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A tribute to Célia P. de Mello

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Abstract

The fruitful and long lasting collaboration of researchers Célia P. de Mello and Aurora Morgana is remembered.

1 Introduction

In a tribute to Célia P. de Mello on the occasion of her birthday, this extended abstract is organized as follows. Section 2 opens the text with Aurora Morgana's speech during LAWCG 2020, followed by sections on research areas of their collaboration: Section 3 on edge coloring of split graphs, Section 4 on well covered graphs, Section 5 on graph sandwich problem, Section 6 on clique operator, and Section 7 on embedding of plane graphs. Section 8 ends the text with an Acknowledgment.

2 Aurora Morgana's speech

Good morning to everyone and many thanks for inviting me to participate in this workshop dedicated to Célia.

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I met Célia more than 20 years ago. I accompanied my husband to Campinas. He had been invited to develop projects at Unicamp's Mathematics Institute. I wanted to meet someone who worked in my field. That's how they introduced me to Célia. I met her in her office, where I never imagined that I would spend a month every year for the next 10 years. It was the best collaboration I've ever had. It was all easy, nice as if we had always studied together. When I was in Campinas or Célia in Rome, we worked hard, sometimes forgetting to eat. And it was Célia who at one point said: "Aurora, don't you want to eat something? It's already two o'clock!".

Then, to our collaboration participated Sheila, Célia's student at that time. She was young, full of enthusiasm and energy and her contribution was fundamental in studying the problem of coloring the edges of split graphs. We spent the day discussing, proposing new ideas to solve the problem, sometimes happy because everything seemed right to unfortunately discover the next day that something was wrong. The day passed quickly and I remember Sheila who, in the late afternoon, sometimes said: "Hi girls, it wouldn't be good to go out for a drink".

We have not been able to prove the conjecture of Meidanis, Figueiredo and Mello, but we have given new evidence that the conjecture may be true.

Rereading these articles now, I think it would be worth trying to prove the conjecture with new ideas.

Other works were made possible with the participation of other friends that I will mention in my brief summary of our collaboration. After I retired I continued to return to Campinas every year and with Célia I have traveled to different places in Brazil, that Brazil that I love. Years later, when I almost forgot about graphs and cliques, I asked Célia if she thought it was worth spending all that time working on problems that I couldn't even remember.

Célia responded by saying: first this job gave us a salary, then it allowed us to teach something to the youngest and it was pleasant, furthermore we would never meet and our great friendship would not have been possible. I was convinced: it was worth it!

One of the merits of the research is that it favors the meeting of people who live in distant countries and allows that the knowledge of the culture and history of those countries make us feel part of the same small, big world that politics often tries to divide.

This year the pandemic prevented me from going to Brazil, but as soon as it ends we will meet again in Campinas or Rome.

Célia, thank you for the wonderful time we spent together and see you soon.

P.S. This year the pandemic virus made impossible to meet all of you in this meeting but, as soon as it will be over, I will come again to visit your wonderful Brazil as it happened in the past 20 years.

3 Edge coloring of split graphs

A *k-edge-coloring* of a graph G is an assignment of one of k colors to each edge of G such that no two edges with the same color are incident to a common vertex. The *chromatic index* of G, $\chi'(G)$, is the minimum k such that G has a k-edge-coloring. An easy lower bound for the chromatic index of a graph G is the maximum vertex degree $\Delta(G)$. A celebrated theorem by Vizing [20] states that, for a simple graph G, the chromatic index is at most $\Delta(G)+1$. It was the origin of the *Classification Problem*, that consists of deciding whether a given graph G has $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. In the first case, we say that G is *Class 1*, otherwise, we say that G is *Class 2*. Despite the powerful restriction imposed by Vizing, it is very hard to compute the chromatic index in general [10].

A graph G with n vertices and m edges is *overfull* if n is odd and $m > \Delta(G)\frac{(n-1)}{2}$. A graph G is *subgraph-overfull* when it has an overfull subgraph H with $\Delta(H) = \Delta(G)$. If the overfull subgraph H is a subgraph induced by the neighborhood of a $\Delta(G)$-vertex, then G is *neighborhood-overfull*. Overfull, subgraph-overfull and neighborhood-overfull graphs are

Class 2 graphs.

A *split graph* is a graph whose vertex set admits a partition into an independent set and a clique. Figueiredo, Meidanis and de Mello [9] proved that every overfull split graph is neighborhood-overfull and posed the following conjecture.

Conjecture 1 (Figueiredo, Meidanis e Mello [9])**.** Every Class 2 split graph is neighborhood-overfull.

A series of papers investigated Conjecture 1 for the class of split graphs. First some wide subclasses of split graphs were considered and then it was given a proof that the graphs belonging to these subclasses are Class 1 if and only if they are not neighborhood-overfull. This result gives another evidence that Conjecture 1 may be true [1, 4]. Polynomial-time algorithms for $\Delta(G)$-edge-coloring non neighborhood-overfull split graphs of these subclasses were additionally presented [1, 4].

Then, a characterization of split graphs that are neighborhood-overfull was presented [2, 3]. If the Conjecture 1 were true they would be the unique split graphs in Class 2.

4 Well covered graphs

A graph G is called *well covered* if every two maximal independent sets of G have the same number of vertices. The maximum number of vertices in a maximal independent set of G is the independence number of G, denoted by $\alpha(G)$. Given a graph G and an integer number $k \geq 0$, to decide if $\alpha(G) \leq k$ is NP-complete. An interesting algorithm property of well covered graphs is that the greedy algorithm for producing a maximal independent set always produces a maximum independent set when applied to well covered graphs. The problem of deciding if a graph is well covered is coNP-complete, this was independently proved by Chvátal and Slater [5] and by Sankaranarayana and Stewart [19]. Based on the primeval decomposition techniques [11], that is a refinement of the modular decomposition, it is possible to construct a recursive algorithm for some special classes of graphs that for each module verifies if it is well covered and computes its independence number; otherwise stops saying that the graph is not well covered.

A graph G is P_4-sparse if every p-connected component of G is isomorphic to a spider and it is P_4-*tidy* if and only if its p-connected components are isomorphic to a P_5 or a C_5 or a spider or a fat spider. A *cograph* is a graph with no induced subgraphs isomorphic to a chordless path on four vertices P_4.

We used the primeval decomposition to decide well coveredness of graphs that belong to the class of cographs, P_4-reducible, P_4-sparse, extended P_4-reducible, extended P_4-sparse graphs, P_4-extensible graphs, P_4-lite graphs, and P_4-tidy graphs [12]. The above classes are partially ordered by inclusion in the class of tidy graphs, as shown in the Figure 1.

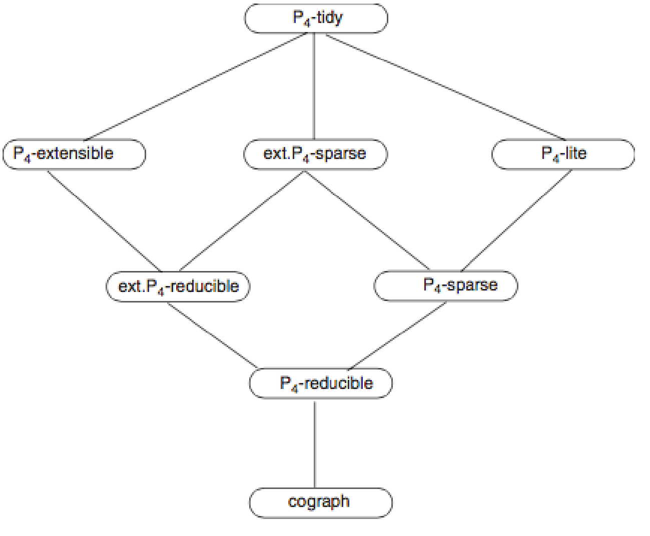

Figure 1: Extensions of the class of cographs

5 Graph Sandwich Problem

The *P_4-sparse Graph Sandwich Problem* asks, given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, whether there exists a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ and G is P_4-sparse. Again the modular decomposition is used to produce a polynomial-time algorithm for solving the Graph Sandwich Problem for P_4-sparse graphs [6, 7].

6 Clique operator

The clique graph of a graph G is the intersection graph $K(G)$ of the (maximal) cliques of G. The iterated clique graphs $K_n(G)$ are denoted by $K_0(G) = G$ and $K_i(G) = K(K_{i-1}(G))$, $i > 0$ and K is the clique operator.

A series of papers investigated the behavior of the clique operator [13, 15, 16, 18]. Graphs behave in a variety of ways under the iteration of the clique operator K, the main distinction being between K-convergence and K-divergence. A graph G is said to be K-divergent if $\lim_{n\to\infty} |V(K_n(G))| = \infty$. If G is not K-divergent, then it is K-convergent. Complete bipartite graphs K_{p_1,p_2} are K-convergent. Complete multipartite graphs $K_{p_1,\ldots,p_n}$, with $n \geq 3$ and $p_i \geq 2$; $i = 1, \ldots, n$ are K-divergent with super exponential growth.

Most of the results on convergence of iterated clique graphs are on the domain of clique-Helly graphs. A graph is *clique-Helly* if its (maximal) cliques satisfy the *Helly property*: each family of mutually intersecting cliques has non-trivial intersection. Clique-Helly graphs are K-convergent and can be recognized in polynomial time [8].

In a first paper [13] the K behavior of cographs was studied. The modular decomposition technique is used to characterize the K-behavior of cographs and to give some partial results for the larger class of serial (i.e. complement-disconnected) graphs. The following was proved:

Theorem 6.1. *[13] A cograph is K-convergent if and only if it is clique-Helly.*

This characterization leads to a polynomial-time algorithm for deciding the K-convergence or K-divergence of any cograph.

In a second paper [15] some natural extensions of the class of cographs were considered, namely the classes of P_4-reducible, P_4-sparse, extended P_4-reducible, extended P_4-sparse graphs, P_4-extensible graphs, P_4-lite graphs and P_4-tidy graphs (See Figure 1). The modular decomposition technique was used again to obtain a characterization of the K-behavior of all the classes under consideration.

If H is a subgraph of G obtained from G by iterated elimination of all dominated vertices in each p-connected component of G, then H is a *strong N-retract* of G. The following was proved:

Theorem 6.2. *[16] A connected P_4-tidy graph G is K-convergent if and only if its strong N-retract H is a clique-Helly graph.*

This characterization leads again to polynomial-time algorithms for deciding the K-behavior of any graph in the class [16].

7 Embedding of plane graphs

By a rectilinear embedding of a plane graph G we shall mean a plane graph with the same infinite face, where vertices are mapped into grid points and an edge uv consists of an alternate sequence of horizontal and vertical grid paths joining u and v on the grid (see Figure 2).

In [14] the following results are proved:

- Every standard plane graph is 3-embeddable.
- Every standard graph, with the only exception of the octahedron is 2-embeddable.

In [17] forbidden configurations for the 1-embedding of a graph are characterized and an algorithm is provided that either detects a forbidden configuration or generates a 1-rectilinear embedding, in $O(n^2)$ time.

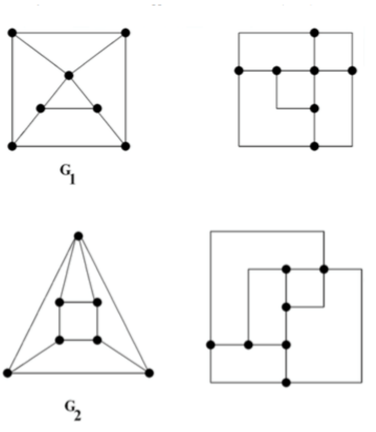

Figure 2: The plane graph G_1 has a 1-rectilinear embedding while G_2 does not have one.

8 Acknowledgment

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