{f} = \mathcal{I} \circ F$ with the Killing vector field $\mathcal{K}$ in $\mathbb{R}^4$ given by $\mathcal{K}(y_0, y_1, y_2, y_3) = (0, -y_3, y_2)$. Thus the tangent component of $\mathcal{K}$ along $\tilde{f}$ is collinear with $\tilde{f}_* e_1 = \mathcal{I}_* F_* e_1$, and hence is a principal direction of $\tilde{f}$. Now notice that $\mathcal{K}$ and the unit vector field $\partial/\partial t$ tangent to the factor $\mathbb{R}$ in $\mathbb{H}^3 \times \mathbb{R}$ are Φ-related, where $\Phi\colon \mathbb{H}^3 \times \mathbb{R} \to \mathbb{R}^4 \setminus \mathbb{R}^2$ be the conformal covering map given by (1.1), that is, $\Phi_*(x,t)\frac{\partial}{\partial t} = \mathcal{K}(\Phi(x,t))$. It follows that $\tilde{f} = \Phi \circ f$, and hence $F = \mathcal{I} \circ \Phi \circ f$, where $f\colon M^3 \to \mathbb{H}^3 \times \mathbb{R}$ is a conformally flat hypersurface with three distinct principal curvatures having the property that the tangent component of the unit vector field $\frac{\partial}{\partial t}$ is a principal direction at any point.

In all three cases above, it follows from Theorem 1 that $f\colon M^3 \to \mathbb{Q}_\epsilon \times \mathbb{R}$ is either given by (1.4) in terms of a linear Weingarten surface $h\colon M^2 \to \mathbb{Q}^3_\epsilon$ and a smooth function $a\colon I \to \mathbb{R}$ given by (1.3), or it is a vertical cylinder over a surface $h\colon M^2 \to \mathbb{Q}^3_\epsilon$ with constant Gauss curvature $K_0 \neq \epsilon$.

We now argue that cases (iv) and (v) can not occur. Let $(y_1, \ldots, y_4)$ be standard coordinates on $\mathbb{R}^4$ and let $\Psi\colon \mathbb{R}^4 \setminus \mathbb{R} \to \mathbb{H}^2 \times \mathbb{S}^2 \subset \mathbb{R}^3_1 \times \mathbb{R}^3$ be the conformal diffeomorphism, with $\mathbb{R} = \{(y_1, \ldots, y_4) \in \mathbb{R}^4 : y_2 = y_3 = y_4 = 0\}$, given by

$$\Psi(y_1, y_2, y_3, y_4) = \frac{1}{\sqrt{y_2^2 + y_3^2 + y_4^2}} \left(e_0 + y_1 e_1 + \left(\sum_{i=1}^4 y_i^2 \right) e_2, (y_2, y_3, y_4) \right),$$

where e_0, e_1, e_2 is a pseudo-orthonormal basis of $\mathbb{R}^3_1$ with $\langle e_0, e_0 \rangle = 0 = \langle e_2, e_2 \rangle$, $\langle e_0, e_2 \rangle = -1/2$ and $\langle e_1, e_j \rangle = \delta_{1j}$, $0 \le j \le 2$.

If either (iv) or (v) holds, then $f = \Psi \circ \mathcal{I} \circ F\colon M^3 \to \mathbb{H}^2 \times \mathbb{S}^2$ maps each integral curve of e_1 into a slice $\mathbb{H}^2 \times \{x\}$ or $\{x\} \times \mathbb{S}^2$ of $\mathbb{H}^2 \times \mathbb{S}^2$, respectively. In the former case, $f(x,s) = (a(s), h(x,s))$ for some smooth maps $a\colon I \to \mathbb{R}$ and $h\colon M^3 \to \mathbb{S}^2$. Since $f_*\partial_s$ is orthogonal to f_*X for any $X \in T_xM^2$, it follows that h does not depend on s. But then $\Phi \circ f = \mathcal{I} \circ F$ would be a rotation hypersurface over the plane curve $s \mapsto \Phi(a(s), h(x))$, for a fixed $x \in M^2$. Therefore $\mathcal{I} \circ F$, and hence also F, would have only two distinct principal curvatures, a contradiction. Arguing in a similar way also rules out case (v). □

Proof of Corollary 3: If $F\colon M^3 \to \mathbb{R}^4$ is a cyclic conformally flat hypersurface, by Theorem 2 it is given by $F = \mathcal{I} \circ \Phi \circ f\colon M^3 \to \mathbb{R}^4$ in terms of a hypersurface $f\colon M^3 \to \mathbb{Q}^3_\epsilon \times \mathbb{R}$ as in the converse statement of Theorem 1, where $\mathcal{I}$ is either the identity map or an inversion with respect to a hypersphere in $\mathbb{R}^4$ and Φ denotes the conformal diffeomorphism $\Phi\colon \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{R}^4 \setminus \{0\}$ if $\epsilon = 1$, the conformal covering map $\Phi\colon \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{R}^4 \setminus \mathbb{R}^2$ if $\epsilon = -1$ or the isometry $\Phi\colon \mathbb{Q}^3_\epsilon \times \mathbb{R} \to \mathbb{R}^4$ if $\epsilon = 0$.

Since Φ is a conformal diffeomorphism, the tangent component of $\partial/\partial t$ is a principal direction of f at any point and $\partial/\partial t$ is Φ-related to either a constant vector field ∂_{x_i}, the radial vector field $\mathcal{R}$ or one of the Killing vector fields $\mathcal{K}_{ij}$ in $\mathbb{R}^4$, according to whether $\epsilon = 0$, $\epsilon = 1$ or $\epsilon = -1$, respectively, then the tangent component of one of those vector fields is a principal direction of $\tilde{f} = \Phi \circ f$ at any point. Finally, if $\mathcal{I}$ is an inversion with respect to a hypersphere in $\mathbb{R}^4$, then (a multiple of) the vector field ∂_{x_i} is $\mathcal{I}$-related to $\mathcal{C}_i$, whereas $\mathcal{R}$ is $\mathcal{I}$-related to (a multiple of) itself. Therefore, the tangent component of either $\mathcal{C}_i$ or $\mathcal{R}$ is a principal direction of $F = \mathcal{I} \circ \Phi \circ f$ at any point.

Conversely, assume that $F\colon M^3 \to \mathbb{R}^4$ is a conformally flat hypersurface with three distinct principal curvatures such that the tangent component of one of the conformal Killing vector fields ∂_{x_i}, $\mathcal{R}$, $\mathcal{K}_{ij}$ or $\mathcal{C}_i$ is a principal direction of $\tilde{f} = \Phi \circ f$ at any point. We argue for $\mathcal{C}_i$, the other cases being similar. Since (a multiple of) the vector field ∂_{x_i} is $\mathcal{I}$-related to $\mathcal{C}_i$, it follows that the tangent component of ∂_{x_i} is a principal direction of $\tilde{f} = \mathcal{I} \circ F$ at any point. Let $\Phi\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^4$ be the isometry given by the orthogonal decomposition of $\mathbb{R}^4$ determined by ∂_{x_i}. Then $F = \mathcal{I} \circ \Phi \circ f$, where $f\colon M^3 \to \mathbb{R}^3 \times \mathbb{R}$ has the property that the tangent component of the unit vector field $\partial/\partial t$ tangent to $\mathbb{R}$ is a principal direction of f at any point. Thus F is a cyclic conformally flat hypersurface by Theorem 2. □

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