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The Strict Terminal Connection Problem on Chordal Bipartite Graphs

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Abstract

A *strict connection tree* T of a graph G for a non-empty subset $W \subseteq V(G)$, called *terminal set*, is a tree subgraph of G whose leaf set coincides with W. A *non-terminal* vertex $v \in V(T) \setminus W$ is called *linker* if its degree in T is exactly 2, and it is called *router* if its degree in T is at least 3. Given a graph G, a terminal set $W \subseteq V(G)$ and two non-negative integers ℓ and r, the STRICT TERMINAL CONNECTION PROBLEM (S-TCP) asks whether G admits a strict connection tree for W with at most ℓ linkers and at most r routers. In the present extended abstract, we prove that S-TCP is NP-complete on chordal bipartite graphs even if ℓ is bounded by a constant.

1 Introduction

STEINER TREE is one of the most fundamental problems in graph theory

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and combinatorial optimization, being related to many real-world applications. In this work, we study the complexity of the so-called STRICT TERMINAL CONNECTION problem, which is a natural variation of STEINER TREE introduced by Dourado et al. [1] motivated by questions in information security, network routing and telecommunication.

Let G be a graph and $W \subseteq V(G)$ be a non-empty set, called *terminal* set. A *strict connection tree* of G for W is a tree subgraph of G whose leaf set is equal to W. A non-terminal vertex of a strict connection tree T is called *linker* if its degree in T is exactly 2, and it is called *router* if its degree in T is at least 3. We remark that the vertex set of every strict connection tree can be partitioned into terminal vertices, linkers and routers. For each strict connection tree T, we let $\mathsf{L}(T)$ denote the linker set of T and $\mathsf{R}(T)$ denote the router set of T. Next, we formally define the STRICT TERMINAL CONNECTION problem.

STRICT TERMINAL CONNECTION (S-TCP)

Input: A graph G, a non-empty terminal set $W \subseteq V(G)$ and two non-negative integers ℓ and r.

Question: Does there exist a strict connection tree T of G for W, such that $|\mathsf{L}(T)| \leq \ell$ and $|\mathsf{R}(T)| \leq r$?

Table 1 summarises the known complexity results of S-TCP with respect to the parameters $\ell, r, \Delta(G)$, and the classes of split graphs and cographs. In addition to these results, in [4], S-TCP was studied from the perspective of disjoint paths and integral commodity flow problems.

	Parameters				
Graph class	–	ℓ	r	ℓ, r	$\ell, r, \Delta(\mathbf{G})$
General	NPC [1]	NPC [1]	P for $r \in \{0,1\}$ [2] but W[2]h [3]	XP [1] but W[2]h [3]	FPT [1, 3] but No-poly kernel [3]
$\Delta = 4$	NPC [3]	NPC[3]	P for $r \in \{0,1\}$ [2]	FPT [1, 3]	FPT [1, 3]
$\Delta = 3$	NPC [3]	XP [3]	P for $r \in \{0,1\}$ [2]	FPT [1, 3]	FPT [1, 3]
Split	NPC [3]	NPC [3]	XP [3] but W[2]h [3]	XP [1, 3] but W[2]h [3]	FPT [1, 3]
Cographs	P [3]	P [3]	P [3]	P [3]	P [3]

Table 1: Computational complexity of S-TCP. (Adapted from [3].)

Contribution. In this work, we prove that S-TCP remains NP-complete when restricted to *chordal bipartite graphs*, even if $\ell \geq 0$ is bounded by a constant.

2 S-TCP on Chordal Bipartite Graphs

A graph G is called *chordal bipartite* if every induced cycle of G has length 4. Equivalently, a graph G is chordal bipartite if G is bipartite and every cycle of G of length at least 6 has a *chord*, i.e. an edge between two non-consecutive vertices of the cycle.

To prove that S-TCP is NP-complete on chordal bipartite graphs, we present a polynomial-time reduction from Vertex Cover, which has as input a graph G and a positive integer k and asks whether there is a subset $S \subseteq V(G)$ such that $|S| \leq k$ and every edge of G has an endpoint in S. The proposed reduction, described next, is based on the polynomial-time reduction given by Müller and Brandstädt [5] so as to prove that Steiner tree is NP-complete on chordal bipartite graphs.

Construction. Let $I = (G, k)$ be an instance of Vertex cover and $c \geq 0$ be a constant. Assume that $V(G) = \{v_1, \ldots, v_n\}$ for some positive integer $n \geq 2$. Moreover, assume that G has at least one edge, i.e. $m = |E(G)| \geq 1$. We let $f(I, c) = (H, W, \ell = c, r)$ be the instance of S-TCP defined as follows.

1. For each $v_i \in V(G)$, create the gadget H_i as illustrated in Figure 1.

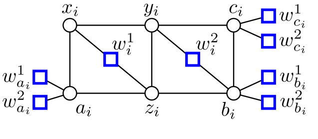

Figure 1: Gadget H_i.

2. Subdivide the edge $w^1_{a_1}a_1$ of H_1 into ℓ new vertices $u_1, u_2, \ldots, u_\ell$, creating the induced path $\langle w^1_{a_1}, u_1, \ldots, u_\ell, a_1\rangle$.

3. For each pair $v_i, v_j \in V(G)$, with $i \neq j$, add the edges x_iy_j and z_iy_j, making the subgraph of H induced by $X \cup Y \cup Z$ a complete bipartite graph with bipartition $(X \cup Z, Y)$, where $X = \{x_i \mid v_i \in V(G)\}$, $Y = \{y_i \mid v_i \in V(G)\}$ and $Z = \{z_i \mid v_i \in V(G)\}$.

4. For each $v_iv_j \in E(G)$, create the gadgets H_{ij} and H_{ji} as illustrated in Figure 2.

(a) H_{ij} (b) H_{ji}

Figure 2: Gadgets H_{ij} and H_{ji}, respectively.

5. Finally, define $W = W_1 \cup W_2 \cup W_3$ and $r = k + 4n + 4m$, where $W_1 = \{w^1_i, w^2_i \mid v_i \in V(G)\}$, $W_2 = \{w^1_{a_i}, w^2_{a_i}, w^1_{b_i}, w^2_{b_i}, w^1_{c_i}, w^2_{c_i} \mid v_i \in V(G)\}$, and $W_3 = \{w^1_{p_{ij}}, w^2_{p_{ij}}, w^1_{q_{ij}}, w^2_{q_{ij}} \mid v_iv_j \in E(G)\}$.

Lemma 2.1. *Let $I = (G, k)$ be an instance of* Vertex Cover*, such that G has at least one edge. For every $c \geq 0$, the graph H of $f(I, c)$ is chordal bipartite.*

Proof. First, we note that H is chordal bipartite if and only if the graph $G' = H - (W_2 \cup W_3)$ is chordal bipartite. Indeed, the vertices belonging to $W_2 \cup W_3$ are vertices of degree 1 of H, and therefore they do not belong to any cycle of H. Consequently, in order to prove this lemma, it is sufficient to show that G' is chordal bipartite. Note that, for every $v_i \in V(G)$, w^1_i is a false twin of a_i in G', i.e. $N_{G'}(w^1_i) = N_{G'}(a_i)$. Similarly, for every $v_i \in V(G)$, w^2_i is a false twin of c_i in G', i.e. $N_{G'}(w^2_i) = N_{G'}(c_i)$. As a result, if w^1_i or w^2_i belongs to an odd cycle or to an induced cycle of length greater than or equal to 6 in G', then certainly a_i or c_i, respectively, also belongs

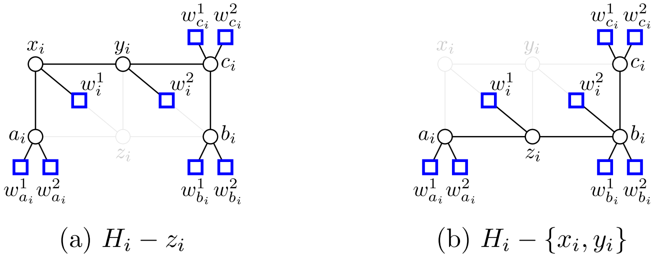

Figure 3: Subgraphs $H_i - z_i$ and $H_i - \{x_i, y_i\}$, respectively.

to an odd cycle or to an induced cycle of length greater than or equal to 6 in G'. Therefore, it follows from the fact that $H - (W_1 \cup W_2 \cup W_3)$ is chordal bipartite [5] that G' (and, thus, H) is chordal bipartite as well. ■

Lemma 2.2. *Let $I = (G, k)$ be an instance of* Vertex Cover*, such that G has at least one edge. For every $c \geq 0$, I is a* yes*-instance of* Vertex cover *if and only if $f(I, c)$ is a* yes*-instance of S-TCP.*

Proof. First, suppose that I is a yes-instance of Vertex cover, and let $S \subseteq V(G)$ be a vertex cover of G such that $|S| \leq k$. Based on S, we construct a strict connection tree T of H for W as described below.

1. For each $v_i \in V(G)$, if $v_i \in S$, then add the subgraph $H_i - z_i$ (see Figure 3a) to T; on the other hand, if $v_i \notin S$, then add the subgraph $H_i - \{x_i, y_i\}$ (see Figure 3b) to T.

2. For each $v_iv_j \in E(G)$ with $v_i \in S$, if $v_j \notin S$ or $i < j$, then add the subgraphs $H_{ij} - y_j$ (see Figure 4a) and $H_{ji} - x_j$ (see Figure 4b) to T. We remark that, possibly, the pairs of vertices x_i and y_j, and y_i and x_j, simultaneously belong to $V(T)$. However, if $x_ip_{ij} \in E(T)$ or $y_iq_{ji} \in E(T)$, then $x_iy_j, y_jq_{ij} \notin E(T)$ and $x_jy_i, x_jp_{ji} \notin E(T)$.

One can verify that, until the last step, T is an acyclic subgraph of H that contains all the terminal vertices belonging to W. Thus, in order to conclude the construction of T, we only need to connect the connected components of T in such a way that the resulting graph is still an acyclic

(a) $H_{ij} - y_j$ (b) $H_{ji} - x_j$

Figure 4: Subgraphs $H_{ij} - y_j$ and $H_{ji} - x_j$, respectively.

subgraph of H. Since $|E(G)| \geq 1$, $|S| \geq 1$. As a result, $Y_S = \{y_i \mid v_i \in S\}$ is non-empty. Then, let y_α be a vertex in Y_S, arbitrarily chosen. It follows from the construction of G that y_α is adjacent in G to all vertices belonging to $X_S \cup Z_S$, where $X_S = \{x_i \mid v_i \in S\}$ and $Z_S = \{z_i \mid v_i \notin S\}$. Moreover, note that all connected components of T necessarily have at least one vertex in $X_S \cup Z_S$. Thus, to conclude the construction of T, we perform the following operation:

3. For each connected component T' of T which does not contain the vertex y_α, select arbitrarily a vertex $v \in (X_S \cup Z_S) \cap V(T')$ and, then, add the edge vy_α to T.

Then, we have finally obtained a subgraph T of H which is a tree and contains all the terminal vertices belonging to W. Moreover, note that $\mathsf{L}(T) = \{u_1, u_2, \ldots, u_\ell\}$ and

$$\begin{aligned}\mathsf{R}(T) = \{x_i, y_i \mid v_i \in S\} &\cup \{z_i \mid v_i \notin S, v_i \in V(G)\} \\ &\cup \{a_i, b_i, c_i \mid v_i \in V(G)\} \cup \{p_{ij}, p_{ji}, q_{ij}, q_{ji} \mid v_iv_j \in E(G)\}.\end{aligned}$$

Hence, T is a strict connection tree of G for W such that $|\mathsf{L}(T)| = \ell$ and $|\mathsf{R}(T)| = 2|S| + (n - |S|) + 3n + 4m \leq k + 4n + 4m = r$. Therefore, $f(I, c)$ is a yes-instance of S-TCP.

Conversely, suppose that G admits a strict connection tree T for W such that $|\mathsf{L}(T)| \leq \ell$ and $|\mathsf{R}(T)| \leq r = k + 4n + 4m$. Note that, by construction of H, the only path in H between the terminal vertices $w^1_{a_1}$

and $w^2_{a_1}$ contains the non-terminal vertices $u_1, u_2, \ldots, u_\ell$. Besides that, $d_H(u_i) = 2$ for every $i \in \{1, \ldots, \ell\}$. As a result, $\mathsf{L}(T) = \{u_1, u_2, \ldots, u_\ell\}$. This implies that all the other non-terminal vertices of T must be routers. In addition, for every $v_i \in V(G)$, we have that $a_i \in \mathsf{R}(T)$, since a_i is the only neighbour in H of the terminal vertices $w^1_{a_i}$ and $w^2_{a_i}$ and, thus, a_i necessarily belongs to $V(T)$. Analogously, we have that $b_i, c_i \in \mathsf{R}(T)$.

Claim 2.1. Let $v_i \in V(G)$ and T_i be the subgraph of T induced by $V(H_i)$. If $y_i \in V(T)$, then we can assume that the degree of y_i in T_i is at least 3.

Proof. First, we note that every path in H between y_i and a_i contains x_i or z_i. Consequently, x_i or z_i must belong to the path P in T between y_i and a_i. It is not hard to verify that, if $x_i \in V(P)$, then we can assume that $x_i, w^2_i, c_i \in N_T(y_i)$. On the other hand, if $z_i \in V(P)$, then we can assume that $z_i, w^2_i, c_i \in N_T(y_i)$. ■

Claim 2.2. Let $v_i \in V(G)$. If $x_i \in V(T)$, then we can assume that $y_i \in V(T)$. Analogously, if $y_i \in V(T)$, then we can assume that $x_i \in V(T)$.

Proof. Suppose that $x_i \in V(T)$ but $y_i \notin V(T)$. The case in which $y_i \in V(T)$ but $x_i \notin V(T)$ is analogous. Note that, the path in T between a_i and c_i must contain z_i, which must be a router of T. Furthermore, by the previous claim, we can assume that, for every vertex $y_j \in N_T(z_i)$ with $j \neq i$, the degree of y_j in T_j is at least 3, where T_j denotes the subgraph of T induced by $V(H_j)$. Thus, let T' be the graph with vertex set $V(T') = V(T) \setminus \{z_i\} \cup \{y_i\}$ and edge set

$$\begin{aligned} E(T') = E(T) \setminus \big(\{vz_i \mid v \in N_T(z_i)\} \cup \{w^2_i b_i\}\big) \\ \cup E(H_i - z_i) \cup \{y_j x_i \mid y_j \in N_T(z_i), j \neq i\}. \end{aligned}$$

One can verify that T' is a strict connection tree of H for W that simultaneously contains the vertices x_i and y_i and satisfies the constraints $|\mathsf{L}(T')| \leq \ell$ and $|\mathsf{R}(T')| \leq r$; more precisely, $\mathsf{L}(T') = \mathsf{L}(T)$ and $\mathsf{R}(T') = (\mathsf{R}(T) \setminus \{z_i\}) \cup \{y_i\}$. ■

Thus, consider the subset $S = \{v_i \in V(G) \mid x_i, y_i \in V(T)\}$. We claim that S is a vertex cover of G. For the sake of contradiction, suppose that there exists an edge $e = v_iv_j \in E(G)$ such that $S \cap \{v_i, v_j\} = \emptyset$. Consequently, $x_i, y_i \notin V(T)$ and $x_j, y_j \notin V(T)$. Moreover, we have that $w^1_{p_{ij}}, w^2_{p_{ij}}, w^1_{q_{ij}}, w^2_{q_{ij}} \notin V(T)$ (as well as $w^1_{p_{ji}}, w^2_{p_{ji}}, w^1_{q_{ji}}, w^2_{q_{ji}} \notin V(T)$), since T is connected and the only path in T between such terminals and any other terminal belonging to W — for example, the terminals in $W_1 \cup W_2$ — necessarily contains x_i or y_j (x_j or y_i, respectively). However, this contradicts the hypothesis that $W_3 \subseteq W \subseteq V(T)$. As a result, such an edge e cannot exist. In other words, S is a vertex cover of G. Finally, note that, if $|S| = k'$, then

$$|\mathsf{R}(T)| = 2k' + (n - k') + 3n + 4m = k' + 4n + 4m \leq r = k + 4n + 4m,$$

which implies $|S| \leq k$. Therefore, I is a yes-instance of Vertex cover. ■

Theorem 2.1. *S-TCP remains* NP*-complete when restricted to chordal bipartite graphs, even if ℓ is bounded by a constant.*

Proof. This result follows from Lemmas 2.1 and 2.2 and from the fact that the construction f can be computed in polynomial-time over the input size of the given instance I of Vertex cover and the parameter ℓ. ■

3 Concluding Remarks

In the present extended abstract, we have proved that S-TCP is NP-complete on chordal bipartite graphs even if ℓ is bounded by a constant. On the other hand, it remains unknown whether S-TCP can be solved in polynomial-time on chordal bipartite graphs if r is bounded by a constant (and ℓ is arbitrarily large). More generally, one of the main questions concerning S-TCP is whether the problem parameterized by r is in XP.

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