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k-Independence on complementary prism graphs

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Abstract

A subset S of vertices of a graph G is k-independent if each vertex in S has degree at most $k-1$. The complementary prism of a graph G, denoted by $G\overline{G}$, is a graph constructed by the disjoint union of G and its complement $\overline{G}$ by adding edges of a perfect matching between the corresponding vertices. We present some results on k-independent sets in complementary prisms, including sharp lower and upper bounds for the 2-independence number. Moreover, we present exact values for α_2 for the complementary prism of some particular graph classes.

1 Introduction

We consider finite, simple, and undirected graphs. For a graph G, the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. We use standard notation and terminology. See [4] for graph-theoretic terms not defined here.

A *dominating set* D is a subset of vertices of graph G such that every vertex that is not in D is adjacent to at least one member of D. An *independent set* of G is a subset $S \subseteq V(G)$ such that its vertices are pairwise non adjacent in G. An independent set S is *maximal* if it is a dominating set, and *maximum* if it has the

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largest possible cardinality. The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G.

Fink and Jacobson [10] generalized the concept of independent set as follows. Let k be a positive integer. A subset S of vertices in a graph G is *k-independent* if the maximum degree of the subgraph induced by S is at most $k-1$. A k-independent set S is maximal if for every vertex $v \in V \setminus S$, $S \cup \{v\}$ is not k-independent. The minimum cardinality of a maximal k-independent set of a graph G is denoted by $i_k(G)$ and its maximum cardinality is denoted by $\alpha_k(G)$, called the k-independence number. A k-independent set of G with maximum cardinality is called an $\alpha_k(G)$-set. In particular, $\alpha_1(G) = \alpha(G)$ is the usual independence number of G. Observe that a j-independent set is also a k-independent set, for $k \geq j$. Moreover, every set with k vertices is k-independent; so $i_k(G) \geq k$ when $|V(G)| \geq k$.

Favaron *et al.* [9] considered k-dependent domination, which corresponds to sets that are $(k+1)$-independent and dominating simultaneously. They establish relationships between k-dependent domination and concepts of classical domination. Blidia *et al.* [3] gave some relations between $\alpha_k(G)$ and $\alpha_j(G)$ and between $i_k(G)$ and $i_j(G)$ for $j \neq k$. They studied two families of extremal graphs for the inequality $i_2(G) \leq i(G) + \alpha(G)$, gave an upper bound on $i_2(G)$, and a lower bound when G is a cactus. A *cactus* is a connected graph in which any two simple cycles have at most one vertex in common. Chellali *et al.* [7] studied graphs such that removing an edge $e \in E(G)$ their k-independence number do not change. That is, $\alpha_k(G-e) = \alpha_k(G)$. For a graph G on n vertices and average degree d, Caro and Hansberg [5] proved that $\alpha_k(G) \geq \frac{kn}{\lceil d \rceil + k}$. This bound was improved by Kogan [12] who proved that $\alpha_k(G) \geq \frac{kn}{d+k}$ and, for $k = 2,3$, he characterized the graphs for which the equality holds. Mao *et al.* [13] considered k-independence on the lexicographic, strong, cartesian, and direct products and presented several upper and lower bounds for these products of graphs. Aram *et al.* [1] studied 2-independence on trees. For more information on k-independence number, see [6].

The *complementary prism* of a graph G, denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of the perfect matching between the corresponding vertices of G and $\overline{G}$. Note that $V(G\overline{G}) = V(G) \cup V(\overline{G})$. The complementary prism of C_5, the graph $C_5\overline{C}_5$, known as Petersen Graph, can be seen in Figure 1.

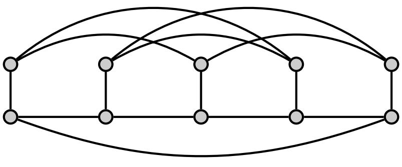


Figure 1: Petersen Graph, the complementary prism of C_5.

Haynes *et al.* [11] investigated several graph theoretic properties of complementary prisms, such as independence, distance and domination. Duarte *et al.* [8] studied algorithmic and computational complexity properties of complementary prisms with respect to cliques, independent sets, domination, and convexity. Barbosa *et al.* [2] presented results for maximal independent sets in complementary prisms whose maximal independent sets are also maximum (known as well-covered).

In this paper, we consider k-independent sets in complementary prisms. We extend some results on independent sets for complementary prisms presented by Haynes *et al.* [11] to α_2-independence and present some general results for α_k. Moreover, we present exact values for α_2 for the complementary prism of some particular graph classes.

To simplify our discussion of complementary prisms, we say simply G and $\overline{G}$ to refer to the subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. Also, for a vertex v of G, we let $\overline{v}$ be the corresponding vertex in $\overline{G}$, and for a set $X \subseteq V(G)$, let $\overline{X}$ be the corresponding set of vertices in $V(\overline{G})$. We denote by $G[S]$ the subgraph of G induced by S. The degree of a vertex v of G is denoted by $d_G(v)$. For a subset A of $V(G)$, we denote by $d_A(v)$ the degree of v in $G[A]$.

2 Main results

For $k \geq 1$ and a given graph G on n vertices, let $S \cup \overline{T}$ be a k-independent set of $G\overline{G}$ such that $S \subseteq V(G)$ and $\overline{T} \subseteq V(\overline{G})$. In the following results we convention that T contains the corresponding vertices of $\overline{T}$ in G and that $X = S \cap T$ with $m = |X|$. Observe that the value of m is bounded by a function of k. In any independent set of a complementary prism it is easy to see that $m = 0$. We show bounds for the degree of a vertex in $G[S]$ and the size of m for each $k \geq 2$.

Proposition 2.1. *For $k \geq 2$ and a given graph G, let $S \cup \overline{T}$ be a k-independent set of $G\overline{G}$, such that $S \subseteq V(G)$ and $\overline{T} \subseteq V(\overline{G})$. If $v \in S \cap T$, then $d_S(v) \leq k-2$.*

Proof. Since $v \in S \cap T$, $\overline{v} \in \overline{T}$. By definition of k-independence, $d_{G\overline{G}}(v) \leq k-1$. Since v has a neighbor in $\overline{T}$, it follows that $d_S(v) \leq k-2$. ∎

Proposition 2.2. *For $k \geq 2$ and a given graph G, let $S \cup \overline{T}$ be a k-independent set of $G\overline{G}$, such that $S \subseteq V(G)$ and $\overline{T} \subseteq V(\overline{G})$ with $S \cap T = X$ and $|X| = m$. If $v \in X$, then $m - (k-1) \leq d_X(v) \leq k-2$.*

Proof. Since $X \subseteq S$, by Proposition 2.1 we have that $d_X(v) \leq k-2$. Moreover, $k-2 \leq m-1$. Analogously, $d_{\overline{X}}(\overline{v}) \leq k-2 \leq m-1$. If $d_X(v) = d$, then $d_{\overline{X}}(\overline{v}) = m-1-d$. Therefore, $m-1-d \leq k-2$, which implies that $m-(k-1) \leq d$, proving the lower bound. ∎

Proposition 2.3. *For $k \geq 2$ and a given graph G, let $S \cup \overline{T}$ a k-independent set of $G\overline{G}$, such that $S \subseteq V(G)$ and $\overline{T} \subseteq V(\overline{G})$. If $S \cap T = X$ and $|X| = m$, then $m \leq 2k-3$, and this bound is sharp.*

Proof. For a contradiction, suppose $S \cup \overline{T}$ as described and $m \geq 2k-2$. By Proposition 2.2, $m-(k-1) \leq d_X(v) \leq k-2$. Since $m \geq 2k-2$, we obtain $d_X(v) \geq (2k-2)-k+1 = k-1$, which is a contradiction. To show that the bound is sharp, let $k = a+2$, for some natural a. It is possible to construct a graph G where $G\overline{G}$ has a k-independent set $S \cup \overline{T}$, such that $S \subseteq V(G)$ and $\overline{T} \subseteq V(\overline{G})$ having $|S \cap T| = m = 2a+1$. In this case, $d_X(v) = d_{\overline{X}}(\overline{v}) = a = k-2$. ∎

Now, we present an upper bound for $\alpha_k(G\overline{G})$ in terms of $\alpha_k(G)$ and $\alpha_k(\overline{G})$.

Theorem 2.1. *For any graph G,* $\alpha_k(G\overline{G}) \leq \alpha_k(G) + \alpha_k(\overline{G})$.

Proof. Let I be an $\alpha_k(G\overline{G})$-set, S be the vertices from I in G and $\overline{T}$ be the vertices from I in $\overline{G}$. As $|S| \leq \alpha_k(G)$ and $|\overline{T}| \leq \alpha_k(\overline{G})$, then $\alpha_k(G\overline{G}) = |S| + |\overline{T}| \leq \alpha_k(G) + \alpha_k(\overline{G})$ and the upper bound holds. ∎

Haynes *et al.* [11] showed that for any graph G, $\alpha(G) + \alpha(\overline{G}) - 1 \leq \alpha(G\overline{G}) \leq \alpha(G) + \alpha(\overline{G})$, and both these bounds are sharp. Furthermore, they characterize the graphs for which the upper bound holds with equality. We extend these results for $\alpha_2(G\overline{G})$.

Theorem 2.2. *For any graph G,* $\alpha(G) + \alpha(\overline{G}) \leq \alpha_2(G\overline{G}) \leq \alpha_2(G) + \alpha_2(\overline{G})$*, and both these bounds are sharp.*

Proof. The upper bound is straightforward from Theorem 2.1. For the lower bound, let S be an $\alpha(G)$-set and $\overline{T}$ be an $\alpha(\overline{G})$-set. The vertices in T induce a clique in G. Thus, at most one vertex in T is a member of S. Hence, the graph $G\overline{G}[S \cup \overline{T}]$ has at most one edge, with implies $S \cup \overline{T}$ is an 2-independent set of $G\overline{G}$. Therefore, $\alpha_2(G\overline{G}) \geq |S \cup \overline{T}| = |S| + |\overline{T}| = \alpha(G) + \alpha(\overline{G})$. The complementary prism in Figure 2 attains the lower bound, and the complementary prism of C_5 (Figure 1) attains the upper bound. ∎

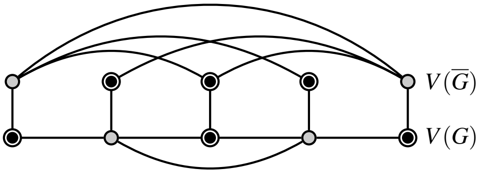


Figure 2: Complementary prism with an $\alpha_2(G\overline{G})$-set (black vertices) that attains the lower bound of Theorem 2.2.

We denote a complete multipartite graph with t parts by $K_{m_1,\ldots,m_t}$, where m_i is the number of vertices in the i^{th} part, and by $K^{\leq k}_{m_1,\ldots,m_t}$ a $K_{m_1,\ldots,m_t}$ graph where $m_i \leq k$, for all $i \in \{1,\ldots,t\}$. Note that, for $k \leq 2$, the subgraph induced by the complement of a k-independent set is isomorphic to $K^{\leq k}_{m_1,\ldots,m_t}$. Let $\theta(G)$ be the number of vertices of a maximum $K^{\leq 2}_{m_1,\dots,m_t}$, which is a subgraph of G. We characterize the graphs that attain the upper bound given in Theorem 2.2.

Theorem 2.3. *A graph G has $\alpha_2(G\overline{G}) = \alpha_2(G) + \alpha_2(\overline{G})$ if and only if there exist vertex sets S and T in $V(G)$ such that S is an $\alpha_2(G)$-set and T induces a maximum $K^{\leq 2}_{m_1,\dots,m_t}$ graph, with $|S \cap T| \leq 1$. If $S \cap T = \{v\}$, then v is isolated in $G[S]$ and it is in a partition of size 1 of $G[T]$.*

Proof. First, assume that sets S and T exist. Thus $\overline{T}$ is an $\alpha_2(\overline{G})$-set. If $|S \cap T| = 0$, then S is an $\alpha_2(G)$-set and $|S \cup \overline{T}| = |S| + |\overline{T}| = \alpha_2(G) + \alpha_2(\overline{G}) = \alpha_2(G\overline{G})$. In case of $|S \cap T| = 1$, with $S \cap T = \{v\}$, by hypothesis, v is isolated in $G[S]$, and it is in a partition of size 1 of $G[T]$, which imply that $d_{S\cup\overline{T}}(v) = 1 = d_{S\cup\overline{T}}(\overline{v})$. Therefore, $S \cup \overline{T}$ is an $\alpha_2(G\overline{G})$-set and $|S \cup \overline{T}| = \alpha_2(G) + \alpha_2(\overline{G}) = \alpha_2(G\overline{G})$.

For the converse, let L be an $\alpha_2(G\overline{G})$-set and $\alpha_2(G\overline{G}) = \alpha_2(G) + \alpha_2(\overline{G})$. Let $S = L \cap V(G)$ and $\overline{T} = L \cap V(\overline{G})$. By Proposition 2.3, at most one vertex from S is adjacent to a vertex of $\overline{T}$. If so, let v be this vertex in S and $\overline{v}$ its neighbor in $\overline{T}$. By degree restriction, vertex v must be isolated in $G[S]$ and also $\overline{v}$ in $\overline{G}[\overline{T}]$. Since the vertices in T induce a maximum $K^{\leq 2}_{m_1,\dots,m_t}$ graph in G, v must be on a partition of size 1 of $G[T]$. Hence $|T| = |\overline{T}| \leq \alpha_2(\overline{G}) = \theta(G)$. Analogously, $|S| = |\overline{S}| \leq \theta(\overline{G}) = \alpha_2(G)$. Therefore, $\alpha_2(G) + \alpha_2(\overline{G}) = |L| = |S| + |\overline{T}| \leq \alpha_2(G) + \alpha_2(\overline{G})$, implying that S is an $\alpha_2(G)$-set and T induces a maximum $K^{\leq 2}_{m_1,\dots,m_t}$ graph. ■

3 Results on some particular graph classes

We study k-independent sets in the complementary prisms of some particular graph classes. It is easy to verify that if $n \geq k$, $\alpha_k(K_n) = k$, $\alpha_k(\overline{K_n}) = n$, and $\alpha_k(K_n\overline{K_n}) = n + k - 1$. Moreover, $\alpha(P_n) = \lceil n/2 \rceil$ and $\alpha(C_n) = \lfloor n/2 \rfloor$. For $k \geq 3$, $\alpha_k(P_n) = \alpha_k(C_n) = n$. We establish the exact values of a maximum 2-independent set in P_n, C_n, and their complements in Proposition 3.1. Furthermore, we show values of $\alpha_2(P_n\overline{P_n})$ and $\alpha_2(C_n\overline{C_n})$ in Proposition 3.2.

Proposition 3.1. *For $n \geq 5$, $\alpha_2(\overline{P_n}) = \alpha_2(\overline{C_n}) = 3$,*

$$\alpha_2(P_n) = \begin{cases} 2\lceil n/3 \rceil - 1, & \text{if } n \equiv 1 \pmod 3, \\ 2\lceil n/3 \rceil, & \text{otherwise,} \end{cases}$$

$$\alpha_2(C_n) = \begin{cases} 2\lfloor n/3 \rfloor + 1, & \textit{if } n \equiv 2 \pmod 3, \\ 2\lfloor n/3 \rfloor, & \textit{otherwise.} \end{cases}$$

Proof. Let $P_n : v_1v_2v_3 \dots v_n$ and $C_n : v_1v_2v_3 \dots v_nv_1$. Consider a graph G and its complement $\overline{G}$ such that $G \in \{P_n, C_n\}$. Let $I' \subseteq V(G)$ with $I' = \{v_i : i \equiv 1,2 \pmod 3\}$. If $G \cong P_n$, then define $I = I'$, and define $I = I' \setminus \{v_n\}$, if $G \cong C_n$. For $\overline{G}$, consider the set $I = \{\overline{v_1}, \overline{v_2}, \overline{v_3}\}$. In every graph, its corresponding set I induces a graph with maximum degree one, and it has the described cardinality. It is easy to verify that I is an α_2-set in its corresponding graph. ■

Proposition 3.2. *For $n \geq 5$,*

$$\alpha_2(P_n\overline{P_n}) = \begin{cases} 2n/3 + 3, & \textit{if } n \equiv 0 \pmod 3, \\ 2\lceil n/3 \rceil + 2, & \textit{otherwise,} \end{cases}$$

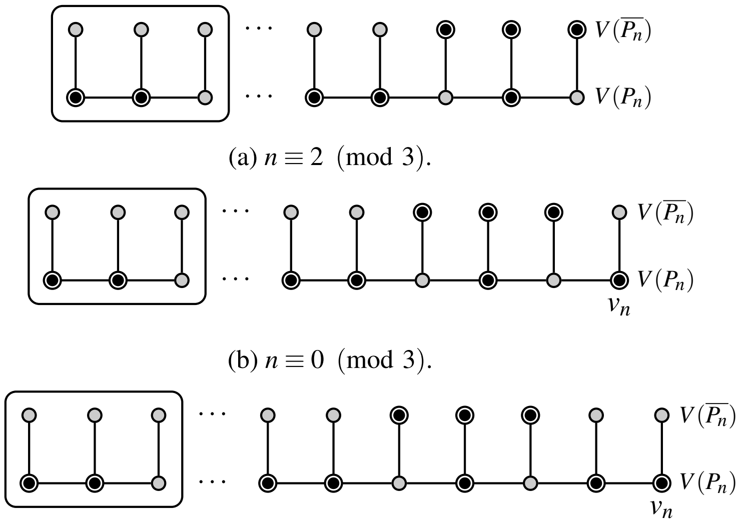
$$\alpha_2(C_n\overline{C_n}) = \begin{cases} 2\lceil n/3 \rceil + 1, & \textit{if } n \equiv 1 \pmod 3, \\ 2\lceil n/3 \rceil + 2, & \textit{otherwise.} \end{cases}$$

Proof. To prove the lower bound, we use the pattern to construct a set I, shown in Figure 3. Note that I is an 2-independent set of $P_n\overline{P_n}$ with the stated cardinality. Moreover, by Proposition 3.1, $|I| = \alpha_2(P_n) + \alpha_2(\overline{P_n}) - 1$ if $n \equiv 2 \pmod 3$, and $|I| = \alpha_2(P_n) + \alpha_2(\overline{P_n})$, otherwise.

Now we prove the upper bound. Since by Theorem 2.2, $\alpha_2(P_n\overline{P_n}) \leq \alpha_2(P_n) + \alpha_2(\overline{P_n})$, we need to prove only the case $n \equiv 2 \pmod 3$. By Theorem 2.3, it follows that $\alpha_2(P_n\overline{P_n}) = \alpha_2(P_n) + \alpha_2(\overline{P_n})$ if and only if there exist vertex sets S and T in $V(P_n)$ such that S is an $\alpha_2(P_n)$-set and T induces a maximum $K^{\leq 2}_{m_1,\dots,m_t}$ graph of P_n, with $|S \cap T| \leq 1$. For the graph P_n, the set T, as described, contains three consecutive vertices, which induces a $K_{1,2}$ graph. When $n \equiv 2 \pmod 3$, there is only one possible choice for the vertices of S. Moreover, for any choice of composition of T, $|S \cap T| = 2$. Therefore, there are no choices of S and T that satisfy Theorem 2.3. Thus, for $n \equiv 2 \pmod 3$, $\alpha_2(P_n\overline{P_n}) \leq \alpha_2(P_n) + \alpha_2(\overline{P_n}) - 1$.

The proof for $C_n\overline{C_n}$ is similar to the previous case. A set I' can be constructed from the set I, shown in Figure 3, by removing the vertex v_n when $n \equiv 0, 1 \pmod 3$. Note that I' has the stated cardinality. Moreover, by Proposition 3.1,

$|I'| = \alpha_2(C_n) + \alpha_2(\overline{C_n}) - 1$ if $n \equiv 0 \pmod 3$, and $|I'| = \alpha_2(C_n) + \alpha_2(\overline{C_n})$, otherwise. If $n \equiv 0 \pmod 3$, $C_n\overline{C_n}$ does not satisfy the conditions of Theorem 2.3, implying that, in this case, $\alpha_2(C_n\overline{C_n}) \leq \alpha_2(C_n) + \alpha_2(\overline{C_n}) - 1$. ■



(a) $n \equiv 2 \pmod 3$.

(b) $n \equiv 0 \pmod 3$.

(c) $n \equiv 1 \pmod 3$.

Figure 3: The black vertices form a set I that is an $\alpha_2(P_n\overline{P_n})$-set, for $n \geq 5$. The first block of each case can be added as many times as desired. The edges of $\overline{P_n}$ are omitted.

4 Future work

Kogan [12] showed that $\alpha_k(G) \geq \frac{kn}{d+k}$, for a graph G on n vertices and average degree d. The complementary prism $G\overline{G}$ of a graph G on n vertices has average degree $d = \frac{n+1}{2}$ and $2n$ vertices. We can apply this lower bound for a complementary prism $G\overline{G}$, resulting in $\alpha_k(G\overline{G}) \geq \frac{4kn}{n+2k+1}$, which is not a tight bound. As immediate future work, we are working on an improvement of this bound. The characterization presented in Theorem 2.3 can not be computed in polynomial time. Another extension would be showing a polynomial-time characterization for the upper and lower bounds of Theorem 2.2.

References

[1] H. Aram, N. Dehgardi, S Sheikholeslami, M. Valinavaz, L. Volkmann, Domination, independent domination number and 2-independence number in trees, Discussiones Mathematicae Graph Theory, Volume 41, 2021.

[2] R. M. Barbosa, M. R. Cappelle, E. M. M. Coelho, Maximal independent sets in complementary prism graphs, Ars Comb., Volume 137, 2018.

[3] M. Blidia, M. Chellali, O. Favaron, N. Meddah, Maximal *k*-independent sets in graphs, Discussiones Mathematicae Graph Theory, Volume 28, 2008.

[4] J. A.Bondy, U. S. R. Murty. Graph theory with applications. Vol. 290. London: Macmillan, 1976.

[5] Y. Caro, A. Hansberg, New Approach to the *k*-Independence Number of a Graph, Electronic Journal of Combinatorics, Volume 20, 2012.

[6] M. Chellali, O. Favaron, A. Hansberg, L. Volkmann, *k*-Domination and *k*-Independence in Graphs: A Survey, Graphs and Combinatorics, Volume 28, 2012.

[7] M. Chellali, T. Haynes, L. Volkmann, *k*-Independence stable graphs upon edge removal, Discussiones Mathematicae. Graph Theory, Volume 2, 2010.

[8] M. A. Duarte, L. Penso, D. Rautenbach, U. S. Souza, Complexity Properties of Complementary Prisms, J. Comb. Optim., Volume 33, 2017.

[9] O. Favaron, S. Hedetniemi, S.T. Hedetniemi, D. Rall, On *k*-dependent domination, Discrete Mathematics, Volume 249, 2002.

[10] J. Fink, M. Jacobson, *n*-domination, *n*-dependence and forbidden subgraphs, Graph Theory with Applications to Algorithms and Computer, 1985.

[11] T. W. Haynes, M. A. Henning, P. J. Slater, L. C. Merwe, The complementary Product of Two Graphs, Bulletin of the Institute of Combinatorics and its Applications, Volume 51, 2007.

[12] S. Kogan, New Results on k-Independence of Graphs, The Electronic Journal of Combinatorics, Volume 24, 2017.

[13] Y. Mao, E. Cheng, Z. Wang, Z. Guo, The k-independence number of graph products, Art Discret. Appl. Math., Volume 1, 2018.

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