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The principal curvature theorem and its applications to constant mean curvature hypersurfaces in Euclidean space

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Dedicated to Professor Renato Tribuzy
on the occasion of his 75th birthday

Abstract. This paper is a survey about some recent results obtained by the authors on the so-called principal curvature theorem and its applications to the study of the curvature of complete oriented hypersurfaces in Euclidean space with constant mean curvature and, more generally, with constant higher order mean curvature. The original results can be found in the following joint papers [1], [2], [3] and [4].

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1 Constant mean curvature surfaces in $\mathbb{R}^3$

In a classical paper, Klotz and Osserman [7] characterized right circular cylinders as the only non-totally umbilical, complete surfaces immersed into the Euclidean 3-space $\mathbb{R}^3$ with constant mean curvature (CMC) $H \neq 0$ and whose Gaussian curvature does not change sign. Specifically, they proved the following.

Theorem 1.1. *Let Σ be a complete surface immersed into the Euclidean space $\mathbb{R}^3$ with constant mean curvature $H \neq 0$. Assume that Σ is not totally umbilical. If its Gaussian curvature K does not change sign then $K = 0$ and Σ is a right circular cylinder $\mathbb{R} \times \mathbb{S}^1(r) \subset \mathbb{R}^3$, with $r > 0$.*

As a nice application of Theorem 1.1, one gets the following consequence for the infimum of the Gaussian curvature of a CMC surface in $\mathbb{R}^3$ (see Theorem 2 of [1] in the case where $c = 0$).

Corollary 1.2. *Let Σ be a complete surface immersed into the Euclidean space $\mathbb{R}^3$ with constant mean curvature $H \neq 0$, and let K stand for its Gaussian curvature. Then*

(i) either $\inf_\Sigma K = H^2$ and Σ is totally umbilical,

(ii) or $\inf_\Sigma K \leq 0$, with equality if and only if Σ is a right circular cylinder $\mathbb{R} \times \mathbb{S}^1(r) \subset \mathbb{R}^3$, with $r > 0$.

Proof. It follows from the Gauss equation of the surface that $K \leq H^2$ on Σ, with equality at the umbilical points of Σ. Therefore, $\inf_\Sigma K \leq H^2$ with equality if and only if Σ is totally umbilical. This proves part (i). Moreover, if $\inf_\Sigma K < H^2$ then it must be $\inf_\Sigma K \leq 0$ necessarily. Otherwise, one would have $K \geq \inf_\Sigma K > 0$ which is not possible by Theorem 1.1, since the non-totally umbilical surfaces are flat. Finally, if equality holds, $\inf_\Sigma K = 0$, then $K \geq 0$ and the result follows from Theorem 1.1. □

Delaunay rotational surfaces show that the estimate in Corollary 1.2 is sharp.

Example 1.3 (Delaunay surfaces: Unduloids with $\inf K = -\varepsilon < 0$)**.** For a given constant $H \neq 0$, consider the family of unduloids in $\mathbb{R}^3$ with constant mean curvature H, given by the parametrization

$$(s, \theta) \mapsto (x_B(s), y_B(s)\cos\theta, y_B(s)\sin\theta)$$

where $0 < B < 1$ and

$$\begin{aligned} x_B(s) &= \int_0^s \frac{1 + B\sin(2Ht)}{\sqrt{1 + B^2 + 2B\sin(2Ht)}} dt \\ y_B(s) &= \frac{\sqrt{1 + B^2 + 2B\sin(2Hs)}}{2|H|}. \end{aligned}$$

The first fundamental form of these surfaces is $ds^2 + y_B(s)^2 d\theta^2$ and the Gaussian curvature is then

$$K_B(s, \theta) = K_B(s) = -\frac{y_B''(s)}{y_B(s)} = \frac{4H^2 B(B + \sin(2Hs))(1 + B\sin(2Hs))}{(1 + B^2 + 2B\sin(2Hs))^2}.$$

Therefore, $\inf K_B = -4H^2 B/(1 - B)^2 < 0$, and for a given $\varepsilon > 0$ there exists $0 < B < 1$ such that $\inf K_B = -\varepsilon < 0$.

As another application of Theorem 1.1, one also gets the following consequence for the supremum of the Gaussian curvature (see Theorem 3 of [2] in the case where $c = 0$).

Corollary 1.4. *Let Σ be a complete surface immersed into the Euclidean space $\mathbb{R}^3$ with constant mean curvature H, and let K stand for its Gaussian curvature. Then*

(i) either $\sup_\Sigma K = H^2$,

(ii) or $0 \leq \sup_\Sigma K < H^2$, with equality $\sup_\Sigma K = 0$ if and only if Σ is a right circular cylinder $\mathbb{R} \times \mathbb{S}^1(r) \subset \mathbb{R}^3$, with $r > 0$.

Proof. We know from the Gauss equation that $K \leq H^2$ on Σ, with equality at the umbilical points. Therefore, $\sup_\Sigma K \leq H^2$. Moreover, if $\sup_\Sigma K < H^2$ then it must be $\sup_\Sigma K \geq 0$ necessarily. Otherwise, it would follow that

$K \leq \sup_\Sigma K < 0$ which is not possible by Theorem 1.1 and the assumption $\sup_\Sigma K < H^2$. This shows that either $\sup_\Sigma K = H^2$ or $0 \leq \sup_\Sigma K < H^2$. Finally, if equality $\sup_\Sigma K = 0$ holds, then $K \leq 0$ and the result follows from Theorem 1.1. □

Once again, Delaunay rotational surfaces show that the estimate in Corollary 1.4 is sharp.

Example 1.5 (Delaunay surfaces: Unduloids with $\sup K = \varepsilon > 0$ and unduloids with $\sup K = H^2 - \varepsilon$)**.** For a given constant $H \neq 0$, the Gaussian curvature of the family of unduloids in $\mathbb{R}^3$ with constant mean curvature H given above is

$$K_B(s,\theta) = K_B(s) = -\frac{y_B''(s)}{y_B(s)} = \frac{4H^2B(B+\sin(2Hs))(1+B\sin(2Hs))}{(1+B^2+2B\sin(2Hs))^2}.$$

Therefore, for these examples,

$$\sup K_B = \frac{4H^2B}{(1+B)^2} > 0,$$

and for a given $\varepsilon > 0$ one may find $0 < B_1, B_2 < 1$ such that

$$\sup K_{B_1} = \varepsilon > 0 \quad \text{and} \quad \sup K_{B_2} = H^2 - \varepsilon$$

respectively.

2 Higher dimensional versions of Corollary 1.2 and Corollary 1.4

The proof of Theorem 1.1 (and hence the proofs of Corollary 1.2 and Corollary 1.4) strongly depends on the conformal structure of the 2-dimensional surface Σ, and it cannot be extended to higher dimensions. Using an alternative approach based on the so called Omori-Yau maximum principle, in [1] we extended Corollary 1.2 to higher dimension $n \geq 3$ as follows (see Corollary 4 of [1] in the case where $c = 0$).

Theorem 2.1. *Let Σ be a complete hypersurface immersed into the Euclidean space $\mathbb{R}^{n+1}$, $n \geq 3$, with constant mean curvature $H \neq 0$, and let R stand for its scalar curvature. Then*

(i) either $\inf_\Sigma R = n(n-1)H^2$ *and Σ is totally umbilical,*

(ii) or $\inf_\Sigma R \leq n^2(n-2)H^2/(n-1)$*. Moreover, the equality holds and this infimum is attained at some point if and only if Σ is a right cylinder* $\mathbb{R} \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$*, with $r > 0$.*

In order to see Theorem 2.1 as an extension of Corollary 1.2 to higher dimensions, recall that when $n = 2$, then $R = 2K$. Similarly and using again the Omori-Yau maximum principle, in [2] Corollary 1.4 was also extended to higher dimensions $n \geq 3$ as follows (see Theorem 4 of [2] in the case where $c = 0$).

Theorem 2.2. *Let Σ be a complete hypersurface which is properly immersed into the Euclidean space $\mathbb{R}^{n+1}$, $n \geq 3$, with constant mean curvature $H \neq 0$ and with two distinct principal curvatures, one of them being simple, and let R stand for its scalar curvature. Then*

$$\sup_\Sigma R \geq 0.$$

Moreover, the equality $\sup_\Sigma R = 0$ *holds and this supremum is attained at some point if and only if Σ is a right circular cylinder* $\mathbb{R}^{n-1} \times \mathbb{S}^1(r) \subset \mathbb{R}^{n+1}$*, with $r > 0$.*

Our objective here is to introduce an alternative approach to Theorem 2.2 which is based on the so called Principal Curvature Theorem (PCT). The PCT is a purely geometric result on the principal curvatures of complete hypersurfaces in $\mathbb{R}^{n+1}$ given by Smyth and Xavier in [14], in their proof of Efimov's theorem in dimension greater than two, and it states as follows.

Theorem 2.3 (Principal Curvature Theorem)**.** *Let Σ be a complete orientable hypersurface immersed into the Euclidean space $\mathbb{R}^{n+1}$, which is not* *a hyperplane, and let A denote its second fundamental form with respect to a global unit normal field. Let $\Lambda \subset \mathbb{R}$ be the set of nonzero values assumed by the eigenvalues of A, and let $\Lambda^{\pm} = \Lambda \cap \mathbb{R}^{\pm}$. Then*

(i) If Λ^+ and Λ^- are both nonempty, then $\inf_\Sigma \Lambda^+ = \sup_\Sigma \Lambda^- = 0$.

(ii) If Λ^+ or Λ^- is empty then the closure $\bar{\Lambda}$ of Λ is connected.

Classical Hilbert's theorem [6] states that the hyperbolic plane cannot be realized isometrically in $\mathbb{R}^3$. In [5] Efimov generalized this result by proving that no complete surface with Gaussian curvature $K \leq -\kappa < 0$ can be isometrically immersed in $\mathbb{R}^3$. As a first application of the PCT, Smyth and Xavier gave the following version of Efimov's theorem for $n \geq 3$.

Theorem 2.4. *Let Σ be a complete hypersurface immersed into the Euclidean space $\mathbb{R}^{n+1}$ with negative Ricci curvature.*

(i) For $n = 3$, $\inf_\Sigma \|A\| = 0$. In particular, $\inf_\Sigma \|\mathrm{Ric}\| = 0$.

(ii) For $n > 3$, if the sectional curvature on Σ does not take every real value, then the Ricci curvature is not bounded away from zero.

As another application of the PCT, they also gave the following.

Theorem 2.5. *Let Σ be a complete hypersurface immersed in $\mathbb{R}^{n+1}$ with constant mean curvature $H \neq 0$. If Σ has non-positive Ricci curvature, $\mathrm{Ric} \leq 0$, then Σ is a right circular cylinder $\mathbb{R}^{n-1} \times \mathbb{S}^1(r) \subset \mathbb{R}^{n+1}$, with $r > 0$.*

3 New applications of the PCT to CMC hypersurfaces

Motivated by these results, and as another application of the PCT, in [2] we obtained the following version of Theorem 3 (see Theorem 11 in [2]).

Theorem 3.1. *Let Σ be a complete oriented hypersurface immersed into the Euclidean space $\mathbb{R}^{n+1}$, $n \geq 3$, with constant mean curvature $H \neq 0$ and with two distinct principal curvatures, one of them being simple, and let R stand for its scalar curvature. Then*

$$\sup_{\Sigma} R \geq 0 \quad \text{or, equivalently,} \quad \inf_{\Sigma} \|A\|^2 \leq n^2 H^2,$$

with equality if and only if Σ is a right circular cylinder $\mathbb{R}^{n-1} \times \mathbb{S}^1(r)$ with $r = 1/n|H| > 0$.

Observe that the equivalence between both inequalities in Theorem 3.1 follows directly from the basic relation between R and $\|A\|^2$ given by the Gauss equation of the hypersurface

$$\|A\|^2 = n^2 H^2 - R.$$

Regarding the condition of having two distinct principal curvatures, it is well known since the pioneering work by Otsuki [12] that if both principal curvatures have multiplicity greater than 1, then the distributions of the space of principal vectors corresponding to each principal curvature are completely integrable and each principal curvature is constant on each of the integral leaves of the corresponding distribution. In particular, if the mean curvature is constant, then the two principal curvatures are also constant and the hypersurface is an isoparametric hypersurface with exactly two constant principal curvatures, with multiplicities m and $n-m$, and $1 < m < n-1$. Then, by the classical results on isoparametric hypersurfaces in Euclidean space [8, 13], the hypersurface must be an open piece of the standard product embedding $\mathbb{R}^m \times \mathbb{S}^{n-m}(r) \subset \mathbb{R}^{n+1}$ with $r > 0$. Therefore, under the condition of having two distinct principal curvatures, the interesting case for studying constant mean curvature hypersurfaces is the case where one of the principal curvatures is simple, that is, with multiplicity 1.

Proof of Theorem 3.1 (following [2]). Choose the orientation so that $H > 0$ and let λ and μ be the two distinct principal curvatures of Σ with multiplicities $(n-1)$ and 1. Then

$$nH = (n-1)\lambda + \mu$$

and

$$\|A\|^2 = (n-1)\lambda^2 + \mu^2,$$

so that by Gauss equation

$$R = n^2H^2 - \|A\|^2 = n(n-1)\lambda(2H-\lambda). \tag{3.1}$$

If $\sup_\Sigma R = -\kappa < 0$ then from (3.1)

$$\lambda^2 - 2H\lambda - \frac{\kappa}{n(n-1)} \geq 0.$$

Therefore, either

$$\lambda \geq H + \sqrt{H^2 + \frac{\kappa}{n(n-1)}} > 0 \tag{3.2}$$

or

$$\lambda \leq H - \sqrt{H^2 + \frac{\kappa}{n(n-1)}} < 0. \tag{3.3}$$

In the first case, by (3.2) we have

$$\mu = nH - (n-1)\lambda < H - \sqrt{H^2 + \frac{\kappa}{n(n-1)}} < 0.$$

In the second case, by (3.3) we get

$$\mu = nH - (n-1)\lambda > H + \sqrt{H^2 + \frac{\kappa}{n(n-1)}} > 0.$$

Therefore, in any case we have that Λ^+ and Λ^- are both non-empty, with $\inf \Lambda^+ > 0$ and $\sup \Lambda^- < 0$, which contradicts the PCT. As a consequence, it must be $\sup_\Sigma R \geq 0$.

Suppose now that $\sup_\Sigma R = 0$. Since $H > 0$, from (3.1) we have $\lambda(2H-\lambda) \leq 0$. This implies that either

$$\lambda \leq 0 \quad \text{or} \quad \lambda \geq 2H > 0.$$

Observe that the second case cannot happen, because it would imply

$$\mu = nH - (n-1)\lambda \leq -(n-2)H < 0,$$

which contradicts again the PCT, since

$$\inf \Lambda^+ \geq 2H > 0 \quad \text{and} \quad \sup \Lambda^- \leq -(n-2)H < 0.$$

Therefore, it must be $\lambda \leq 0$ and hence

$$\mu = nH - (n-1)\lambda \geq nH > 0.$$

This implies that $\inf \Lambda^+ \geq nH > 0$ and hence, again by the PCT, Λ^- must be empty, which means that $\lambda = \text{constant} = 0$. Hence, $\mu = \text{constant} = nH > 0$ is also constant and, by the classical results on isoparametric hypersurfaces in Euclidean space we conclude that Σ is a circular cylinder $\mathbb{R}^{n-1} \times \mathbb{S}^1(r) \subset \mathbb{R}^{n+1}$, with $r = 1/nH > 0$. □

4 The case of higher order mean curvatures

Let Σ be an oriented hypersurface immersed into $\mathbb{R}^{n+1}$ and let A denote its second fundamental form with respect to a global unit normal field. As is well known, A is self-adjoint and its eigenvalues $\kappa_1, \dots, \kappa_n$ are the principal curvatures of the hypersurface. Associated to A there are n algebraic invariants given by

$$S_k = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \dots \kappa_{i_k}, \quad 1 \leq k \leq n.$$

The k-th mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = S_k, \quad 1 \leq k \leq n.$$

In particular, when $k = 1$, $H_1 = \frac{1}{n}\sum_{i=1}^n \kappa_i = \frac{1}{n}\mathrm{tr}(A) = H$ is nothing but the mean curvature of Σ. On the other hand, for $k = 2$, H_2 is (up to a factor) the scalar curvature of Σ, since

$$R = n^2H^2 - \|A\|^2 = \left(\sum_{i=1}^n \kappa_i\right)^2 - \sum_{i=1}^n \kappa_i^2 = n(n-1)H_2.$$

More generally, when k is odd the curvature H_k is extrinsic (and its sign depends on the chosen orientation), while when k is even the curvature H_k is intrinsic and its value does not depend on the chosen orientation. Finally, for $k = n$, $H_n = \det(A)$ is called the Gauss-Kronecker curvature of Σ and usually denoted by K.

As another application of the PCT, in [3] we extended Theorem 3.1 to the case of constant higher order mean curvature as follows (see Theorem 2 in [3]).

Theorem 4.1. *Let $n \geq 3$ and $2 \leq k < n$. Let Σ be a complete oriented hypersurface immersed in $\mathbb{R}^{n+1}$ with constant k-th mean curvature $H_k > 0$ and two distinct principal curvatures, one of them being simple. Then*

$$\inf_{\Sigma} \|A\|^2 \leq (n-1) \left(\frac{nH_k}{n-k} \right)^{2/k}$$

with equality if and only if Σ is a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, with $r = ((n-k)/nH_k)^{1/k} > 0$.

On the other hand, regarding the supremum of $\|A\|^2$ we have obtained the following result, which holds even for $k = 1$ (see Theorem 3 in [3]).

Theorem 4.2. *Let $n \geq 3$ and $1 \leq k < n$. Let Σ be a complete oriented hypersurface immersed in $\mathbb{R}^{n+1}$ with constant k-th mean curvature $H_k > 0$ and two distinct principal curvatures, one of them being simple. Then*

$$\sup_{\Sigma} \|A\|^2 \geq (n-1) \left(\frac{nH_k}{n-k} \right)^{2/k}$$

with equality if and only if Σ is a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, with $r = ((n-k)/nH_k)^{1/k} > 0$.

Proof of Theorem 4.1 (following [3]). Let λ and μ be the two distinct principal curvatures of Σ with multiplicities $(n-1)$ and 1, respectively. Then

$$nH_k = \lambda^{k-1}((n-k)\lambda + k\mu)$$

and

$$\|A\|^2 = (n-1)\lambda^2 + \mu^2.$$

Since H_k is a positive constant and $k > 1$, then $\lambda \neq 0$ and one gets

$$\mu = \mu(\lambda) = \frac{nH_k}{k\lambda^{k-1}} - \frac{n-k}{k}\lambda, \tag{4.1}$$

and

$$\|A\|^2 = \|A\|^2(\lambda) = (n-1)\lambda^2 + \left(\frac{nH_k}{k\lambda^{k-1}} - \frac{(n-k)\lambda}{k}\right)^2 \tag{4.2}$$

When k is even, changing the orientation if necessary, we may assume without loss of generality that $\lambda > 0$. On the other hand, when k is odd we will see that the case $\lambda < 0$ cannot occur. Actually, if k is odd it follows from (4.1) that the global minimum of $\mu(\lambda)$ for $\lambda < 0$ is attained at

$$\lambda_0 = -\left(\frac{n(k-1)H_k}{(n-k)}\right)^{1/k},$$

with

$$\min_{\lambda<0} \mu(\lambda) = \mu(\lambda_0) > 0.$$

Therefore, if $\lambda < 0$ on Σ we get that Λ^+ and Λ^- are both non-empty with $\inf_\Sigma \Lambda^+ \geq \mu(\lambda_0) > 0$, which contradicts the PCT. Therefore, we may always assume $\lambda > 0$.

It is easy to see from (4.2) that the global minimum of $\|A\|^2(\lambda)$ for $\lambda > 0$ is attained at $\lambda = H_k^{1/k}$, with

$$\min_{\lambda>0} \|A\|^2(\lambda) = \|A\|^2(H_k^{1/k}) = nH_k^{2/k}.$$

Observe that, independently of the value of $H_k > 0$, the equation

$$\|A\|^2(\lambda) = (n-1)\left(\frac{nH_k}{n-k}\right)^{2/k}$$

has exactly two positive roots λ_1 and λ_2 satisfying

$$0 < \lambda_1 < H_k^{1/k} < \lambda_2$$

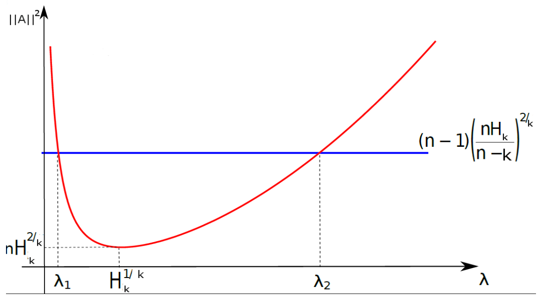

Figure 4.1: Graph of the function $\|A\|^2(\lambda)$ given in (4.2) and the constant function $\|A\|^2 \equiv (n-1)\left(\frac{nH_k}{n-k}\right)^{2/k}$, for $k > 1$. Note that the global minimum of $\|A\|^2$ for $\lambda > 0$ is attained at $\lambda = H_k^{1/k}$.

with

$$\lambda_2 = \left(\frac{nH_k}{n-k}\right)^{1/k} \quad \text{and} \quad \mu(\lambda_2) = 0. \tag{4.3}$$

Suppose now that

$$\inf_{\Sigma} \|A\|^2 > (n-1)\left(\frac{nH_k}{n-k}\right)^{2/k}$$

This implies that either

$$0 < \lambda < \lambda_1, \quad \text{or} \quad \lambda > \lambda_2.$$

If $0 < \lambda < \lambda_1$, since $\mu(\lambda)$ is decreasing when $\lambda > 0$, we obtain

$$0 < \lambda < \lambda_1 < H_k^{1/k} < \mu(\lambda_1) < \mu(\lambda). \tag{4.4}$$

Then, Λ^- is empty and, by the PCT, $\bar{\Lambda}$ should be connected. Also from (4.4), there exists $\varepsilon > 0$ such that intervals $(-\varepsilon, H_k^{1/k} - \varepsilon)$ and $(H_k^{1/k} + \varepsilon, +\infty)$ are disjoint and

$$\bar{\Lambda} \subset (-\varepsilon, H_k^{1/k} - \varepsilon) \cup (H_k^{1/k} + \varepsilon, +\infty)$$

which contradicts that $\bar{\Lambda}$ is connected. Thus, this case cannot occur.

On the other hand, if $\lambda > \lambda_2$, then from (4.3) we have

$$\mu(\lambda) < \mu(\lambda_2) = 0.$$

Therefore, Λ^+ and Λ^- are both non-empty with $\inf_\Sigma \Lambda^+ \geq \lambda_2 > 0$. But this contradicts the PCT. Thus, this case cannot occur neither.

This proves the inequality

$$\inf_\Sigma \|A\|^2 \leq (n-1)\left(\frac{nH_k}{n-k}\right)^{2/k}$$

Suppose now that equality holds. This implies that either

$$0 < \lambda \leq \lambda_1, \quad \text{or} \quad \lambda \geq \lambda_2 = \left(\frac{nH_k}{n-k}\right)^{1/k}.$$

The first case cannot happen. Actually, if $0 < \lambda \leq \lambda_1$ then

$$0 < \lambda \leq \lambda_1 < H_k^{1/k} < \mu(\lambda_1) \leq \mu(\lambda). \tag{4.5}$$

Thus, Λ^- would be empty and, by the PCT, $\bar{\Lambda}$ would be connected. But from (4.5), there exists $\varepsilon > 0$ such that intervals $(-\varepsilon, H_k^{1/k} - \varepsilon)$ and $(H_k^{1/k} + \varepsilon, +\infty)$ are disjoint and

$$\bar{\Lambda} \subset (-\varepsilon, H_k^{1/k} - \varepsilon) \cup (H_k^{1/k} + \varepsilon, +\infty)$$

which contradicts that $\bar{\Lambda}$ is connected. Therefore, it must be necessarily

$$\lambda \geq \lambda_2 = \left(\frac{nH_k}{n-k}\right)^{1/k}.$$

From (4.3), since $\mu(\lambda)$ is decreasing when $\lambda > 0$, we obtain $\mu(\lambda) \leq 0$. This implies that $\inf_\Sigma \Lambda^+ \geq \lambda_2 > 0$ and hence, again by the PCT, Λ^- must be empty, which means that $\mu =$ constant $= 0$. Hence, $\lambda =$ constant $= \left(\frac{nH_k}{n-k}\right)^{1/k}$ is also constant and, by the classical results on isoparametric hypersurfaces in Euclidean space, Σ is a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, with $r = \left(\frac{n-k}{nH_k}\right)^{1/k}$. □

The proof of Theorem 4.2 follows similar ideas. For the details, we refer the reader to the original paper [3]. Finally, another application of the PCT gives the following extension of Theorem 1.1 to the case of hypersurfaces with constant higher order mean curvature, recently given in [4].

Theorem 4.3. *Let $n \geq 3$ and $2 \leq k < n$. Let Σ be a complete oriented hypersurface immersed in $\mathbb{R}^{n+1}$ with constant k-th mean curvature $H_k \neq 0$ and two distinct principal curvatures, one of them being simple. If the Gauss-Kronecker curvature K of Σ does not change sign, then $K = 0$ and Σ is a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, with $r = ((n-k)/nH_k)^{1/k} > 0$.*

5 Further developments and open problems

Related to the main results of this paper, that is, Theorems 3.1, 4.1, 4.2 and 4.3, it would be very interesting to study the validity of them for the case of hypersurfaces with an arbitrary number of distinct principal curvatures. In that case, our technique of applying the PCT can still work but it certainly gives more work. For instance, we refer the reader to the paper [11] for a recent application of the PCT to the study of complete hypersurfaces with constant mean and scalar curvatures in Euclidean spaces with no restriction on the number of distinct principal curvatures. In this direction, we notice that the hypersurfaces considered in Theorems 4.1 and 4.2 are just rotational hypersurfaces, or equivalently, $O(n)$-invariant hypersurfaces (see Corollary 2.2 in [10]). Hence it is natural to analyze hypersurfaces which are invariant under other isometry groups of Euclidean space. Thus, we would like to propose the following

Conjecture 5.1. *Let Σ be a complete hypersurface immersed in $\mathbb{R}^{2n}$, with $n \geq 2$, invariant under the group $O(n) \times O(n)$ with constant mean curvature $H > 0$. Then*

$$\inf_{\Sigma} \|A\|^2 \leq \frac{(2n-1)^2 H^2}{n-1} \leq \sup_{\Sigma} \|A\|^2. \tag{5.1}$$

Moreover, the equality holds in either side of (5.1) *if and only if* Σ *is isometric to* $\mathbb{R}^n \times \mathbb{S}^{n-1}(r)$ *with* $r = (n-1)/(2n-1)H$*.*

Another interesting open problem related to the main results of this paper is to consider them from a local point of view, that is, without the completeness assumption. In that case, one cannot apply the PCT, which is valid under the completeness assumption, and it is necessary to look for alternatives approaches.

On the other hand, and as a extension of our results to the case of linear Weingarten hypersurfaces, we refer the reader to [9] for a recent application of the PCT to the study of complete Weingarten hypersurfaces in Euclidean space with two distinct principal curvatures, one of them being simple.

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