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A unified approach to Bäcklund type theorems for surfaces in 3-dimensional pseudo-euclidean space

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Dedicated to Professor Renato Tribuzy on the occasion of his 75th birthday

Abstract. We provide a unified geometric proof for the six cases of Bäcklund type theorem and the integrability theorem for space-like and time-like surfaces in the pseudo-euclidean space $\mathbb{R}^3_s$ with $s = 0, 1$, without requiring the additional assumption on the shape operator considered for the surfaces in the Lorentzian space by McNertney, Palmer, Tian and Gu-Hu-Inogushi. In each case, the surfaces have non zero constant Gaussian curvature. By assuming that the surfaces have two distinct real principal curvatures, we provide the analytic interpretation of these theorems in terms of solutions of the associated differential equations: the sine-Gordon and the elliptic sine-Gordon, the sinh-Gordon and the elliptic sinh-Gordon equations.

Keywords: Bäcklund transformation, pseudo-Riemannian surfaces, sine-Gordon, sinh-Gordon, elliptic sine-Gordon, elliptic sinh-Gordon.

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1 Introduction

The so called Bäcklund theorem was originally proved by A. V. Bäcklund [2, 3]. One considers a line congruence, called *pseudo-spherical line congruence*, in $\mathbb{R}^3$ between two focal surfaces M and $\bar{M}$. This is a smooth two-parameter family of straight lines such that, for each pair of points $p \in M$ and $\bar{p} \in \bar{M}$, the vector from p to $\bar{p}$ lays in a unique straight line of the line congruence and it has constant length independent of p. Moreover, this line is tangent to M and $\bar{M}$ at p and $\bar{p}$ and the angle between the normal vectors at p and $\bar{p}$ is a constant independent of p. Bäcklund proved that a pseudo-spherical line congruence exists only between surfaces whose Gaussian curvatures are both equal to the same negative constant (this justifies the denomination of pseudo-spherical line congruence). Moreover, he showed that given a surface M of constant negative curvature one obtains a two-parameter family of surfaces with the same constant negative curvature, associated by a pseudo-spherical line congruence to M.

A natural question to ask is whether similar theorems are realized for surfaces in the Lorentz-Minkowski space $\mathbb{R}^3_1$. In [16], Palmer proved that a space-like line congruence between a time-like and a space-like focal surfaces in $\mathbb{R}^3_1$ exists only between surfaces whose Gaussian curvatures are both equal to the same positive constant. In [15], McNertney considered a space-like and a time-like line congruence between two time-like focal surfaces in $\mathbb{R}^3_1$. Under the assumption that the shape operator of the time-like surface is either diagonalizable over the field $\mathbb{R}$ or it is not diagonalizable over $\mathbb{C}$ and has one single light-like eigenvector, she proved that both time-like surfaces must have the same positive Gaussian curvature. In [12], Gu-Hu-Inoguchi considered a space-like and a time-like line congruence between two time-like focal surfaces in $\mathbb{R}^3_1$, under the assumption that the shape operator of the time-like surface is diagonalizable over $\mathbb{C}$, but not over $\mathbb{R}$. These assumptions were required in order to consider some special local coordinates (such as asymptotic Chebyshev coordinates), so that the existence of the geometric line congruence between surfaces would be equivalent to the Bäcklund transformation of solutions to some special partial differential equation.

In this paper, we present a unified approach to treat all possible line congruences in $\mathbb{R}^3_s$ (there are six cases), which include the classical results ($s = 0$) and also the analogue results in the Lorentz-Minkowski 3-space ($s = 1$), without any extra condition on the shape operator. The proof, which is based on a coordinate-free approach, proves a Bäcklund type Theorem, which states that a necessary condition for the existence of a Bäcklund type line congruence in $\mathbb{R}^3_s$, between two focal surfaces M^2_r and $\bar{M}^2_{\bar{r}}$, where r and $\bar{r}$ are the indices of the pseudo-Riemannian surfaces M^2 and $\bar{M}^2$ respectively, is that both surfaces have the same non zero constant Gaussian curvature, and that the constant may be positive or negative depending on the indices r and $\bar{r}$. Then we prove the Integrability Theorem which shows that starting with a space-like or time-like surface $M \subset \mathbb{R}^3_s$, there exists a 2-parameter family of surfaces associated to M by a Bäcklund type line congruence.

In order to obtain the analytic interpretation of the geometric theory, in terms of solutions of partial differential equations, we assume that the surfaces have two real distinct principal curvatures. Under this assumption, we consider appropriate local coordinate systems for the surfaces and we show that locally a space-like or time-like surface in $\mathbb{R}^3_s$ with non zero constant Gaussian curvature corresponds to a solution of one of the following partial differential equation: the sine-Gordon equation, the sinh-Gordon equation, the elliptic sine-Gordon equation and the elliptic sinh-Gordon equation. The analytic interpretation of the Bäcklund type Theorem and of the Integrability Theorem in each case provides a Bäcklund type transformation for the differential equations. In particular, we obtain transformations (complementary in terms of the parameter) that provide new solutions for the sine-Gordon equation (see Theorems 7.1 and 7.2) and for the sinh-Gordon equation (see Theorems 7.3 and 7.4), from a given solution of each equation. While in the case of the elliptic equations, we obtain a transformation that provides solutions to the elliptic sine-Gordon equation (resp. elliptic sinh-Gordon equation) from a given solution of the elliptic sinh-Gordon equation (resp. elliptic sine-Gordon equation), given by Theorem 7.5 (resp. Theorem 7.6).

The commutativity property for these analytic Bäcklund type transformations i.e., the corresponding superposition formula for the differential equations, will be given in a forthcoming paper.

This paper is organized as follows. In Section 2, we introduce some preliminaries which include the structure equations for a moving frame adapted to surfaces in $\mathbb{R}^3_s$. In Section 3, we include the definition of a Bäcklund type line congruence in $\mathbb{R}^3_s$ and we prove the Bäcklund type Theorem. Section 4 contains the Integrability Theorem. In Section 5 and 6, considering surfaces which admit two real distinct principal curvatures, we consider the differential equations associated to the surfaces of non zero constant Gaussian curvature and we give an analytic interpretation of the Integrability Theorem in terms of solutions of the differential equations. In Section 7, we explicitly give each one of the six cases. In Section 8, we conclude mentioning some generalizations of the theory described above, including higher dimensional results.

2 Structure Equations in $\mathbb{R}^3_s$

We denote by $M^2_r \hookrightarrow \mathbb{R}^3_s$ a semi-Riemannian surface of index r isometrically immersed in $\left(\mathbb{R}^3_s, \langle,\rangle = dx_1^2 + dx_2^2 + (-1)^s dx_3^2\right)$, where $r,\ s \in \{0,1\}$, $0 \le r \le s \le 1$. We consider $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3_s$ to be a local parametrization of $M^2_r \subset \mathbb{R}^3_s$, where U is an open subset of $\mathbb{R}^2$.

Given a vector $v \in \mathbb{R}^3_s$, we say that v is a *unit vector* if the absolute value $|\langle v,v\rangle| = 1$. We say that v is *space-like* if $\langle v,v\rangle > 0$ or $v = 0$, *time-like* if $\langle v,v\rangle < 0$ and *light-like* if $\langle v,v\rangle = 0$ and $v \neq 0$. We say that M^2_r is a *space-like* (resp. *time-like*) surface if $r = 0$ (resp. $r = 1$).

Definition 2.1. An *adapted moving frame* on $M^2_r \subset \mathbb{R}^3_s$ is a set of orthonormal vector fields $\{e_1, e_2, e_3\}$, where e_1 and e_2 are tangent to M^2_r and e_3 is normal to M^2_r. We denote by $\epsilon_i = \langle e_i, e_i\rangle \in \{-1, 1\}$ the *sign* of the vector e_i.

Consider a local parametrization $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3_s$ of M^2_r and a moving frame $\{e_1, e_2, e_3\}$ adapted to $X(U)$ and let ω_i, $1 \le i \le 3$, be the dual frame. Observe that $\omega_3 = 0$ when restricted to $X(U)$ and

$$dX = \omega_1 e_1 + \omega_2 e_2. \tag{2.1}$$

Consider the connection forms defined by

$$de_i = \sum_{j=1}^{3} \epsilon_j \omega_{ij} e_j, \qquad 1 \leq i \leq 3. \tag{2.2}$$

The following *structure equations* hold (see for example [17]),

$$\omega_{ij} = -\omega_{ji}, \tag{2.3}$$

$$d\omega_1 = \epsilon_1 \omega_2 \wedge \omega_{21}, \tag{2.4}$$

$$d\omega_2 = \epsilon_2 \omega_1 \wedge \omega_{12}, \tag{2.5}$$

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0, \tag{2.6}$$

$$d\omega_{12} = \epsilon_3 \omega_{13} \wedge \omega_{32} = -K \epsilon_1 \epsilon_2 \omega_1 \wedge \omega_2, \tag{2.7}$$

$$d\omega_{13} = \epsilon_2 \omega_{12} \wedge \omega_{23}, \tag{2.8}$$

$$d\omega_{23} = \epsilon_1 \omega_{21} \wedge \omega_{13}, \tag{2.9}$$

where K is called the *Gaussian curvature* of $M_r^2 \subset \mathbb{R}_s^3$.

We can write the fundamental forms of the immersion $M_r^2 \hookrightarrow \mathbb{R}_s^3$ as follows

$$I = \langle dX, dX \rangle = \epsilon_1 \omega_1 \otimes \omega_1 + \epsilon_2 \omega_2 \otimes \omega_2,$$
$$II = \langle -de_3, dX \rangle = \omega_1 \otimes \omega_{13} + \omega_2 \otimes \omega_{23},$$

where $\otimes$ denotes the tensor product.

3 Bäcklund type Theorem

In this section, we introduce the definition of a Bäcklund type line congruence in $\mathbb{R}_s^3$ and provide a unified proof for the Bäcklund Theorem, for space-like or time-like surfaces in $\mathbb{R}_s^3$.

Definition 3.1. Let $M_r^2, \bar{M}_{\bar{r}}^2 \hookrightarrow \mathbb{R}_s^3$ be two semi-Riemannian surfaces that are isometrically immersed in $\mathbb{R}_s^3$ with $0 \leq r, \bar{r} \leq s \leq 1$. Let $\psi : M_r \to \bar{M}_{\bar{r}}$ be a diffeomorphism such that for each $p \in M$, $\bar{p} = \psi(p) \neq p$ and let $v(p) \in \mathbb{R}_s^3$ be the vector from p to $\bar{p}$. We say that ψ is a *Bäcklund type Line Congruence* (*BLC*) if there exist constants $\lambda > 0$ and $\Lambda \geq 0$, such that:

1. The line in $\mathbb{R}^3_s$ determined by $v(p)$ is tangent to M and $\bar{M}$ at p and $\bar{p}$ respectively.

2. For all $p \in M$, $|v(p)| = \sqrt{|\langle v(p), v(p)\rangle|} = \lambda$.

3. The normals $N(p)$ and $\bar{N}(\bar{p})$ are not collinear and $\langle N(p), \bar{N}(\bar{p})\rangle = \Lambda$.

The constant λ is called the *distance of the line congruence*. We say that the BLC in $\mathbb{R}^3_s$ is *space-like* (resp. *time-like*) if for all $p \in M$ the vector $v(p)$ is space-like (resp. time-like), i.e.,

$$\epsilon := \left\langle \frac{v(p)}{\lambda}, \frac{v(p)}{\lambda} \right\rangle = 1 \quad (\text{resp.} -1).$$

Possible cases of BLCs. By definition six cases may occur, namely:
1. A Euclidian BLC between Euclidian surfaces ($s = 0$, $\epsilon = 1$, $r = \bar{r} = 0$).
2. A space-like BLC between space-like surfaces ($s = 1$, $\epsilon = 1$, $r = \bar{r} = 0$).
3. A space-like BLC between time-like surfaces ($s = 1$, $\epsilon = 1$, $r = \bar{r} = 1$).
4. A time-like BLC between time-like surfaces ($s = 1$, $\epsilon = -1$, $r = \bar{r} = 1$).
5. A space-like BLC where M is a space-like surface and $\bar{M}$ is a time-like surface ($s = 1$, $\epsilon = 1$, $r = 0$, $\bar{r} = 1$).
6. A space-like BLC where M is a time-like surface and $\bar{M}$ is a space-like surface ($s = 1$, $\epsilon = 1$, $r = 1$, $\bar{r} = 0$).

Remark 3.2. The first condition of Definition 3.1 implies that there are no time-like BLC when either M or $\bar{M}$ is a space-like surface, i.e. $\epsilon = -1$ implies that $s = r = \bar{r} = 1$. Moreover, the cases 5 and 6 above are equivalent since they correspond to switching the surfaces M and $\bar{M}$. However, we will keep both cases due to the analytic interpretation that will be given in Section 6.

Let us introduce some notation. Consider $\xi \in \{1, -1\}$ and denote

$$C_\xi(\phi) = \begin{cases} \cos\phi & \text{if } \xi = 1, \\ \cosh\phi & \text{if } \xi = -1, \end{cases} \quad \text{and} \quad S_\xi(\phi) = \begin{cases} \sin\phi & \text{if } \xi = 1, \\ \sinh\phi & \text{if } \xi = -1, \end{cases} \tag{3.1}$$

where $\phi \in (0, \pi)$ if $\xi = 1$ or $\phi \in [0, +\infty)$ if $\xi = -1$. Then

$$C_\xi^2(\phi) + \xi S_\xi^2(\phi) = 1.$$

The next lemma allows one to determine an *angle* between a pair of independent vectors in $\mathbb{R}^3_s$.

Lemma 3.3. *Let $N, \bar{N} \in \mathbb{R}^3_s$, $s \in \{0,1\}$, be two linearly independent and unitary vectors in $\mathbb{R}^3_s$, let $\epsilon = \pm 1$ be the sign of the vector normal to the plane spanned by N and $\bar{N}$. Suppose that $\langle N, \bar{N}\rangle \geq 0$. Then there exists a unique angle ϕ, such that*

$$\langle N, \bar{N}\rangle = \begin{cases} C_{\epsilon(-1)^s}(\phi), & \text{if } \langle N, N\rangle = \langle \bar{N}, \bar{N}\rangle, \\ S_{-1}(\phi), & \text{if } \langle N, N\rangle \neq \langle \bar{N}, \bar{N}\rangle. \end{cases} \tag{3.2}$$

Proof. We consider three cases, depending on the normal vectors.
i) Suppose that N and $\bar{N}$ span a Riemannian plane then, $\epsilon(-1)^s = 1$ for all $s \in \{0,1\}$. Moreover, the *Cauchy-Schwarz inequality*, $|\langle N, \bar{N}\rangle| \leq |N||\bar{N}|$, implies that there exists $\phi \in (0, \pi)$ such that $\langle N, \bar{N}\rangle = \cos\phi$.
ii) Suppose that N and $\bar{N}$ span a semi-Riemannian plane and $\langle N, N\rangle = \langle \bar{N}, \bar{N}\rangle$, then $s = 1$, $\epsilon = 1$ and the *backwards Cauchy-Schwarz inequality* [14] holds, i.e., $|\langle N, \bar{N}\rangle| \geq |N||\bar{N}|$. Thus, there exists an angle $\phi \in (0, +\infty)$ such that $\langle N, \bar{N}\rangle = \cosh\phi$.
iii) Suppose that N and $\bar{N}$ span a semi-Riemannian plane and $\langle N, N\rangle \neq \langle \bar{N}, \bar{N}\rangle$, then $s = 1$, $\epsilon = 1$ and $\langle N, \bar{N}\rangle$ can be any real number, hence there exists $\phi \in [0, +\infty)$ such that $\langle N, \bar{N}\rangle = \sinh\phi$.

Hence, in each case the assertion of equation (3.2) holds. □

In a BLC between the surfaces M and $\bar{M}$, the linearly independent normal vectors (N and $\bar{N}$) span a plane, whose normal direction in $\mathbb{R}^3_s$ is given by $v(p)$. On the other hand, up to orientation of the surfaces we can always have $\langle N, \bar{N}\rangle \geq 0$. Hence, Lemma 3.3 allows one to consider an angle, as in (3.2), between the normal vectors in a BLC, which depends on the type of congruence, the index of the surfaces and the index of the ambient space $\mathbb{R}^3_s$.

Now we consider a Bäcklund type Theorem for surfaces in $\mathbb{R}^3_s$ which gives a necessary condition, on the Gaussian curvature, for the existence of a BLC in $\mathbb{R}^3_s$ between two surfaces in $\mathbb{R}^3_s$. We state and present a unified proof that deals with all possible cases of BLCs in $\mathbb{R}^3_s$.

Theorem 3.4 (Bäcklund type Theorem)**.** *Let $M^2_r, \bar{M}^2_{\bar{r}} \hookrightarrow \mathbb{R}^3_s$, with $0 \leq r, \bar{r} \leq s \leq 1$, be pseudo-Riemannian surfaces of $\mathbb{R}^3_s$. Suppose that M*

and $\bar{M}$ are related by a Bäcklund type line congruence in $\mathbb{R}^3_s$. Then both surfaces have the same constant Gaussian curvature, $K = \bar{K}$, given by

$$K = \begin{cases} (-1)^{s+1} \frac{S^2_{\epsilon(-1)^s}(\phi)}{\lambda^2}, & \text{if } r = \bar{r}, \\ -\frac{C^2_{-1}(\phi)}{\lambda^2}, & \text{if } r \neq \bar{r}, \end{cases} \tag{3.3}$$

where λ is the distance of the line congruence, ϕ is the angle between the normals and $\epsilon = 1$ if the line congruence is space-like or $\epsilon = -1$ if the congruence is time-like. In case $\epsilon = -1$, then $s = r = \bar{r} = 1$.

Proof. Suppose that M and $\bar{M}$ are related by a BLC. Let $\{e_1, e_2, e_3\}$ be a local moving frame adapted to M such that, $e_3 = N$ is normal to M, and for each $p \in M$, $e_1(p) = \frac{v(p)}{\lambda}$, where $v(p) \in \mathbb{R}^3_s$ is the vector from p to $\bar{p}$. Let ϵ_i denote the sign of e_i, hence

$$\epsilon_1 = \epsilon.$$

Note that as observed in Remark 3.2, when $\epsilon = -1$, then $s = r = \bar{r} = 1$. We shall consider a local moving frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$, adapted to $\bar{M}$, such that $\bar{e}_1 = -e_1$ and $\bar{e}_3 = \bar{N}$ is the normal to $\bar{M}$. We denote by $\bar{\epsilon}_i$ the sign of $\bar{e}_i$. In order to express $\{\bar{e}_i\}$ in terms of the orthonormal basis $\{e_i\}$, we must find functions μ_i, $i = 1, 2, 3$, such that $\bar{e}_3 = \sum_{j=1}^{3} \mu_j e_j$. It follows from the third condition of Definition 3.1 that we must have $\mu_3 = \epsilon_3 \Lambda$, where $\Lambda = \langle N, \bar{N} \rangle$. The orthogonal condition $\langle \bar{e}_3, \bar{e}_1 \rangle = 0$ implies that μ_1=0. Since $(-1)^s = \epsilon\epsilon_2\epsilon_3$, the relation $\bar{\epsilon}_3 = \langle \bar{e}_3, \bar{e}_3 \rangle$ implies that

$$\mu_2^2 = \epsilon(-1)^s \left(\epsilon_3 \bar{\epsilon}_3 - \Lambda^2 \right). \tag{3.4}$$

Observe that the right hand side of (3.4) is positive due to (3.1) and (3.2). Finally, we choose $\bar{e}_2$ to be orthornomal to $\bar{e}_1$ and $\bar{e}_3$ in $\mathbb{R}^3_s$. So we have,

$$\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \epsilon(-1)^s \Lambda & -\epsilon_3 \mu_2 \\ 0 & \mu_2 & \epsilon_3 \Lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{3.5}$$

where μ_2 satisfies (3.4).

Let X be a local parametrization of $M^2_r \subset \mathbb{R}^3_s$. Then

$$\bar{X} = X + \lambda e_1$$

is a local parametrization of $\bar{M}^2_{\bar{r}} \subset \mathbb{R}^3$. Taking the exterior differentiation we have

$$d\bar{X} = dX + \lambda de_1.$$

Considering the dual frame, and using equations (2.1) and (2.2), this expression may be writen as

$$\bar{\omega}_1\bar{e}_1 + \bar{\omega}_2\bar{e}_2 = \omega_1 e_1 + \omega_2 e_2 + \lambda\left(\epsilon_2\omega_{12}e_2 + \epsilon_3\omega_{13}e_3\right).$$

Using the relation (3.5), this is equivalent to the system of one-forms,

$$\begin{cases} \omega_1 = -\bar{\omega}_1, \\ \omega_2 + \epsilon_2\lambda\omega_{12} = \epsilon(-1)^s\Lambda\bar{\omega}_2, \\ \lambda\omega_{13} = -\mu_2\bar{\omega}_2. \end{cases} \tag{3.6}$$

Solving the last two equations for $\bar{\omega}_2$ we have,

$$-\epsilon_2\lambda^{-1}\mu_2\omega_2 = \epsilon_3\Lambda\omega_{13} + \mu_2\omega_{12}. \tag{3.7}$$

Now we are going to express the connections $\bar{\omega}_{ij}$ in terms of the coframe ω_i. Using (2.2), we have $\bar{\omega}_{13} = \langle d\bar{e}_1, \bar{e}_3\rangle$ and by using equations (3.5) and (3.7), we can check that

$$\bar{\omega}_{13} = \epsilon_2\lambda^{-1}\mu_2\omega_2,$$

holds. In a similar procedure, by using (3.4), we have

$$\bar{\omega}_{23} = \epsilon\bar{\epsilon}_3(-1)^s\omega_{23}.$$

Finally, using these two relations and the structure equations (2.6) on the right-hand side of the expression $d\bar{\omega}_{12} = \bar{\epsilon}_3\bar{\omega}_{13}\wedge\bar{\omega}_{32}$ we have,

$$d\bar{\omega}_{12} = \epsilon_3\lambda^{-1}\mu_2\omega_1\wedge\omega_{13}.$$

Thus, by (3.6)

$$d\bar{\omega}_{12} = \epsilon_3\lambda^{-2}\mu_2^2\bar{\omega}_1\wedge\bar{\omega}_2.$$

Comparing this equation to the Gauss Equation (2.7) and using the fact that $\bar{\epsilon}_1\bar{\epsilon}_2\bar{\epsilon}_3 = (-1)^s$, we have that $\bar{K} = -\epsilon_3\bar{\epsilon}_3(-1)^s\lambda^{-2}\mu_2^2$. Furthermore, considering (3.4) and the identities $\epsilon_3\bar{\epsilon}_3 = (-1)^{r+\bar{r}}$ we conclude that,

$$\bar{K} = \epsilon\frac{(-1)^{r+\bar{r}}\Lambda^2 - 1}{\lambda^2}.$$

By the symmetry of this expression, with respect to r and $\bar{r}$, we conclude that $K = \bar{K}$. Finally it follows from Lemma 3.3 that we may set

$$\Lambda = \langle N, \bar{N} \rangle = \begin{cases} C_{\epsilon(-1)^s}(\phi), & \text{if } r = \bar{r}, \\ S_{\epsilon(-1)^s}(\phi), & \text{if } r \neq \bar{r}, \end{cases}$$

and using the identity $C_\xi^2(\phi) + \xi S_\xi^2(\phi) = 1$ for $\xi = \pm 1$, we may write the curvature as

$$K = \begin{cases} (-1)^{s+1} \dfrac{S^2_{\epsilon(-1)^s}(\phi)}{\lambda^2}, & \text{if } r = \bar{r}, \\ -\dfrac{C^2_{-1}(\phi)}{\lambda^2}, & \text{if } r \neq \bar{r}. \end{cases}$$

This completes the proof. □

We summarize the Bäcklund type Theorem for all possible types of BLC in Table 3.1.

Bäcklund type Theorem in $\mathbb{R}^3_s$					
	$\mathbb{R}^3_s$	BLC	1st surface	2nd surface	Gaussian curvature
1	$s=0$	Euclidian $\epsilon = 1$	Euclidian $r=0$	Euclidian $\bar{r}=0$	$K = \bar{K} = -\frac{\sin^2\phi}{\lambda^2}$, $\phi \in (0,\pi)$
2	$s=1$	space-like $\epsilon = 1$	space-like $r=0$	space-like $\bar{r}=0$	$K = \bar{K} = \frac{\sinh^2\phi}{\lambda^2}$, $\phi \in (0,+\infty)$
3	$s=1$	space-like $\epsilon = 1$	time-like $r=1$	time-like $\bar{r}=1$	$K = \bar{K} = \frac{\sinh^2\phi}{\lambda^2}$, $\phi \in (0,+\infty)$
4	$s=1$	time-like $\epsilon = -1$	time-like $r=1$	time-like $\bar{r}=1$	$K = \bar{K} = \frac{\sin^2\phi}{\lambda^2}$, $\phi \in (0,\pi)$
5	$s=1$	space-like $\epsilon = 1$	space-like $r=0$	time-like $\bar{r}=1$	$K = \bar{K} = -\frac{\cosh^2\phi}{\lambda^2}$, $\phi \in [0,+\infty)$
6	$s=1$	space-like $\epsilon = 1$	time-like $r=1$	space-like $\bar{r}=0$	$K = \bar{K} = -\frac{\cosh^2\phi}{\lambda^2}$, $\phi \in [0,+\infty)$

Table 3.1: Parameters and curvature for each kind of BLC.

4 Integrability Theorem

In the previous section, we obtained the Bäcklund type Theorem (Theorem 3.4) which gives a necessary condition on the Gaussian curvature for the existence of a BLC between two surfaces in $\mathbb{R}^3_s$. In this section, we prove the so called Integrability Theorem, which shows that by starting with a given surface M_r in $\mathbb{R}^3_s$ of nonzero constant Gaussian curvature, one can construct new surfaces $\bar{M}_{\bar{r}}$ in $\mathbb{R}^3_s$, which are locally related to M by a BLC in $\mathbb{R}^3_s$.

From now on, without loss of generality, we will consider $M^2_r \subset \mathbb{R}^3_s$ to be a surface with normalized constant Gaussian curvature $K = \delta \in \{-1, 1\}$. Therefore, the parameters ϕ and λ are related by the expression (3.3), with $K = \delta$.

Theorem 4.1 (Integrability Theorem)**.** *Let $M^2_r(\delta) \hookrightarrow \mathbb{R}^3_s$ be a surface with constant Gaussian curvature $K = \delta = \pm 1$, where $r, s \in \{0, 1\}$ and*

$$\begin{cases} 0 \le r \le s = 1 & \text{if} \quad \delta = 1, \\ 0 \le r \le s \le 1 & \text{if} \quad \delta = -1. \end{cases} \tag{4.1}$$

Let $p_0 \in M^2_r$ be a non-umbilic point. Consider a unit tangent vector $v_0 \in T_{p_0}M^2_r$ which is not a principal direction, with $\langle v_0, v_0 \rangle = \epsilon$, such that

$$\epsilon = \begin{cases} \pm 1 & \text{if} \quad \delta = 1, \\ 1 & \text{if} \quad \delta = -1, \end{cases} \tag{4.2}$$

and a constant ϕ such that

$$\phi \in \begin{cases} (0, \pi) & \text{if} \quad \epsilon(-1)^s = 1, \\ [0, +\infty) & \text{if} \quad \epsilon(-1)^s = -1 \quad \delta = -1, \\ (0, +\infty) & \text{if} \quad \epsilon(-1)^s = -1 \quad \delta = 1. \end{cases} \tag{4.3}$$

Then, there exists a surface $\bar{M}^2_{\bar{r}} \subset \mathbb{R}^3_s$, with index $\bar{r}$ satisfying

$$(-1)^{\bar{r}} = \delta(-1)^{s+r+1}, \tag{4.4}$$

which is related by a BLC to some open neighborhood of $p_0 \in M^2_r$, such that the line of the congruence at p_0 is the direction of v_0 and the distance of the congruence is given by

$$\lambda = \begin{cases} S_{-\epsilon}(\phi) & \text{if } \delta = 1, \\ S_1(\phi) & \text{if } \delta = -1 \quad \text{and } s = 0, \\ C_{-1}(\phi) & \text{if } \delta = -1 \quad \text{and } s = 1. \end{cases} \tag{4.5}$$

Moreover, ϕ is the angle between the normals at corresponding points and the inner product between the normals is

$$\Lambda = \begin{cases} C_{-\epsilon}(\phi) & \text{if } \delta = 1, \\ C_1(\phi) & \text{if } \delta = -1 \quad \text{and } s = 0, \\ S_{-1}(\phi) & \text{if } \delta = -1 \quad \text{and } s = 1. \end{cases} \tag{4.6}$$

In particular, λ and Λ are related by the relation

$$\delta \left[(-1)^{s+1} \Lambda^2 - \lambda^2 \epsilon \right] = 1. \tag{4.7}$$

Proof. Fix δ, r, s, ϵ and ϕ satisfying (4.1) - (4.3). Define λ and Λ by the relations (4.5) and (4.6) respectively. Notice that $\lambda \neq 0$ and the trigonometric identity $C_\xi^2(\phi) + \xi S_\xi^2(\phi) = 1$, with $\xi = \pm 1$, implies that λ and Λ are related as follows: $-\epsilon\lambda^2 + \Lambda^2 = 1$ if $\delta = 1$ and $\lambda^2 + (-1)^s \Lambda^2 = 1$ if $\delta = -1$. When $\delta = 1$ we have $s = 1$, hence we can rewrite these relations as in (4.7).

We look for an orthonormal moving frame $\{e_1, e_2, e_3\}$ adapted to M_r^2, with $\epsilon_i = \langle e_i, e_i \rangle$ such that

$$\epsilon_1 = \epsilon, \ \epsilon_2 = \epsilon_1 (-1)^r \text{ and } \epsilon_3 = \begin{cases} (-1)^{r+1}, & \text{if } \delta = 1, \\ (-1)^{r+s}, & \text{if } \delta = -1, \end{cases} \quad \text{i.e. } \epsilon_3 = (-1)^{r+s}, \tag{4.8}$$

such that $e_1(p_0) = v_0$ and the relation

$$\omega_{12} = -\lambda^{-1} (\epsilon_2 \omega_2 + \epsilon_3 \Lambda \omega_{13}) \tag{4.9}$$

holds, where λ and Λ are related by (4.7).

Let $\mathcal{J}$ denote the ideal generated by the 1-form

$$\gamma = \omega_{12} + \lambda^{-1} (\epsilon_2 \omega_2 + \epsilon_3 \Lambda \omega_{13}). \tag{4.10}$$

Observe that $\mathcal{J}$ is closed under exterior differentiation. In fact, taking the exterior differentiation and using the structure equations (2.7), (2.4) and (2.8) (where $K = \delta$ and ϵ_i is given by (4.8)) we have

$$d\gamma = -\delta \epsilon_1 \epsilon_2 \omega_1 \wedge \omega_2 + \omega_{12} \wedge \left(-\lambda^{-1} \omega_1 + \epsilon_2 \epsilon_3 \lambda^{-1} \Lambda \omega_{23} \right).$$

Hence, using (4.10) to substitute ω_{12}, it follows from the structure equations (2.6) and (2.7) that

$$d\gamma \equiv -\frac{\epsilon_2}{\lambda^2}\left[\delta\epsilon\lambda^2 + 1 + (-1)^s\delta\Lambda^2\right]\omega_1 \wedge \omega_2, \qquad (\mathrm{mod}\mathcal{J}).$$

The relation for λ and Λ given by (4.7) implies that $d\gamma \equiv 0(\mathrm{mod}\mathcal{J})$. It follows from Frobenius Theorem (see for example [13]) that, on a neighborhood W of p_0, there exists a moving frame $\{e_1, e_2, e_3\}$ with initial condition $e_1(p_0) = v_0$ such that (4.9) is satisfied. We may consider a subset of W, if necessary, so that e_1 is not a principal direction at each point of W. Observe that the vector fields e_2 and e_3 are determined by e_1 and $\gamma = 0$ is a differential equation for the vector field e_1, which can be written as

$$\langle de_1, e_2 + \lambda^{-1}\epsilon_3\Lambda e_3\rangle + \lambda^{-1}\epsilon_2\omega_2 = 0,$$

where ω_2 is dual to e_2. This justifies the existence of the moving frame.

Now, consider $\bar{X}$ on W given by

$$\bar{X} = X + \lambda e_1, \tag{4.11}$$

where $X : M \to \mathbb{R}^3_s$ is a parametrization of M. Note that $\bar{X}$ is regular. In fact, taking the exterior differentiation of (4.11) and using equations (2.1), (2.2), (4.9) and (4.8) we have

$$d\bar{X} = \omega_1 e_1 + \omega_{13}\left((-1)^{s+1}\epsilon\Lambda e_2 + (-1)^{r+s}\lambda e_3\right),$$

where we used the fact that $\epsilon\epsilon_2\epsilon_3 = (-1)^s$. Since e_1 is not a principal direction of M^2_r, ω_1 and ω_{13} are linearly independent 1-forms, thus $\bar{M}^2_{\bar{r}} := \bar{X}(V)$ is regular. In order to determine the index $\bar{r}$, we choose a moving frame adapted to $\bar{M}^2_{\bar{r}}$ as follows:

$$\begin{aligned} \bar{e}_1 &= -e_1, \\ \bar{e}_2 &= (-1)^{s+1}\epsilon\Lambda e_2 + \lambda(-1)^{r+s}e_3 \\ \bar{e}_3 &= \lambda e_2 + (-1)^{r+s}\Lambda e_3. \end{aligned} \tag{4.12}$$

Hence the index $\bar{r}$ of $\bar{M}^2_{\bar{r}}$ may be computed by the relation $(-1)^{\bar{r}} = \bar{\epsilon}_1\bar{\epsilon}_2 = \epsilon\bar{\epsilon}_2$. Equations (4.12) and (4.8) imply that $\epsilon\bar{\epsilon}_2 = \Lambda^2\epsilon\epsilon_2 + \epsilon\lambda^2(-1)^{r+s}$ and $\epsilon\epsilon_2 = (-1)^r$. Therefore,

$$(-1)^{\bar{r}} = (-1)^r\left[\Lambda^2 + \epsilon\lambda^2(-1)^s\right].$$

Thus, from (4.7) we conclude that $\bar{r}$ satisfies (4.4). Moreover, (4.11) implies that

$$\sqrt{|\langle X-\bar{X}, X-\bar{X}\rangle|} = \lambda,$$

and (4.12) with (4.8) imply that $\langle N, \bar{N}\rangle = \Lambda$, where $N = e_3$ and $\bar{N} = \bar{e}_3$ are normal vector fields to the surfaces M_r^2 and $\bar{M}_{\bar{r}}^2$ respectively. Thus, $W \subset M_r^2$ and $\bar{M}_{\bar{r}}^2$ are related by a BLC in $\mathbb{R}_s^3$ with constants λ, Λ and the line of the congruence is in the direction of e_1. This concludes the proof. □

Remark 4.2. The proof of the Integrability Theorem shows that given a surface $M_r^2(\delta) \subset \mathbb{R}_s^3$, of constant Gaussian curvature $K = \delta = \pm 1$, where r, s and δ satisfy (4.1), there exists a two-parameter family of surfaces $\bar{M}$ which is related locally to M by BLC in $\mathbb{R}_s^3$. The two parameters are ϕ (or equivalently the constants λ and Λ related by (4.7)) and the one that corresponds to choosing the unit tangent vector v_0, where $\langle v_0, v_0\rangle = \epsilon = \pm 1$. Then, Theorem 3.4 implies that each surface $\bar{M}$ has Gaussian curvature $\bar{K} = \delta$. Moreover, the proof shows that $\bar{M}$ is obtained by integrating (4.9). This procedure is called a *Bäcklund Transformation.* Observe that when $\delta = 1$ and $r = 1$ then one can choose the unit vector v_0 to be spacelike or timelike i.e., $\epsilon = \pm 1$. Otherwise, v_0 is spacelike i.e., $\epsilon = 1$ (see Table 3.1).

5 Associated partial differential equations

In this section, by considering an appropriate local coordinate system on surfaces which admit two real distinct principal curvatures, we show that locally a space-like or time-like surface in $\mathbb{R}_s^3$, with non zero constant Gaussian curvature corresponds to a solution of a partial differential equation.

Theorem 5.1. *Let $M_r^2(\delta) \subset \mathbb{R}_s^3$ be a surface with constant Gaussian curvature $\delta = \pm 1$, $0 \le r \le s \le 1$, such that (4.1) holds. Suppose that M_r^2 has two distinct real principal curvatures. Then, there exists a local parametrization $X(x_1, x_2)$ of M_r^2 such that for $i \in \{1, 2\}$, $\left\langle \frac{X_{x_i}}{|X_{x_i}|}, \frac{X_{x_i}}{|X_{x_i}|}\right\rangle =$*

$(-1)^{(i-1)r}\epsilon$, $\epsilon=\pm 1$ and a differentiable function $\alpha(x_1,x_2)$ that is a solution of the differential equation

$$\epsilon[-\delta\alpha_{x_1x_1}+(-1)^{s+1}\alpha_{x_2x_2}]=S_l(\alpha), \quad \textit{where} \quad l=(-1)^{r+s+1}\delta. \tag{5.1}$$

Moreover, in these coordinates, the first and second fundamental forms of M_r^2 are given, up to orientation $\tau=\pm 1$, respectively, by

$$\begin{aligned} I &= \epsilon C_l^2(\frac{\alpha}{2})dx_1^2+(-1)^r\epsilon S_l^2(\frac{\alpha}{2})dx_2^2,\\ II &= \tau S_l(\frac{\alpha}{2})C_l(\frac{\alpha}{2})\ \left[\epsilon dx_1^2-(-1)^r\epsilon l dx_2^2\right]. \end{aligned} \tag{5.2}$$

Conversely, for fixed integers $\delta=\pm 1$, $r,s\in\{0,1\}$, $\epsilon=\pm 1$ such that (4.1) - (4.3) are satisfied, consider $l=(-1)^{r+s+1}\delta$ and let $\alpha(x_1,x_2)$ be a non zero solution of (5.1). Then, there exists a surface $M_r^2\subset\mathbb{R}_s^3$ with Gaussian curvature $K=\delta$, whose fundamental forms are given by (5.2).

Proof. Let $X(u,v)$, $(u,v)\in W\subset\mathbb{R}^2$ be a local parametrization of M_r^2 such that the coordinate curves are lines of curvature. Consider a moving frame $\{e_1,e_2,e_3\}$ such that e_1 and e_2 are principal directions and denote by k_1 and k_2 the distinct principal curvatures.

In order to obtain the dual frame and the connection forms in the basis $\{du,dv\}$, one can use equations (2.1), (2.2) and the fundamental form, $I=\langle dX,dX\rangle$ and $II=\langle -de_3,dX\rangle$, respectively. These 1-forms can be expressed as follows

$$\begin{aligned} &\omega_1=Adu, &&\omega_2=Bdv,\\ &\omega_{31}=-\epsilon k_1Adu, &&\omega_{32}=-(-1)^r\epsilon k_2Bdv, \end{aligned} \tag{5.3}$$

where $A=\epsilon\langle X_u,e_1\rangle$ and $B=(-1)^r\epsilon\langle X_v,e_2\rangle$. Once we know the dual frame, then the connection form ω_{12} is uniquely determined by equations (2.3), (2.4) and (2.5), and it is given by

$$\omega_{12}=-\epsilon\frac{A_v}{B}du+(-1)^r\epsilon\frac{B_u}{A}dv. \tag{5.4}$$

We notice that the structure equations (2.8) and (2.9) expressed in the basis $\{du,dv\}$ are equivalent, respectively, to the following equations.

$$\begin{aligned} \frac{(k_1)_v}{k_1-k_2}+\frac{A_v}{A}=0,\\ \frac{(k_2)_u}{k_2-k_1}+\frac{B_u}{B}=0. \end{aligned}$$

Let α be a function such that $k_1 = \tau S_l(\frac{\alpha}{2})/C_l(\frac{\alpha}{2})$, where $l = -\epsilon_3\delta$, $\epsilon_3 = \langle e_3, e_3\rangle$ and $\tau = \text{sign } k_1$. Since $\epsilon_3 = (-1)^{r+s}$, it follows that $l = (-1)^{r+s+1}\delta$. Then Gauss Equation (2.7) implies that $k_1k_2 = (-1)^{r+s}\delta$ and hence $k_2 = -l\tau C_l(\frac{\alpha}{2})/S_l(\frac{\alpha}{2})$. Therefore, this system is equivalent to

$$\frac{\partial}{\partial v}\ln A - \frac{\partial}{\partial v}\ln C_l(\frac{\alpha}{2}) = 0,$$
$$\frac{\partial}{\partial u}\ln B - \frac{\partial}{\partial u}\ln S_l(\frac{\alpha}{2}) = 0.$$

Hence, there exist functions $U(u), V(v) > 0$ such that $A = C_l(\frac{\alpha}{2})U(u)$ and $B = S_l(\frac{\alpha}{2})V(v)$. This allows us to consider a reparametrization of X with parameters (x_1, x_2) such that $dx_1 = U(u)du$ and $dx_2 = V(v)dv$. Thus, it follows from (5.3) and (5.4), that the dual frame and the connection forms are given by

$$\begin{aligned} \omega_1 = C_l(\tfrac{\alpha}{2})dx_1, \qquad \omega_2 = S_l(\tfrac{\alpha}{2})dx_2, \\ \omega_{12} = \frac{l\epsilon}{2}\alpha_{x_2}dx_1 + \frac{(-1)^r\epsilon}{2}\alpha_{x_1}dx_2, \\ \omega_{31} = -\epsilon\tau S_l(\tfrac{\alpha}{2})dx_1, \qquad \omega_{32} = (-1)^r\epsilon\tau l C_l(\tfrac{\alpha}{2})dx_2. \end{aligned} \tag{5.5}$$

Therefore, the first and second fundamental forms of $X(x_1, x_2)$ are

$$I = \epsilon C_l^2(\frac{\alpha}{2})dx_1{}^2 + (-1)^r\epsilon S_l^2(\frac{\alpha}{2})dx_2{}^2,$$
$$II = \tau S_l(\frac{\alpha}{2})C_l(\frac{\alpha}{2})\left(\epsilon dx_1{}^2 - (-1)^r\epsilon f dx_2{}^2\right).$$

Moreover, it follows from (2.7) and (5.5) that the function α satisfies

$$-\delta(-1)^r\epsilon\alpha_{x_1x_1} + l\delta\epsilon\alpha_{x_2x_2} = (-1)^r S_l(\alpha).$$

Therefore, $\alpha(x_1, x_2)$ is a solution of (5.1).

Conversely, for fixed integers $\delta = \pm 1$, $r, s \in \{0, 1\}$, $\epsilon = \pm 1$ such that (4.1) - (4.3) are satisfied, consider $l = (-1)^{r+s+1}\delta$ and let $\alpha(x_1, x_2)$ be a non trivial solution of (5.1) defined on an open subset of $\mathbb{R}^2$. We define the 1-forms ω_1, ω_2, ω_{12}, ω_{31} and ω_{32} as in (5.5). A straightforward computation shows that these forms satisfy the structure equations (2.3)-(2.9), with $K = \delta$, i.e., the Gauss and Codazzi equations are satisfied. Therefore, the fundamental theorem of surfaces implies that there exists a surface $M_r^2 \subset \mathbb{R}_s^3$ with Gaussian curvature $K = \delta$ and whose fundamental forms are given by (5.2).

□

In Theorem 5.1, in order to obtain the differential equation (5.1), we considered an appropriate local parametrization for surfaces which admit two real distinct principal curvatures. We observe that there exist time-like surfaces with positive constant Gaussian curvatures that are non-umbilic and do not satisfy this condition. For instance, one can have a *time-like totally quasi-umbilic* surface [10] (this is, a time-like surface whose shape operator is non-diagonalizable over $\mathbb{C}$) or, a time-like surface whose shape operator is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$ [12].

6 Bäcklund Transformations in Local Coordinates and Analytic Interpretation

In this section, we provide the analytic interpretation of the Integrability Theorem (Theorem 4.1), in terms of a local coordinate system on the surface. Then, we show how to use the Integrability Theorem to explicitly construct the two-parameter families of surfaces with the same Gaussian curvature mentioned in Remark 4.2. Finally, we provide the analytic version of the integrability theorem by considering the associated differential equations and their corresponding Bäcklund transformations.

Theorem 6.1. *Under the hypothesis of Theorem 4.1 consider a surface $M_r^2(\delta) \subset \mathbb{R}_s^3$, with curvature $K = \delta = \pm 1$, where $r, s \in \{0, 1\}$, $r \leq s$, satisfy (4.1). Let $p_0 \in M$ and $v_0 \in T_{p_0}M$ be a unit tangent vector at the point p_0, such that $\epsilon =< v_0, v_0 >$ satisfies (4.2). Assume that M_r^2 has two real distinct principal curvatures and let $X(x_1, x_2)$ be a local parametrization of M, containing $p_0 = (x_1^0, x_2^0)$, as in Theorem 5.1 and let τ be the sign of the principal curvature k_1 associated to the direction X_{x_1}. Let $\alpha(x_1, x_2)$ be a solution of the differential equation (5.1) corresponding to the surface $X(x_1, x_2)$. Then the following system for the function α' is* *integrable*

$$\left\{ \begin{array}{l} \alpha'_{x_1} - (-1)^s \delta \alpha_{x_2} = \frac{2}{\lambda} S_{(-1)^r}(\frac{\alpha'}{2}) C_l(\frac{\alpha}{2}) - \frac{2(-1)^s \tau \Lambda}{\lambda} C_{(-1)^r}(\frac{\alpha'}{2}) S_l(\frac{\alpha}{2}) \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\lambda} C_{(-1)^r}(\frac{\alpha'}{2}) S_l(\frac{\alpha}{2}) - \frac{2\delta\tau\Lambda}{\lambda} S_{(-1)^r}(\frac{\alpha'}{2}) C_l(\frac{\alpha}{2}), \end{array} \right. \tag{6.1}$$

where S_ξ, C_ξ are given by (3.1) and the constants λ and Λ satisfy (4.7). Moreover,

$$\bar{X} = X + \lambda \left(\frac{C_{(-1)^r}(\frac{\alpha'}{2})}{C_l(\frac{\alpha}{2})} X_{x_1} + \frac{S_{(-1)^r}(\frac{\alpha'}{2})}{S_l(\frac{\alpha}{2})} X_{x_2} \right), \tag{6.2}$$

is related to X by a BLC in $\mathbb{R}^3_s$.

Proof. As in Theorem 5.1, the vector fields

$$\bar{e}_1 = \frac{X_{x_1}}{C_l(\frac{\alpha}{2})}, \qquad \bar{e}_2 = \frac{X_{x_2}}{S_l(\frac{\alpha}{2})},$$

are the principal directions of M, where $l = (-1)^{r+s+1}\delta$, for $i = 1, 2$, $< \bar{e_i}, \bar{e_i} >= (-1)^{(i-1)r}\epsilon$, $\epsilon = \pm 1$ and $\alpha(x_1, x_2)$ is a solution of (5.1). The 1-forms associated to this frame are given by (5.5).

It follows from the Integrability Theorem that there exists a moving frame e_1, e_2, e_3 adapted to M, on a neighborhood of p_0, such that $e_1(x_1^0, x_2^0) = v^0$ and equation (4.9) is satisfied for the 1-forms ω_1, ω_2 and ω_{12}, where Λ and λ are related by (4.7). Hence, in order to determine the vector field e_1, we look for a function α' such that

$$\begin{aligned} e_1 &= C_{(-1)^r}(\frac{\alpha'}{2})\bar{e}_1 + S_{(-1)^r}(\frac{\alpha'}{2})\bar{e}_2, \\ e_2 &= (-1)^{r+1} S_{(-1)^r}(\frac{\alpha'}{2})\bar{e}_1 + C_{(-1)^r}(\frac{\alpha'}{2})\bar{e}_2, \end{aligned}$$

and the associated 1-forms

$$\begin{gathered} \omega_1 = C_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_1 + (-1)^r S_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_2, \\ \omega_2 = -S_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_1 + C_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_2, \\ \omega_{12} = (-1)^r \epsilon \, d(\tfrac{\alpha'}{2}) + \bar{\omega}_{12}, \\ \omega_{13} = C_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_{13} + S_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_{23}, \\ \omega_{23} = (-1)^{r+1} S_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_{13} + C_{(-1)^r}(\tfrac{\alpha'}{2})\bar{\omega}_{23}, \end{gathered}$$

satisfy the Bäcklund Transformation (4.9). Observe that from the expression of e_1, we get $\langle e_1, e_1\rangle = \epsilon$ and $\langle e_2, e_2\rangle = (-1)^r\epsilon$.

By substituting these 1-forms into (4.9) and using the linear independence of dx_1 and dx_2 we obtain that α' satisfies the system of differential equations given by (6.1). The Integrability Theorem implies that there exists α' which is a solution to this system and moreover that $\bar{X}$ given by (6.2) is related to X by a BLC in $\mathbb{R}^3_s$. □

In the proof of Theorem 6.1, one can see that the system of equations (6.1) is equivalent to (4.9), since it was obtained from this equation by considering an appropriate local parametrization $X(x_1, x_2)$ for surfaces which admit two distinct real principal curvatures. For the analytical interpretation of the Integrability Theorem for time-like surfaces with positive constant Gaussian curvature which do not admit such a parametrization we refer the reader to [15], when the shape operator of the surface is non-diagonalizable over $\mathbb{C}$ and to [12], when the shape operator is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$.

Now each surface considered in Theorem 6.1 corresponds to a solution of equation (5.1) and the system of equations (6.1) provides a transformation between the functions α and α'. Therefore we can rewrite the theorem in its purely analytic version.

Theorem 6.2. *Consider fixed constants $\delta = \pm 1$, $\epsilon = \pm 1$, $r, s \in \{0, 1\}$, $r \leq s$, satisfying (4.1) and (4.2). Let $\alpha(x_1, x_2)$ be a solution of the differential equation (5.1) i.e.*

$$\epsilon[-\delta\alpha_{x_1x_1} + (-1)^{s+1}\alpha_{x_2x_2}] = S_l(\alpha), \tag{6.3}$$

where $l = (-1)^{r+s+1}\delta$ and S_l is given by (3.1). Then the system of equations (6.1) for $\alpha'(x_1, x_2)$ is integrable, with $\tau = \pm 1$, and α' is a solution of

$$\epsilon[-\delta\alpha'_{x_1x_1} + (-1)^{s+1}\alpha'_{x_2x_2}] = S_{(-1)^r}(\alpha'). \tag{6.4}$$

Although this theorem is just the analytic version of Theorem 6.1, one can also prove it directly as we will see below. Before proving Theorem 6.2, we point out that whenever $l = (-1)^r$ (i.e. $\delta = (-1)^{s+1}$), then (6.3) and (6.4) coincide and (6.1) is a two parameter self-Bäcklund transformation, where one of the parameters is determined by λ and Λ satisfying (4.7) and the other is the initial condition of α' at a point. It provides a procedure of obtaining new solutions of (6.3) from a given one, by integrating (6.1). Otherwise, when $l \neq (-1)^r$ (i.e., $\delta \neq (-1)^{s+1}$), then Theorem 6.2 shows that (6.1) is a two-parameter Bäcklund transformation from solutions of the equation (6.3) into solutions of a different equation, namely (6.4) where $(-1)^r = -l$.

Proof. The proof of Theorem 6.2 is a straightforward computation. In fact, considering the system (6.1), we obtain the derivative of the first (resp. second) equation with respect to x_2 (resp. x_1), where we substitute the expressions of α'_{x_1} and α'_{x_2} given by the system. Assuming that $\alpha(x_1, x_2)$ is a solution of (5.1) and using the relation between λ and Λ given by (4.7), the expression of l and the identities

$$
\begin{aligned}
C_\xi^2(\beta) + \xi S_\xi^2(\beta) &= 1, \qquad & 2C_\xi(\beta)S_\xi(\beta) &= S_\xi(2\beta),\\
\frac{dC_\xi}{d\beta} &= -\xi S_\xi(\beta), \qquad & \frac{dS_\xi}{d\beta} &= C_\xi(\beta),
\end{aligned}
$$

we conclude that $\alpha'_{x_1 x_2} = \alpha'_{x_2 x_1}$ i.e. that (6.1) is integrable.

Now considering the system (6.1), we obtain the derivative of the first (resp. second) equation with respect to x_1 (resp. x_2), where we substitute α'_{x_1} and α'_{x_2}. Then using the identities above, the relation (4.7) and the expression of l, we conclude that $\alpha'(x_1, x_2)$ satisfies the equation

$$
-\delta\alpha'_{x_1 x_1} + (-1)^{s+1}\alpha'_{x_2 x_2} = \epsilon S_{(-1)^r}(\alpha').
$$

□

7 Six cases

As an immediate consequence of the results proved in Sections 5 and 6, we get the following results which are obtained by restricting Theorems 5.1 and Theorem 6.2 to each one of the cases considered in Table 3.1. Recall that without loss of generality, we have normalized the Gaussian curvature to be $K = \delta = \pm 1$.

Before stating each case, we observe that the first four cases give an analytic Bäcklund transformation which provides new solutions of the associated differential equation from a given solution of the same equation. However, cases 5 and 6 give a transformation which provides new solutions of the elliptic sine-Gordon equation (resp. elliptic sinh-Gordon equation) from a given solution of the elliptic sinh-Gordon equation (resp. elliptic sine-Gordon equation).

Case 1. Considering $s = 0$, $\epsilon = 1$, $r = 0$ and $\delta = -1$ then $l = 1$ and (4.7) implies that $\Lambda^2 + \lambda^2 = 1$ hence, $\Lambda = \cos\phi$ and $\lambda = \sin\phi$.

Theorem 7.1.

a) Non zero solutions $\alpha(x_1, x_2)$ of the sine-Gordon equation

$$\alpha_{x_1x_1} - \alpha_{x_2x_2} = \sin\alpha, \tag{7.1}$$

are in correspondence, up to rigid motions, with parametrized surfaces $X(x_1, x_2)$ in $\mathbb{R}^3$ with Gaussian curvature $K = -1$ and fundamental forms

$$I = \cos^2(\frac{\alpha}{2})dx_1^2 + \sin^2(\frac{\alpha}{2})dx_2^2, \qquad II = \tau\sin(\frac{\alpha}{2})\ \cos(\frac{\alpha}{2})(dx_1^2 - dx_2^2),$$

where $\tau = \pm 1$.

b) For any solution $\alpha(x_1, x_2)$ of (7.1), the following system of differential equations for $\alpha'(x_1, x_2)$, where $\phi \in (0, \pi)$, is integrable

$$\begin{cases} \alpha'_{x_1} + \alpha_{x_2} = \frac{2}{\sin\phi}\sin(\frac{\alpha'}{2})\cos(\frac{\alpha}{2}) - \frac{2\tau\cos\phi}{\sin\phi}\cos(\frac{\alpha'}{2})\sin(\frac{\alpha}{2}), \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\sin\phi}\cos(\frac{\alpha'}{2})\sin(\frac{\alpha}{2}) + \frac{2\tau\cos\phi}{\sin\phi}\sin(\frac{\alpha'}{2})\cos(\frac{\alpha}{2}). \end{cases} \tag{7.2}$$

Moreover, α' is also a solution of (7.1).

This is the classical Bäcklund transformation for the sine-Gordon equation, when $\tau = -1$, found for example in [17], Theorem 5.1, page 44.

Case 2. Considering $s = 1$, $\epsilon = 1$, $r = 0$ and $\delta = 1$ then $l = 1$ and (4.7) implies that $\Lambda^2 - \lambda^2 = 1$, hence $\Lambda = \cosh\phi$ and $\lambda = \sinh\phi$. Then Theorems 5.1 and 6.2 reduce to

Theorem 7.2.

a) Non zero solutions $\alpha(x_1, x_2)$ of the sine-Gordon equation

$$\alpha_{x_1x_1} - \alpha_{x_2x_2} = -\sin\alpha, \tag{7.3}$$

are in correspondence, up to rigid motions, with parametrized space-like surfaces $X(x_1, x_2)$, in $\mathbb{R}^3_1$ with Gaussian curvature $K = 1$ and fundamental forms

$$I = \cos^2(\frac{\alpha}{2})dx_1^2 + \sin^2(\frac{\alpha}{2})dx_2^2, \qquad II = \tau\sin(\frac{\alpha}{2})\ \cos(\frac{\alpha}{2})(dx_1^2 - dx_2^2),$$

where $\tau = \pm 1$.

b) For any solution $\alpha(x_1, x_2)$ of (7.3), the following system of differential equations for $\alpha'(x_1, x_2)$, $\phi \in (0, \infty)$, is integrable

$$\begin{cases} \alpha'_{x_1} + \alpha_{x_2} = \frac{2}{\sinh\phi}\sin(\frac{\alpha'}{2})\cos(\frac{\alpha}{2}) + \frac{2\tau\cosh\phi}{\sinh\phi}\cos(\frac{\alpha'}{2})\sin(\frac{\alpha}{2}), \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\sinh\phi}\cos(\frac{\alpha'}{2})\sin(\frac{\alpha}{2}) - \frac{2\tau\cosh\phi}{\sinh\phi}\sin(\frac{\alpha'}{2})\cos(\frac{\alpha}{2}). \end{cases} \tag{7.4}$$

Moreover, α' is also a solution of (7.3).

Case 3. Considering $s = 1$, $\epsilon = 1$, $r = 1$ and $\delta = 1$ then $l = -1$ and (4.7) implies that $\Lambda^2 - \lambda^2 = 1$, hence $\Lambda = \cosh\phi$ and $\lambda = \sinh\phi$. Then Theorems 5.1 and 6.2 reduce to

Theorem 7.3.

a) Non zero solutions $\alpha(x_1, x_2)$ of the sinh-Gordon equation

$$\alpha_{x_1x_1} - \alpha_{x_2x_2} = -\sinh\alpha, \tag{7.5}$$

are in correspondence, up to rigid motions, with parametrized time-like surfaces $X(x_1, x_2)$ in $\mathbb{R}^3_1$ with Gaussian curvature $K = 1$ and fundamental forms given by

$$I = \cosh^2(\frac{\alpha}{2})dx_1^2 - \sinh^2(\frac{\alpha}{2})dx_2^2, \quad II = \tau\sinh(\frac{\alpha}{2})\ \cosh(\frac{\alpha}{2})(dx_1^2 - dx_2^2),$$

where $\tau = \pm 1$.

b) *For any solution $\alpha(x_1, x_2)$ of (7.5) the following system of differential equations for $\alpha'(x_1, x_2)$, where $\phi \in (0, \infty)$, is integrable*

$$\begin{cases} \alpha'_{x_1} + \alpha_{x_2} = \frac{2}{\sinh\phi}\sinh(\frac{\alpha'}{2})\cosh(\frac{\alpha}{2}) + \frac{2\tau\cosh\phi}{\sinh\phi}\cosh(\frac{\alpha'}{2})\sinh(\frac{\alpha}{2}), \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\sinh\phi}\cosh(\frac{\alpha'}{2})\sinh(\frac{\alpha}{2}) - \frac{2\tau\cosh\phi}{\sinh\phi}\sinh(\frac{\alpha'}{2})\cosh(\frac{\alpha}{2}). \end{cases} \tag{7.6}$$

Moreover, α' is also a solution of (7.5).

Case 4. Considering $s = 1$, $\epsilon = -1$, $r = 1$ and $\delta = 1$ then $l = -1$ and (4.7) implies that $\Lambda^2 + \lambda^2 = 1$, hence $\Lambda = \cos\phi$ and $\lambda = \sin\phi$. Then Theorems 5.1 and 6.2 reduce to

Theorem 7.4.

a) *Non zero solutions $\alpha(x_1, x_2)$ of the sinh-Gordon equation*

$$\alpha_{x_1x_1} - \alpha_{x_2x_2} = \sinh\alpha, \tag{7.7}$$

are in correspondence, up to rigid motions, with parametrized time-like surfaces $X(x_1, x_2)$ in $\mathbb{R}^3_1$ with Gaussian curvature $K = 1$ and fundamental forms given by

$$I = -\cosh^2(\tfrac{\alpha}{2})dx_1^2 + \sinh^2(\tfrac{\alpha}{2})dx_2^2,$$
$$II = \tau\sinh(\tfrac{\alpha}{2})\ \cosh(\tfrac{\alpha}{2})(-dx_1^2 + dx_2^2),$$

where $\tau = \pm 1$.

b) *For any solution $\alpha(x_1, x_2)$ of (7.7)*

$$\alpha_{x_1x_1} - \alpha_{x_2x_2} = \sinh\alpha. \tag{7.8}$$

Then the following system of differential equations for $\alpha'(x_1, x_2)$, where $\phi \in (0, \pi)$, is integrable

$$\begin{cases} \alpha'_{x_1} + \alpha_{x_2} = \frac{2}{\sin\phi}\sinh(\frac{\alpha'}{2})\cosh(\frac{\alpha}{2}) + \frac{2\tau\cos\phi}{\sin\phi}\cosh(\frac{\alpha'}{2})\sinh(\frac{\alpha}{2}), \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\sin\phi}\cosh(\frac{\alpha'}{2})\sinh(\frac{\alpha}{2}) - \frac{2\tau\cosh\phi}{\sin\phi}\sinh(\frac{\alpha'}{2})\cosh(\frac{\alpha}{2}). \end{cases} \tag{7.9}$$

Moreover, α' is also a solution of (7.7).

The Bäcklund transformation above, with $\tau = -1$, was obtained in [15].

Case 5. Considering $s = 1$, $\epsilon = 1$, $r = 0$ and $\delta = -1$ then $l = -1$ and (4.7) implies that $-\Lambda^2 + \lambda^2 = 1$, hence $\Lambda = \sinh\phi$ and $\lambda = \cosh\phi$. Then Theorems 5.1 and 6.2 reduce to

Theorem 7.5.

a) Non zero solutions $\alpha(x_1, x_2)$ of the elliptic sinh-Gordon equation

$$\alpha_{x_1x_1} + \alpha_{x_2x_2} = \sinh\alpha, \tag{7.10}$$

are in correspondence, up to rigid motions, with parametrized space-like surfaces $X(x_1, x_2)$ in $\mathbb{R}^3_1$ with Gaussian curvature $K = -1$ and fundamental forms given by

$$I = \cosh^2(\frac{\alpha}{2})dx_1^2+\sinh^2(\frac{\alpha}{2})dx_2^2, \quad II = \tau\sinh(\frac{\alpha}{2})\ \cosh(\frac{\alpha}{2})(dx_1^2+dx_2^2),$$

where $\tau = \pm 1$.

b) For any solution $\alpha(x_1, x_2)$ of (7.10) the following system of differential equations for $\alpha'(x_1, x_2)$, where $\phi \in [0, \infty)$, is integrable

$$\left\{ \begin{array}{l} \alpha'_{x_1} - \alpha_{x_2} = \frac{2}{\cosh\phi}\sin(\frac{\alpha'}{2})\cosh(\frac{\alpha}{2}) + \frac{2\tau\sinh\phi}{\cosh\phi}\cos(\frac{\alpha'}{2})\sinh(\frac{\alpha}{2}), \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\cosh\phi}\cos(\frac{\alpha'}{2})\sinh(\frac{\alpha}{2}) + \frac{2\tau\sinh\phi}{\cosh\phi}\sin(\frac{\alpha'}{2})\cosh(\frac{\alpha}{2}). \end{array} \right. \tag{7.11}$$

Moreover, α' is a solution of the elliptic sine-Gordon equation

$$\alpha'_{x_1x_1} + \alpha'_{x_2x_2} = \sin\alpha'. \tag{7.12}$$

The system of equations (7.11) is a Bäcklund transformation that for a given solution α of the elliptic Sinh-Gordon equation (7.10), it provides a solution α' for the elliptic sine-Gordon equation (7.12). This transformation with $\tau = -1$ was obtained by Tian [20].

Case 6. Finally, considering $s = 1$, $\epsilon = 1$, $r = 1$ and $\delta = -1$ then $l = 1$ and (4.7) implies that $-\Lambda^2 + \lambda^2 = 1$, hence $\Lambda = \sinh\phi$ and $\lambda = \cosh\phi$. Then Theorems 5.1 and 6.2 reduce to

Theorem 7.6.

a) Non zero solutions $\alpha(x_1, x_2)$ of the elliptic sine-Gordon equation (7.12) are in correspondence, up to rigid motions, with parametrized time-like surfaces $X(x_1, x_2)$ in $\mathbb{R}^3_1$ with Gaussian curvature $K = -1$ and fundamental forms given by

$$I = \cos^2(\frac{\alpha}{2})dx_1^2 - \sin^2(\frac{\alpha}{2})dx_2^2, \qquad II = \tau \sin(\frac{\alpha}{2}) \ \cos(\frac{\alpha}{2})(dx_1^2 + dx_2^2),$$

where $\tau = \pm 1$.

b) For any solution $\alpha(x_1, x_2)$ of (7.12) the following system of differential equations for $\alpha'(x_1, x_2)$, where $\phi \in [0, \infty)$, is integrable

$$\left\{ \begin{array}{l} \alpha'_{x_1} - \alpha_{x_2} = \frac{2}{\cosh\phi}\sinh(\frac{\alpha'}{2})\cos(\frac{\alpha}{2}) + \frac{2\tau\sinh\phi}{\cosh\phi}\cosh(\frac{\alpha'}{2})\sin(\frac{\alpha}{2}), \\ \alpha'_{x_2} + \alpha_{x_1} = -\frac{2}{\cosh\phi}\cosh(\frac{\alpha'}{2})\sin(\frac{\alpha}{2}) + \frac{2\tau\sinh\phi}{\cosh\phi}\sinh(\frac{\alpha'}{2})\cos(\frac{\alpha}{2}). \end{array} \right. \tag{7.13}$$

Moreover, α' is a solution of the elliptic sinh-Gordon equation (7.10).

As an application of Theorem 6.2, we conclude this section by providing some examples of solutions of the differential equations. The Bäcklund transformation (6.1) between the differential equations (6.3) and (6.4) was obtained from a BLC between two surfaces. However, the Bäcklund transformation (6.1) holds for every initial solution α of (6.3). In particular for the solution $\alpha = 0$, which is not associated to any surface.

Example 7.7. Consider $\alpha = 0$ a trivial solution of (6.3). Then (6.1) reduces to

$$\left\{ \begin{array}{l} \alpha'_{x_1} = \frac{2}{\lambda} S_{(-1)^r}(\frac{\alpha'}{2}), \\ \alpha'_{x_2} = -\frac{2\delta\tau\Lambda}{\lambda} S_{(-1)^r}(\frac{\alpha'}{2}), \end{array} \right. \tag{7.14}$$

whose solutions are given by

$$\alpha'(x_1, x_2) = \left\{ \begin{array}{ll} 4\arctan\exp\xi(x_1, x_2), & \text{if } (-1)^r = 1, \\ 2\ln\tanh\left(-\frac{1}{2}\xi(x_1, x_2)\right), & \text{if } (-1)^r = -1, \end{array} \right. \tag{7.15}$$

where

$$\xi(x_1, x_2) = \frac{1}{\lambda}x_1 - \delta\tau\frac{\Lambda}{\lambda}x_2 + c,$$

and c is a constant that depends on a given initial condition.

Case 1. Given $c \in \mathbb{R}$ and $\phi \in (0, \pi)$, then

$$\alpha'(x_1, x_2) = 4 \arctan \exp\left(\frac{1}{\sin\phi} x_1 + \tau \frac{\cos\phi}{\sin\phi} x_2 + c\right), \qquad \tau = \pm 1,$$

is a solution of $\alpha_{x_1x_1} - \alpha_{x_2x_2} = \sin\alpha$.

Case 2. Given $c \in \mathbb{R}$ and $\phi \in (0, \infty)$, then

$$\alpha'(x_1, x_2) = 4 \arctan \exp\left(\frac{1}{\sinh\phi} x_1 - \tau \frac{\cosh\phi}{\sinh\phi} x_2 + c\right), \qquad \tau = \pm 1,$$

is a solution of $\alpha_{x_1x_1} - \alpha_{x_2x_2} = -\sin\alpha$.

Case 3. Given $c \in \mathbb{R}$ and $\phi \in (0, \infty)$, then

$$\alpha'(x_1, x_2) = 2 \ln \tanh\left(-\frac{1}{2\sinh\phi} x_1 + \tau \frac{\cosh\phi}{2\sinh\phi} x_2 + c\right), \qquad \tau = \pm 1,$$

is a solution of $\alpha_{x_1x_1} - \alpha_{x_2x_2} = -\sinh\alpha$.

Case 4. Given $c \in \mathbb{R}$ and $\phi \in (0, \pi)$, then

$$\alpha'(x_1, x_2) = 2 \ln \tanh\left(-\frac{1}{2\sin\phi} x_1 + \tau \frac{\cos\phi}{2\sin\phi} x_2 + c\right), \qquad \tau = \pm 1,$$

is a solution of $\alpha_{x_1x_1} - \alpha_{x_2x_2} = \sinh\alpha$.

Case 5. Given $c \in \mathbb{R}$ and $\phi \in [0, \infty)$, then

$$\alpha'(x_1, x_2) = 4 \arctan \exp\left(\frac{1}{\cosh\phi} x_1 + \tau \frac{\sinh\phi}{\cosh\phi} x_2 + c\right), \qquad \tau = \pm 1,$$

is a solution of $\alpha_{x_1x_1} + \alpha_{x_2x_2} = \sin\alpha$.

Case 6. Given $c \in \mathbb{R}$ and $\phi \in [0, \infty)$, then

$$\alpha'(x_1, x_2) = 2 \ln \tanh\left(-\frac{1}{2\cosh\phi} x_1 - \tau \frac{\sinh\phi}{2\cosh\phi} x_2 + c\right), \qquad \tau = \pm 1,$$

is a solution of $\alpha_{x_1x_1} + \alpha_{x_2x_2} = \sinh\alpha$.

We observe that explicit solutions of the sine-Gordon equation and the corresponding surfaces can also be found for example in [17].

8 Some generalizations

We conclude by pointing out some generalizations of the Bäcklund type Line Congruence. Buyske (1994), in [7], dropped the tangential condition of Line Congruence and obtained a Bäcklund type of transformation between Linear Weingarten surfaces. However, this result was originally obtained by Bianchi see [6], who considered the lines of the congruence to have a constant angle (not necessarily orthogonal) with the normal vector field of each surface. He also obtained similar results for surfaces in 3-dimensional space forms. Zuo, Chen and Cheng considered line congruence in the de Sitter and anti-de Sitter Spaces [21].

Tenenblat and Terng (1980) generalized the line congruence for n-dimensional sub-manifolds M^n of Euclidian space $\mathbb{R}^{2n-1}$ in [18], obtaining a higher dimensional version of the classical Bäcklund theorem. The dimension $2n-1$ is due to results of Cartan [8], which show that $n-1$ is the smallest codimension for such an immersion to exist, when the sectional curvature of M is $K = -1$. Such immersions are associated to a system of differential equations for an orthogonal matrix valued function with n independent variables, which is called the *Generalized sine-Gordon equation* (GSGE), whose Bäcklund transformation was obtained as the analytic interpretation of the geometric theory.

The notion of a geodesic congruence in a space form was considered by Tenenblat (1985) in [19], for manifolds M^n with constant sectional curvature K immersed in a space form $\bar{M}^{2n-1}$ of constant sectional curvature $\bar{K}$, with $K < \bar{K}$. Such submanifolds are associated to a system of differential equations for an $O(n)$ valued matrix function which is the GSGE when $K \neq 0$ and it is called the *Generalized wave equation* (GWE), when $K = 0$. The analytic interpretation of the Bäcklund theorem provides a Bäcklund transformation for the solutions of the GWE. We observe that for $n > 2$, the GWE is a system of nonlinear differential equations and it reduces to the wave equation when $n = 2$. The Bäcklund transformation provides a method of obtaining, from a given solution of the GSGE (resp. GWE), new solutions of the same equation.

In [9], Campos-Tenenblat (1994) considered pseudo-spherical geodesic congruences for n-dimensional Riemannian manifolds M^n of constant sectional curvature K, isometrically immersed in a $(2n-1)$-dimensional pseudo-Riemannian manifold $\bar{M}_s^{2n-1}$ of constant sectional curvature $\bar{K}$, $K \neq \bar{K}$. Assuming the normal bundle is flat and the principal normal curvatures are different from $K - \bar{K}$, in [4] Barbosa-Ferreira-Tenenblat showed that such immersions are associated to a system of partial differential equations satisfied by an $O(n-q, q)$ matrix valued function. This system was initially called the Generalized equation and later *the Generating equation* in [17] (see Chapter II). This denomination is due to the fact that by choosing the constants K, s and q, $0 \leq q \leq n-1$, it generates distinct systems of differential equations. In particular, when $q = s = 0$ and $K < \bar{K}$ it reduces to the GSGE when $K \neq 0$ and to the GWE when $K = 0$. Similarly, when $q = n-1$, $s = 0$ and $K > \bar{K}$ the generating equation reduces to the generalized elliptic sinh-Gordon equation and to the generalized Laplace equation when $K \neq 0$ and $K = 0$ respectively. The Bäcklund transformation for the Generating equation was obtained by Campos-Tenenblat in [9] (see also Chapter IV in [17]). The generalized elliptic sinh-Gordon and Laplace equations were also investigated by Dajczer-Tojeiro in [11] for submanifolds M^n of the Riemannian space forms $\bar{M}$, with $\bar{K} > K$.

One should mention that in 1986, the linearization of the Bäcklund transformation of the GSGE and the GWE was the key factor used by Ablowitz-Beals-Tenenblat in [1], in order to apply the inverse scattering method to solve an initial boundary-value problem for both equations (see also Chapter III in [17]).

If we restrict ourselves to the differential equations that are satisfied by the metric of the manifold M (i.e. the Gauss equation), one has a system of equations for a unit vector valued function v defined on an open subset of $\mathbb{R}^n$, which is called the *Generating Intrinsic equation* introduced in 1998 (see chapter V in [17]). The generic solutions of this system (i.e. solutions v whose coordinate functions do not vanish) are in correspondence with metrics of constant sectional curvature, on subsets of $\mathbb{R}^n$. Moreover, solutions of the Generating Intrinsic equation are in correspondence with the solutions of the Generating equation. In fact, any manifold M^n of constant sectional curvature K, contained in a space form $\bar{M}_s^{2n-1}$, with curvature $\bar{K}$, $K \neq \bar{K}$ provides a solution of the Generating equation and hence of the Generating Intrinsic equation which is satisfied by its metric. Conversely, given a generic solution of the the Intrinsic Generating equation, there exists a unique submanifold associated to the given solution, which is determined by its metric up to rigid motion of $\bar{M}$. Therefore, any such manifold is determined by its metric satisfying the Gauss equation, which is the Generating Intrinsic equation.

We observe that the intrinsic generalization for the wave and the sine-Gordon equations was originally introduced by Beals-Tenenblat in 1991 [5], where the authors obtained transformations which provide new solutions for these equations from a given one. Moreover, they showed that solutions for these equations can be obtained by applying the inverse scattering method.

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