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A survey on critical metrics of the volume functional

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Dedicated to Professor Renato Tribuzy on the occasion of his 75th birthday

Abstract. In this article, we present some recent results on the geometry of critical metrics of the volume functional on an n-dimensional compact manifold with (possibly disconnected) boundary. We present some rigidity results and geometric inequalities obtained in the last years in order to classify the classical examples.

Keywords: Volume functional; critical metrics; compact manifolds with boundary; prescribed metrics.

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1 Introduction

A promising way to find canonical metrics on a given manifold is to investigate critical metrics which arise as solutions of the Euler-Lagrange equations for curvature functionals. An useful tool in this direction is to analyze the critical points of the total scalar curvature functional. More

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precisely, let M^n be a compact oriented smooth manifold and $\mathcal{M}$ the set of smooth Riemannian structures on M^n of unitary volume. Given a metric $g \in \mathcal{M}$, we define the total scalar curvature, or Einstein-Hilbert functional, $\mathcal{S} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{S}(g) = \int_M R_g dV_g, \tag{1.1}$$

where R_g and dV_g stand, respectively, for the scalar curvature of M^n and the volume form of $(M^n,\, g)$. Einstein and Hilbert showed that the critical points of the functional $\mathcal{S}$ are Einstein, for more details see Theorem 4.21 in [16]. The Einstein-Hilbert functional restricted to a given conformal class is just the Yamabe functional, whose critical points are constant scalar curvature metrics in that class.

A result obtained by combining results due to Aubin [2], Schoen [45], Trudinger [47] and Yamabe [48] gives the existence of a constant scalar curvature metric in every conformal class of Riemannian metrics on a compact manifold M^n. Therefore, it is interesting to consider the set $\mathcal{C} = \{g \in \mathcal{M};\, R_g \text{ is constant}\}$. In [33], Koiso showed that, under generic condition, $\mathcal{C}$ is an infinite dimensional manifold (cf. Theorem 4.44 in [16] p. 127). It is well-known that the linearization $\mathfrak{L}_g$ of the scalar curvature operator at g is given by

$$\mathfrak{L}_g(h) = -\Delta_g(tr_g h) + div_g(div_g(h)) - \langle h,\, Ric_g\rangle_g,$$

where h is a symmetric 2-form on M. Besides, the formal L^2-adjoint $\mathfrak{L}_g^*$ of the linearization of the scalar curvature operator at g is

$$\mathfrak{L}_g^*(f) = -(\Delta_g f)g + Hess_g f - fRic_g, \tag{1.2}$$

where f is a smooth function on M^n. Moreover, for any given metric g in the space of the metrics with constant scalar curvature, the map $\mathfrak{L}_g^*$ defined from C^∞ to $\mathcal{M}$ is an over determined-elliptic operator.

In a similar context, Miao and Tam [40, 41] and Corvino, Eichmair and Miao [23] investigated the modified problem of finding stationary points for the volume functional

$$V(g) = \int_M dV_g \tag{1.3}$$

on the space of metrics whose scalar curvature is equal to a given constant.

This situation is also motived by a result obtained by Fan, Shi and Tam [27]. There one considers an asymptotically flat 3-dimensional (M^3, g) with a given end and nonnegative scalar curvature. Then, they applied the Positive Mass Theorem to prove that for sufficiently large r one has

$$V(r) \geq V_0,$$

where $V(r)$ is the volume of the region enclosed by a coordinate sphere $S_r = \{x \in M;\ |x| = r\}$ and $V(0)$ is the volume of the region enclosed by S_r isometrically embedded in the Euclidean space $\mathbb{R}^3$ as a strictly convex hypersurface; for more details, see [27, 40, 49]. Therefore, it is natural to ask whether there exist related results on compact manifolds with boundary.

In order to make this approach more understandable, we need to fix some terminology (see [13, 15]).

Definition 1.1. Let $(M^n,\, g)$ be a connected compact Riemannian manifold with boundary ∂M. We say that g is, for brevity, a *Miao-Tam critical metric* (or simply, *critical metric*), if there is a nonnegative smooth function f on M^n such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

$$\mathfrak{L}_g^*(f) = -(\Delta_g f)g + Hess_g f - fRic_g = g. \tag{1.4}$$

Here, $\mathfrak{L}_g^*$ is the formal L^2-adjoint of the linearization of the scalar curvature operator $\mathfrak{L}_g$. Moreover, Ric, Δ and $Hess$ stand for the Ricci tensor, the Laplacian operator and the Hessian on M^n, respectively.

Miao and Tam [40] showed that such critical metrics defined in (1.4) arise as critical points of the volume functional on M^n when restricted to the class of metrics g with prescribed constant scalar curvature such that $g_{|_{T\partial M}} = \gamma$ for a prescribed Riemannian metric γ on the boundary; see also [23]. Afterward, Corvino, Eichmair and Miao [23] studied the modified problem of finding stationary points for the volume functional on the space of metrics whose scalar curvature is equal to a given constant. Interestingly, it follows from [40, Theorem 7] that connected Riemannian manifolds satisfying the critical metric equation (1.4) have necessarily constant scalar curvature R. Such metrics are effectively relevant in understanding the influence of the scalar curvature in controlling the volume of a given manifold. In this context, Corvino, Eichmair and Miao [23] were able to establish a deformation result which suggests that the information of the scalar curvature is not sufficient in giving volume comparison. This is in fact important because the volume comparison results are often used to explore gemetrical and topological properties of a given manifold. Classical examples in this direction are various volume comparison theorems which turned out to be fruitful in Riemannian geometry.

Remark 1.2. In [46], Schoen conjectured: *Let* $(M^n, \overline{g})$ *be a closed hyperbolic manifold and let* g *be another metric on* M^n *with scalar curvature* $R(g) \geq R(\overline{g})$, *then* $Vol(g) \geq Vol(\overline{g})$. The 3-dimensional case followed as a consequence of the works of Hamilton on nonsingular Ricci flow and Perelman on geometrization of 3-manifolds. For higher dimensions, Besson, Courtois and Gallot [17] verified it for metrics C^2-close to the canonical metric. One of the consequences of the works [40, 23] is that Schoen's conjecture on closed manifolds does not generalize directly to manifolds with boundary if only the Dirichlet boundary condition is imposed. More precisely, they gave a negative answer on the geodesic balls in 3-dimensional hyperbolic space using the variation of the volume with scalar curvature constraints.

Explicit examples of critical metrics can be found in [40, 41]. They include the spatial Schwarzschild metrics and AdS-Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres. Besides, standard metrics on geodesic balls in space forms are critical metrics. For the sake of completeness, it is important to underline ones.

Let us start with the Euclidean space $(\mathbb{R}^n, g)$, where g is its canonical metric.

Example 1.3 ([40])**.** We consider Ω to be a Euclidean ball in $\mathbb{R}^n$ of radius

r. Suppose that

$$f=-\frac{1}{2(n-1)}|x|^2+\frac{1}{2(n-1)}r^2,$$

where $x\in\mathbb{R}^n$. Under these conditions, it is not hard to check that $(\Omega,\, g,\, f)$ is a Miao-Tam critical metric.

At the same time, we present a similar example as before on the standard sphere $(\mathbb{S}^n,\, g_0)$, where g_0 is its canonical metric.

Example 1.4 ([40])**.** Let Ω be a geodesic ball in $\mathbb{S}^n\subset\mathbb{R}^{n+1}$ with radius $r_0\neq\frac{\pi}{2}$. Suppose that

$$f=\frac{1}{n-1}\Big(\frac{\cos r}{\cos r_0}-1\Big),$$

where r is the geodesic distance from the point $(0,...,0,1)$. Therefore, $f=0$ on the boundary of Ω and f satisfies (1.4). Moreover, if Ω is contained in a hemisphere, then $(\Omega,\, g_0,\, f)$ is also a Miao-Tam critical metric.

Reasoning as in the spherical case it is not difficult to build a similar example on the hyperbolic space $\mathbb{H}^n$.

Example 1.5 ([40])**.** Regarding the hyperbolic space $\mathbb{H}^n$ embed in $\mathbb{R}^{n,1}$, the Minkowski space, with standard metric $dx_1^2+...+dx_n^2-dt^2$ such that

$$\mathbb{H}^n=\{(x_1,...,x_n,t)\,|\,x_1^2+...+x_n^2-t^2=-1,\, t>0\}.$$

We assume that Ω is a geodesic ball in $\mathbb{H}^n$ with center at $(0,...,0,1)$ and geodesic radius r_0. Suppose that

$$f=\frac{1}{n-1}\Big(1-\frac{\cosh r}{\cosh r_0}\Big),$$

where r is the geodesic distance from the point $(0,...,0,1)$. Similarly, one sees that $f=0$ on the boundary of Ω and f satisfies (1.4).

It is natural to ask whether these examples are the only critical metrics on simply connected compact manifolds with connected boundary. Here, we call attention to the paragraph where Miao and Tam [41] wrote:

"we want to know if there exist non-constant sectional curvature critical metrics on a compact manifold whose boundary is isometric to standard sphere. If yes, what can we say about the structure of such metrics?"

Before to discuss some rigidity results related to this question, we recall some special tensors in the study of curvature for a Riemannian manifold $(M^n,\, g)$ of dimension $n \geq 3$. The first one is the Weyl tensor W which is defined by the following decomposition formula

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) \\ &\quad - \frac{R}{(n-1)(n-2)}\big(g_{jl}g_{ik} - g_{il}g_{jk}\big), \end{aligned} \tag{1.5}$$

where R_{ijkl} denotes the Riemann curvature tensor. The second one is the Cotton tensor C given by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}\big(\nabla_i R g_{jk} - \nabla_j R g_{ik}\big). \tag{1.6}$$

Notice that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices. The Weyl and Cotton tensors are related as follows

$$C_{ijk} = -\frac{(n-2)}{(n-3)}\nabla_l W_{ijkl}, \tag{1.7}$$

for $n \geq 4$. In particular, it is well known that, for $n = 3$, W_{ijkl} vanishes identically, while $C_{ijk} = 0$ if and only if $(M^3,\, g)$ is locally conformally flat; for $n \geq 4$, $W_{ijkl} = 0$ if and only if $(M^n,\, g)$ is locally conformally flat.

The Bach tensor on a Riemannian manifold (M^n, g), $n \geq 4$, is defined in terms of the components of the Weyl tensor W_{ikjl} as follows

$$B_{ij} = \frac{1}{n-3}\nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}. \tag{1.8}$$

For $n = 3$, it is given by

$$B_{ij} = \nabla_k C_{kij}. \tag{1.9}$$

We say that $(M^n,\, g)$ is Bach-flat when $B_{ij} = 0$. In particular, either locally conformally flat or Einstein metrics are necessarily Bach-flat. For $n \geq 4$, it is known among the experts (see [22, Lemma 5.1]) the following formula for the divergence of the Bach tensor

$$\nabla_i B_{ij} = \frac{n-4}{(n-2)^2} C_{jks} R_{ks}. \tag{1.10}$$

Returning to the question on rigidity of critical metrics, we are going to present some of the known rigidity results. Firstly, inspired by ideas developed by Kobayashi [35], Kobayashi and Obata [36], Miao and Tam [41] gave a partial answer to the rigidity problem. To be precise, they proved the following result.

Theorem 1.6 (Miao-Tam, [41])**.** *Let (M^n, g, f) be a compact locally conformally flat critical metric with boundary isometric to a standard sphere $\mathbb{S}^{n-1}$. Then M^n must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$.*

Four-dimensional manifolds display fascinating and peculiar features, for this reason much attention has been given to this specific dimension. Many peculiar features on oriented 4-manifolds directly rely on the fact that the bundle of 2-forms Λ^2 can be invariantly decomposed as a direct sum

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{1.11}$$

where $\Lambda^\pm$ is the (± 1)-eigenspace of the Hodge star operator $*$, respectively. For dimension $n = 4$, it is well-known that half-conformally flat ($W^+ = 0$ or $W^- = 0$) or locally conformally to an Einstein manifold implies Bach-flat. However, Leistner and Nurowski [38] obtained a large class of Bach-flat examples which are not conformally Einstein.

In [13], Barros, Diógenes and Ribeiro investigated Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. More precisely, they replaced the assumption of locally conformally flat in Theorem 1.6 by the Bach-flat condition, which is weaker that the former.

Theorem 1.7 (Barros-Diógenes-Ribeiro, [13]). *Let (M^4, g, f) be a compact Bach-flat critical metric with boundary isometric to a standard sphere $\mathbb{S}^3$. Then M^4 is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$.*

They also proved that the result even is true in dimension three by replacing the Bach-flat condition by the weaker assumption that M^3 has divergence-free Bach tensor. We remark that the proof of Theorem 1.7 was inspired in the trend developed by Cao and Chen in [22].

The other interesting assumption involving the Weyl tensor is the harmonic Weyl tensor condition, i.e., $div\,(W) = 0$. It is well known that Einstein metrics have harmonic Weyl tensor. But, the converse it is not necessarily true (see [16]). In this context, Kim and Shin [37, Theorem 10.2] showed the following result.

Theorem 1.8 (Kim-Shin, [37]). *Let (M^4, g, f) be a compact critical metric with harmonic Weyl tensor and boundary isometric to a standard sphere $\mathbb{S}^3$. Then M^4 is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$.*

Remark 1.9. The conclusion obtained in Theorem 1.8 also holds by replacing the harmonic Weyl tensor condition by the *second order divergence-free Weyl tensor* condition (i.e., $div^2(W) = \nabla_l \nabla_i W_{ijkl} = 0$); see [11, Theorem 1].

Remark 1.10. Recently, Baltazar, Batista and Bezerra [4] extended Theorems 1.7 and 1.8 for arbitrary dimension $n \geq 4$.

Remark 1.11. It is interesting to show that Theorem 1.8 also holds by replacing the harmonic Weyl tensor condition by the *harmonic self-dual Weyl tensor assumption*, i.e., $div(W^+) = 0$.

At the same time, by assuming the Einstein condition, Miao and Tam (cf. Theorem 1.1 in [41]) were able to remove the condition of boundary isometric to a standard sphere. More precisely, they proved the following result.

Theorem 1.12 (Miao-Tam, [41])**.** *Let (M^n, g, f) be a connected, compact Einstein critical metric with smooth boundary ∂M. Then (M^n, g) is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$.*

It is easy to check that the Einstein condition implies the Ricci parallel condition. However, there are examples of metrics with parallel Ricci tensor which are not Einstein. For instance, we can take the product metric given by two Einstein manifolds with different Einstein constants. Baltazar and Ribeiro [6] improved Theorem 1.12 by replacing the assumption of Einstein in the Miao-Tam result by the parallel Ricci tensor condition.

Theorem 1.13 (Baltazar-Ribeiro, [6])**.** *Let (M^n, g, f) be a connected, compact critical metric with Ricci parallel and smooth boundary ∂M. Then M^n is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$.*

To state the next result, it is fundamental to remember that a Riemannian manifold $(M^n,\, g)$ has *zero radial Weyl curvature* when the interior product $i_{\nabla f}W = 0$, i.e.,

$$W(\cdot,\, \cdot,\, \cdot, \nabla f) = 0, \tag{1.12}$$

for a suitable potential function f on M^n. This class of manifolds clearly includes the case of locally conformally flat manifolds. As observed in [31, Table 2], there exist examples of $(n \geq 5)$-dimensional manifolds which have zero radial Weyl curvature but are not locally conformally flat. In [9], Baltazar and Ribeiro showed that a compact $(n \geq 3)$-dimensional critical metric of the volume functional $(M^n,\, g)$ with smooth boundary ∂M, nonnegative sectional curvature and zero radial Weyl curvature is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$ or $\mathbb{S}^n$. In dimension $n = 3$, Huiya He [32] improved this result obtained by Baltazar and Ribeiro by replacing the nonnegative sectional curvature by the nonnegative Ricci curvature condition. Moreover, Huiya He obtained a similar result for 3-dimensional static spaces (see also [1]).

In recent years, the zero radial Weyl curvature condition has been combined with other geometric assumptions in order to obtain new classification results for gradient Ricci solitons, quasi-Einstein manifolds and critical metrics by using pointwise arguments; see, for instance, [26, 24, 30, 31]. By adopting this condition, Baltazar, Barros, Batista and Viana [3] proved the following result by using a different approach.

Theorem 1.14 (Baltazar et al. [3])**.** *Let* (M^n, g, f), $n \geq 5$, *be a connected, compact critical metric with zero radial Weyl curvature and boundary isometric to a standard sphere* $\mathbb{S}^{n-1}$*. Then* M^n *is isometric to a geodesic ball in a simply connected space form* $\mathbb{R}^n$*,* $\mathbb{H}^n$ *or* $\mathbb{S}^n$*.*

A more general version of Theorem 1.14 including the case of disconnected boudary was also obtained in [3, Theorem 3].

There are several others uniqueness results in the literature. For more details on this subject see, e.g., [3, 7, 8, 9, 11, 13, 14, 15, 23, 28, 32, 37, 40, 41, 44, 49].

Boundary estimates are classical objects of study in geometry and physics. Besides being interesting on their own, such estimates are useful in proving new classification results and discarding some possible new examples of special metrics on a given manifold. Among the contributions that motivated the results that follow, we primarily mention the classical isoperimetric inequality and a result due to Shen [43] and Boucher, Gibbons and Horowitz [21] which asserts that the boundary ∂M of a compact three-dimensional oriented static space with connected boundary and scalar curvature 6 must be a 2-sphere whose area satisfies the inequality $|\partial M| \leq 4\pi$, with equality holding if and only if M^3 is equivalent to the standard hemisphere.

In [23], Corvino, Eichmair and Miao showed that the area of the boundary ∂M of an n-dimensional scalar flat Miao-Tam critical metric must have an upper bound depending on the volume of M^n (see [23, Proposition 2.5]).

Theorem 1.15 (Corvino, Eichmair and Miao, [23])**.** *Let* (M^n, g, f) *be a compact critical metric with zero scalar curvature and connected boundary*

∂M. Then we have:

1.
$$\int_{\partial M} R^{\partial M} dS > 0,$$
where $R^{\partial M}$ stands for the scalar curvature of ∂M.

2.
$$Vol(M) \geq \frac{\sqrt{(n-2)(n-1)}}{n} \left(\int_{\partial M} R^{\partial M} dS \right)^{-\frac{1}{2}} |\partial M|^{\frac{3}{2}}.$$
Equality holds if and only if (M^n, g) is isometric to a standard ball in $\mathbb{R}^n$.

3. *When $n = 3$, one has*
$$Vol(M) \geq \frac{|\partial M|^{\frac{3}{2}}}{6\sqrt{\pi}}.$$

Afterward, motivated by the quoted result due to Shen [43] and Boucher, Gibbons and Horowitz [21] and Theorem 1.15, Batista et al. [15] obtained a sharp boundary estimate for 3-dimensional oriented Miao-Tam critical metric (M^3, g) with connected boundary and nonnegative scalar curvature. To be precise, they proved the following result.

Theorem 1.16 (Batista et al. [15])**.** *Let (M^3, g, f) be a compact, oriented, critical metric with connected boundary ∂M and nonnegative scalar curvature. Then ∂M is a 2-sphere and*
$$|\partial M| \leq \frac{4\pi}{C(R)},$$
where $C(R) = \frac{R}{6} + \frac{1}{4|\nabla f|^2} > 1$ is constant. Moreover, equality holds if and only if (M^3, g) is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^3$ or $\mathbb{S}^3$.

This result also holds for negative scalar curvature, provided that the mean curvature of the boundary satisfies $H > 2$, as was proven in [12]; see also [14]. Moreover, it was established a general version of Theorem 1.16 by Barros and Silva [14]. To be precise, they proved the following result.

Theorem 1.17 (Barros-Silva, [14]). *Let $(M^n,\, g,\, f)$, $n \geq 4$, be a compact, oriented, critical metric with connected boundary ∂M, and scalar curvature $R = n(n-1)\varepsilon$, where $\varepsilon = -1, 0, 1$. Suppose that the boundary ∂M is an Einstein manifold with positive scalar curvature $R^{\partial M}$. In addition, if $\varepsilon = -1$, assume that the mean curvature of ∂M satisfies $H > n-1$. Then we have*

$$|\partial M|^{\frac{2}{n-1}} \leq \frac{Y(\mathbb{S}^{n-1}, [g_{can}])}{C(R)}, \tag{1.13}$$

where $C(R) = \frac{n-2}{n}R + \frac{n-2}{n-1}H^2$ is a positive constant and $Y(\mathbb{S}^{n-1}, [g_{can}])$ is the Yamabe constant of the standard sphere $\mathbb{S}^{n-1}$. Moreover, equality holds in (1.13) if and only if (M^n, g) is isometric to a geodesic ball in a simply connected space form $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$.

In summary, it is important to present an estimate to the area of the boundary ∂M of an n-dimensional Miao-Tam critical metric which was obtained by contributions by [12, 15, 23].

Theorem 1.18 ([12, 15] and [23]). *Let $(M^n,\, g,\, f)$ be an n-dimensional compact critical metric with connected boundary ∂M. For the case of negative scalar curvature we assume in addition that $H^2 > -\frac{n-1}{n}R$. Then the area of the boundary ∂M satisfies*

$$|\partial M| \leq \frac{1}{\left(\frac{n-2}{n}R + \frac{n-2}{n-1}H^2\right)} \int_{\partial M} R^{\partial M} dS, \tag{1.14}$$

where $R^{\partial M}$ stands for the scalar curvature of $(\partial M, g_{|_{\partial M}})$. Moreover, equality holds if and only if M^n is isometric to a geodesic ball in $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$.

Isoperimetric problem is a classical topic in mathematics. Generally speaking, the isoperimetric inequality is a geometric inequality involving the surface area of a set and its volume. The isoperimetric inequality on the plane states that the length L of a closed curve on $\mathbb{R}^2$ and the area A of the planar region that it encloses must to satisfy

$$L^2 \geq 4\pi A.$$

Moreover, equality holds if and only if the curve is a circle. In $\mathbb{R}^n$, the classical isoperimetric inequality asserts that if $M \subset \mathbb{R}^n$ is a compact domain with smooth boundary ∂M, then

$$\frac{|\partial M|}{|\partial \mathbb{B}_1^n|} \geq \frac{\mathrm{Vol}(M)^{\frac{n-1}{n}}}{\mathrm{Vol}(\mathbb{B}_1^n)^{\frac{n-1}{n}}}, \tag{1.15}$$

where $|\partial M|$ denotes the $(n-1)$-dimensional volume of ∂M, $\mathrm{Vol}(M)$ is the volume of M and $\mathbb{B}_1^n$ denotes the unit Euclidean ball. For sufficiently smooth domains, the n-dimensional isoperimetric inequality is equivalent to the Sobolev inequality on $\mathbb{R}^n$. In the spirit of these quoted results and stimulated by the isoperimetric problem, it was proved in [10] the following isoperimetric type inequality.

Theorem 1.19 (Baltazar-Diógenes-Ribeiro, [10])**.** *Let (M^n, g, f) be a compact, oriented, critical metric with connected boundary ∂M and non-negative scalar curvature. Then we have:*

$$|\partial M| \geq (C_{R,H})^{\frac{1}{n}} \mathrm{Vol}(M)^{\frac{n-1}{n}}, \tag{1.16}$$

where $C_{R,H}$ is a positive constant depending on the dimension, the scalar curvature R and the mean curvature H of the boundary ∂M with respect to the outward unit normal. Moreover, equality holds if and only if M^n is isometric to a geodesic ball in $\mathbb{R}^n$.

Remark 1.20. It would be interesting to see if the constant $C_{R,H}$ in (1.16) can be improved to only depend on the dimension and the volume of the unit Euclidean ball.

In the last years several progress have been made on boundary and volume estimates for critical metrics of volume functional (see, e.g., [14, 12, 15, 23, 49]), however, only few results are known for possibly disconnected boundary case. Motivated by this context, Barbosa, Freitas and Lima [12] used a generalized Pohozaev-Schoen identity to obtain rigidity results for critical metrics and generalized solitons. In particular, they proved the following result.

Theorem 1.21 (Barbosa-Freitas-Lima, [12])**.** *Let (M^n, g) be compact, oriented, critical metric with boundary $\partial M = \bigcup_{i=1}^{l} \partial M_i$, and $R_g = \varepsilon n(n-1)$, where $\varepsilon = -1, 0, 1$. Then the following integral identity holds:*

$$\int_M f|\overset{\circ}{\mathrm{Ric}}_g|^2 dM = -\sum_i \kappa_i \int_{\partial M_i} (\mathrm{Ric}_g(\nu,\nu) - \varepsilon(n-1))\, dS_i, \qquad (1.17)$$

where κ_i is the restriction of $|\nabla_g f|$ to ∂M_i. In particular,

$$\sum_i \kappa_i \int_{\partial M_i} (\mathrm{Ric}_g(\nu,\nu) - \varepsilon(n-1))\, dS_i \leq 0, \qquad (1.18)$$

with the equality holding if and only if (M^n, g) is isometric to a geodesic ball in a simply connected space form.

In the sequel, mainly inspired by the works of Borghini and Mazzieri [19, 20] on the uniqueness result for the de Sitter solution, we shall present a sharp estimate to the mean curvature H_i of the boundary components ∂M_i of a critical metric of the volume functional on an n-dimensional compact manifold. More precisely, it was established the following result.

Theorem 1.22 (Baltazar-Batista-Ribeiro, [5])**.** *Let (M^n, g, f) be an n-dimensional compact critical metric with boundary ∂M (possibly disconnected). Then we have:*

$$\min H_i \leq \sqrt{\frac{n(n-1)}{R(f_{\max})^2 + 2nf_{\max}}} \quad on \;\; \partial M, \qquad (1.19)$$

where $f_{\max}$ is the maximum value of f. Moreover, equality holds if and only if M^n is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$.

A relevant observation is that the mean curvature of the boundary of geodesic balls in space form satisfies $H^2 = \frac{n(n-1)}{R(f_{\max})^2+2nf_{\max}}$. Therefore, it follows from Theorem 1.22 that the mean curvature of the boundary of a geodesic ball is the maximum possible among all Miao-Tam critical metrics on compact manifolds with connected boundary. Moreover, no restriction on the sign of the scalar curvature is assumed.

A key ingredient to establish the proof of Theorem 1.22 is a Robinson-Shen type identity which is essentially inspired by [42, 43]. Recently, a similar idea was used by Diógenes, Gadelha and Ribeiro [25] to obtain a sharp geometric inequality for compact quasi-Einstein manifolds with boundary.

As a consequence of Theorem 1.22 we obtain the following corollary for scalar flat critical metrics.

Corollary 1.23 (Baltazar-Batista-Ribeiro, [5])**.** *Let $(M^n,\, g,\, f)$ be an n-dimensional compact scalar flat critical metric with connected boundary ∂M. Then we have:*

$$|\partial M| \leq \sqrt{\frac{n^2}{2(n-1)f_{\max}}}Vol(M). \tag{1.20}$$

Moreover, equality holds if and only if M^n is isometric to a geodesic ball in $\mathbb{R}^n$ of radius $\sqrt{2(n-1)f_{\max}}$.

References

[1] L. Ambrozio, On static three-manifolds with positive scalar curvature. *J. Diff. Geom.* 107 (2017) 1-45.

[2] T. Aubin, Équation différentielles non-linéares et probléme de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.* 55 (1976) 269-296.

[3] H. Baltazar, A. Barros, R. Batista, E. Viana, On static manifolds and related critical spaces with zero radial Weyl curvature. *Monatsh. Math.* 191 (2020) 449-463.

[4] H. Baltazar, R. Batista, K. Bezerra, On the volume functional of compact manifolds with harmonic Weyl tensor. *Math. Nachr.* (2021), accepted.

[5] H. Baltazar, R. Batista, E. Ribeiro Jr., Geometric inequalities for critical metrics of the volume functional. *Annali di Matematica* (2021) https://doi.org/10.1007/s10231-021-01164-9.

[6] H. Baltazar, E. Ribeiro Jr., Critical metrics of the volume functional on manifolds with boundary. *Proc. Amer. Math. Soc.* 145 (2017) 3513-3523.

[7] H. Baltazar, A. Silva, On static manifolds and related critical spaces with cyclic parallel Ricci tensor. *Adv. Geom.* (2021) DOI: 10.1515/advgeom-2020-0021.

[8] H. Baltazar, A. Silva, F. Oliveira, Weakly Einstein critical metrics of the volume functional on compact manifolds with boundary. *J. Math. Anal. App.* 487 (2020) 124013.

[9] H. Baltazar, E. Ribeiro Jr., Remarks on critical metrics of the scalar curvature and volume functionals on compact manifolds with boundary. *Pacific J. Math.* 297 (2018) 29-45.

[10] H. Baltazar, R. Diógenes, E. Ribeiro Jr., Isoperimetric inequality and Weitzenböck type formula for critical metrics of the volume. *Israel J. Math.* 234 (2019), no. 1, 309-329.

[11] H. Baltazar, R. Diógenes, R. , E. Ribeiro Jr., Volume functional of compact 4-manifolds with a prescribed boundary metric. *J. Geom. Anal.* 31 (2021) 4703-4720.

[12] E. Barbosa, L. Lima, A. Freitas, The generalized Pohozaev-Schoen identity and some geometric applications. *Commun. Anal. Geom.* 28 (2020) 223-242.

[13] A. Barros, R. Diógenes, E. Ribeiro Jr., Bach-Flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. *J. Geom. Anal.* 25 (2015) 2698-2715.

[14] A. Barros, A. Da Silva, Rigidity for critical metrics of the volume functional. *Math. Nachr.* 291 (2019) 709-719.

[15] R. Batista, R. Diógentes, M.Ranieri, E. Ribeiro Jr., Critical metrics of the volume functional on compact three-manifolds with smooth boundary. *J. Geom. Anal.* 27 (2017) 1530-1547.

[16] A. Besse, *Einstein manifolds.* Springer-Verlag, Berlin Heidelberg (1987).

[17] G. Besson, G. Courtois, S. Gallot, Volume et entropie minimale des espaces localement symétriques, *Invent. Math.* 103 (1991) 417-445.

[18] S. Borghini, *On the characterization of static spacetimes with positive cosmological constant.*, PhD thesis, Scuola Normale Superiore. Classe di Scienze Matem. Nat. (2018).

[19] S. Borghini, L. Mazzieri, On the mass of static metrics with positive cosmological constant - I. *Class. Quantum Grav.* 35 (2018) 125001.

[20] S. Borghini, L. Mazzieri, On the Mass of Static Metrics with Positive Cosmological Constant: II. *Commun. Math. Phys.* 377 (2020) 2079-2158.

[21] W. Boucher, G. Gibbons, G. Horowitz, Uniqueness theorem for anti-de Sitter spacetime. *Phys. Rev. D* (3) 30 (1984) 2447-2451.

[22] H.-D. Cao, Q. Chen, On Bach-flat gradient shrinking Ricci solitons. *Duke Math. J.* 162 (2013) 1149-1169.

[23] J. Corvino, M.Eichmair, P. Miao, Deformation of scalar curvature and volume. *Math. Annalen.* 357 (2013) 551-584.

[24] G. Catino, Generalized quasi-Einstein manifolds with harmonic Weyl tensor. *Math. Z.* 271, 3-4 (2012) 751-756.

[25] R. Diógentes, T. Gadelha, E. Ribeiro Jr., Geometry of compact quasi-Einstein manifolds with boundary. *Manuscripta Math.* (2021) https://doi.org/10.1007/s00229-021-01340-4.

[26] R. Diógentes, T. Gadelha, E. Ribeiro Jr., Remarks on compact quasi-Einstein manifolds with boundary. *Proc. Amer. Math. Soc* 150 (2022) 351-363.

[27] X.-Q. Fan, Y.-G. Shi., L.-F. Tam, Large-sphere and small-sphere limits of the Brown-York mass. *Comm. Anal. Geom.* 17 (2009) 37-72.

[28] Y. Fang, W. Yuan, Brown-York mass and positive scalar curvature II: Besse's conjecture and related problems. *Ann. Glob. Anal. Geom.* 56 (2019) 1-15.

[29] P. Feehan, M. Maridakis, Lojasiewicz-Simon gradient inequalities for analytic and Morse-Bott functionals on banach spaces and applications to harmonic maps. *ArXiv:1510.03817 [math.DG]* (2015).

[30] C. He, P. Petersen, W.Wylie, On the classification of gradient Ricci solitons. *Geom. Topol.* 14 (2010) 2277-2300.

[31] C. He, P. Petersen, W.Wylie, On the classification of warped product Einstein metrics. *Commun. Anal. Geometry.* 20 (2012) 271-311.

[32] H. He, Critical metrics of the volume functional on three-dimensional manifolds. *ArXiv:2101.05621 [math.DG]*, (2021).

[33] N. Koiso, A decomposition of the space of Riemannian metrics on a manifold. *Osaka J. Math.* 16 (1979) 423-429.

[34] G. Krantz, H. Parks, *A primer of real analytic functions.* Birkhauser Advanced Texts: Basler Lehrbucher. Birkhauser Boston, Inc. Boston, MA, (2002).

[35] O. Kobayashi, A differential equation arising from scalar curvature function. *J. Math. Soc. Japan.* 34 (1982) 665-675.

[36] O. Kobayashi, M. Obata, Conformally-flatness and static space-time, Manifolds and Lie-Groups. (Notre Dame,Ind.,1980). Progr. Math., 14, Birkhäuser, Boston, MA, (1981) 197-206.

[37] J. Kim, J. Shin, Four dimensional static and related critical spaces with harmonic curvature. *Pacific J. Math.* 295 (2018) 429-462.

[38] T. Leistner, P. Nurowski, Ambient metrics for n-dimensional pp-waves. *Comm. Math. Phys.* 296(3) (2010) 881-898

[39] S. Lojasiewicz, Una propriété topologique des sous-ensembles analystiques réels. In *Les Équations aux Dérivées Parialles* (Paris, 1962), pages 87-89. Éditions du Centre national de la Recherche Sientifique, Paris, 1963.

[40] P. Miao, L.-F. Tam, On the volume functional of compact manifolds with boundary with constant scalar curvature. *Calc. Var. PDE.* 36 (2009) 141-171.

[41] P. Miao, L.-F. Tam, Einstein and conformally flat critical metrics of the volume functional. *Trans. Amer. Math. Soc.* 363 (2011) 2907-2937.

[42] D. Robinson, A simple proof of the generalization of Israel's Theorem. *Gen. Relat. Grav.* 8 (1977) 695-698.

[43] Y. Shen, A note on Fischer-Marsden's conjecture. *Proc. Amer. Math. Soc.* 125 (1997) 901-905.

[44] W. Sheng, L. Wang, Critical metrics with cyclic parallel Ricci tensor for volume functional on manifolds with boundary. *Geom. Dedic.*, 201 (2019) 243-251.

[45] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.* 20 (1984) 479-495.

[46] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. *Topics in calculus of variations. Lecture Notes in Math.* 1365, 120-154 (1989).

[47] N. Trudinger, Remarks concerning the conformal deformations of Riemannian structures on compact manifolds. *Annali Scuola Norm. Sup. Pisa.* 22 (1968) 265-274.

[48] H. Yamabe, On deformation of Riemannian structures on compact manifolds. *Osaka J. Math.* 12 (1960), 21-37.

[49] W. Yuan, Volume comparison with respect to scalar curvature. *ArXiv:1609.08849v3 [math.DG]* (2016).