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Equitable Partition of Graphs into Independent Sets and Cliques

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Abstract

A graph is $(\boldsymbol{k}, \boldsymbol{\ell})$ if its vertex set can be partitioned into k independent sets and l cliques. Deciding if a graph is $(\boldsymbol{k}, \boldsymbol{\ell})$ can be seen as a generalization of coloring, since deciding if a graph belongs to $(\boldsymbol{k}, \mathbf{0})$ corresponds to deciding if a graph is k-colorable. A coloring is equitable if the cardinalities of color classes differ by at most 1. In this paper, we generalize the equitable coloring problem, by showing that deciding whether a given graph can be equitably partitioned into k independent sets and ℓ cliques is solvable in polynomial time if $\max(k, \ell) \leq 2$, and NP-complete otherwise.

1 Introduction

The *Vertex Coloring Problem* (VCP) consists of partitioning the vertex set of a graph into k disjoint sets, such that no edges has both endpoints in the same set. This problem is known to be NP-complete even for fixed $k \geq 3$ [1]. A widely studied variation of this problem is the *Equitable Coloring Problem* (ECP), in which we want the cardinalities of the sets

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to differ by at most 1. The latter is also NP-Complete, since it can be reduced from VCP by adding sufficiently many isolated vertices.

Let $\boldsymbol{(k,\ell)}$ be the set of graphs that can be partitioned into k independent sets and ℓ cliques. Recognizing $\boldsymbol{(k,\ell)}$ graphs can be seen as a generalization of the VCP, since deciding if a graph belongs to $\boldsymbol{(k,0)}$ corresponds to deciding if the graph is k-colorable. This problem was shown to be solvable in polynomial time if $\max(k,l) \leq 2$, and NP-complete otherwise [2, 3, 4]. Besides the recognition problem, $\boldsymbol{(k,\ell)}$ graphs and their subclasses have been the focus of recent research for a variety of problems [5, 6, 7].

In this paper we introduce the equitable version of the $\boldsymbol{(k,\ell)}$ problem: we want the cardinalities of the sets of the partition to be as close as possible. For that, we define $\boldsymbol{(k,\ell)_E}$ as the set of graphs that can be equitably partitioned into k independent sets and ℓ cliques, that is, the cardinalities of the sets differ by at most 1. It is easy to see that this is a generalization of ECP.

If $\overline{G}$ is the complement graph of G, then $G \in \boldsymbol{(k,\ell)_E}$ if and only if $\overline{G} \in \boldsymbol{(\ell,k)_E}$. Checking whether $G \in \boldsymbol{(1,0)_E}$ is to check if G has no edges, and to check whether $G \in \boldsymbol{(0,1)_E}$ is to check if G is complete. Both can be easily done in linear time.

2 Recognition of $(1,1)_E$ Graphs

Let $G = (V,E)$ be a graph, and $x \in V$ be one of its vertices. Let $N(x)$ be the neighborhood of x and $\overline{N}(x)$ be the set of non-neighbors of x. We will also denote by $N[x]$ the set $N(x) \cup x$ and by $\overline{N}[x]$ the set $\overline{N}(x) \cup x$. We say that a set $S \subset V$ is an *independent set* if there is no edge between its vertices, and a *clique* if there is an edge between any pair of its vertices.

Initially we try to find a partition of V into an independent set I and a clique C. For any $x \in V$, if $x \in I$, then it does not have a neighbor in I. Therefore, $x \in I \Rightarrow N(x) \subseteq C$, that is, $N(x) \in \boldsymbol{(0,1)}$. Similarly, if $x \in C$, then it is adjacent to all other vertices in C: $x \in C \Rightarrow \overline{N}(x) \in \boldsymbol{(1,0)}$.

So we can check, for every vertex x from G, if **1)** $N(x) \in \mathbf{(0,1)}$ and **2)** $\overline{N}(x) \in \mathbf{(1,0)}$.

If there is a vertex that does not satisfy any of the previous conditions, then $G \notin \mathbf{(1,1)_{\mathit{E}}}$. If there is a vertex x such that both **1** and **2** hold, then we can observe that $\overline{N}[x] \in \mathbf{(1,0)}$, and $N[x] \in \mathbf{(0,1)}$. Now it is enough to insert x in the set that has fewer elements, because two vertices from $N[x]$ cannot be at the same time in an independent set, and, similarly, two vertices from $\overline{N}[x]$ cannot be at the same time in a clique. By doing this, we get the sets $I = \overline{N}[x]$ and $C = N(x)$, or $I = \overline{N}(x)$ and $C = N[x]$. If the partition is equitable, then $G \in \mathbf{(1,1)_{\mathit{E}}}$. Otherwise, $G \notin \mathbf{(1,1)_{\mathit{E}}}$.

If every vertex satisfy exactly one of the conditions, we can construct the sets I and C in the following manner: for each vertex $x \in G$, if x satisfies condition **1**, insert it in I. If x satisfies condition **2**, insert it in C. If at the end of the construction I induces an empty subgraph, C induces a complete subgraph and $||I| - |C|| \leq 1$, then $G \in \mathbf{(1,1)_{\mathit{E}}}$. Otherwise, $G \notin \mathbf{(1,1)_{\mathit{E}}}$.

Checking conditions **1** and **2** can be easily done in $\mathcal{O}(|V|^2)$ time. Hence, deciding if $G \in \mathbf{(1,1)_{\mathit{E}}}$ can be solved in $\mathcal{O}(|V|^3)$ time.

It should be noted that $\mathbf{(1,1)}$ graphs are also known as split graphs. A more efficient recognition of $\mathbf{(1,1)_{\mathit{E}}}$ graphs can be found using well-known algorithms for recognizing split graphs, and finding a suitable partition in linear time [8]. Since our goal is just to show that it can be done in polynomial time, we decided to use a less efficient algorithm for the sake of self-containment and simplicity.

3 Recognition of $(2,0)_E$ Graphs

Deciding whether a graph $G = (V,E) \in \mathbf{(2,0)_{\mathit{E}}}$ is the same as deciding if G is equitably bipartite. It if folklore that this can be done in polynomial time, but for the sake of completeness we will prove this result here.

If G is connected we only have to check if it is bipartite, and if the partition is equitable, since it is unique. Otherwise, we initially have two possibilities of partitioning for each connected component. In this case, we first find a bipartition for each of the c connected components of G. If any one of them is not bipartite, then $G \notin \mathbf{(2,0)_E}$. Let x_i and y_i be the number of vertices in each side of the bipartition of the i-th connected component of G, assuming they are labeled from 1 to c.

Let G_i be the subgraph of G induced by components with label at least i. We define the function $f(i, S)$, that equals TRUE if there is a partition of $V(G_i)$ into two independent sets such that the number of vertices in one of the two sets is equal to S. Otherwise, $f(i, S)$ equals FALSE. Now we can notice that $f(1, \lfloor \frac{|V|}{2} \rfloor)$ answers if $G \in \mathbf{(2,0)_E}$, because at least one of the partitions must have size $\lfloor \frac{|V|}{2} \rfloor$. We show that this can be computed in polynomial time in Theorem 3.1.

Theorem 3.1. *Deciding if* $G = (V, E) \in \mathbf{(2,0)_E}$ *can be solved in* $\mathcal{O}(|V|^2)$ *time.*

Proof. To prove that $f(1, \lfloor \frac{|V|}{2} \rfloor)$ can be computed in $\mathcal{O}(|V|^2)$ time, we will show that we only have to compute $\mathcal{O}(|V|^2)$ values of f, and that each of them can be computed in constant time using previously computed quantities.

To compute $f(i, S)$, we check both possibilities of partition of the i-th connected component. Depending on which one we choose, the number of vertices we will need starting from the $(i+1)$-th connected component is either $S - x_i$ or $S - y_i$. The full algorithm to compute $f(i, S)$ is shown in Algorithm 1.

Since the values in the first parameter of f varies from 1 to c, which is $\mathcal{O}(|V|)$, and the second parameter varies from $-\lceil \frac{|V|}{2} \rceil$ to $\lfloor \frac{|V|}{2} \rfloor$, which is also $\mathcal{O}(|V|)$, computing $f(1, \lfloor \frac{|V|}{2} \rfloor)$ depends on $\mathcal{O}(|V|^2)$ previous values of f. Moreover, the computation of f takes $\mathcal{O}(1)$ time assuming the values it depends on are already computed. Therefore, as long as values are not recalculated, computing $f(1, \lfloor \frac{|V|}{2} \rfloor)$ can be done in $\mathcal{O}(|V|^2)$ time. ■

Algorithm 1 Equitable Bipartite Recognition

Input: number c of connected components; $(x_1, y_1), \ldots, (x_c, y_c)$, number of vertices in each set of the partition of each connected component.

```
function f(i, S)
    if i = c then
        if S = x_c ∨ S = y_c then return TRUE
        else return FALSE
    else return f(i + 1, S − x_i) ∨ f(i + 1, S − y_i)
```

4 Recognition of $(\mathbf{2,1})_{\boldsymbol{E}}$ Graphs

For the recognition of $(\mathbf{2,1})_{\boldsymbol{E}}$ graphs, we will use an algorithm to find a $(\mathbf{2,1})$ partition of G [2, 3, 4]. If $G \notin (\mathbf{2,1})$, then $G \notin (\mathbf{2,1})_{\boldsymbol{E}}$. Let $\{I_1, I_2, C\}$ be a partition of V such that I_1 and I_2 are independent sets and C is a clique. Let I_1', I_2', and C' be respectively the independent sets and the clique that equitably partition G, which are the sets we want to find. Since I_1 is an independent set, at most one vertex from I_1 can belong to C', because that C' is a clique. Similarly, at most two vertices of C can belong to $I_1' \cup I_2'$, each one in one of the sets.

Now, we look at all the possibilities of inserting I_1, I_2, and C in $I_1' \cup I_2'$ and C'. For each possibility, we check if $I_1' \cup I_2' \in (\mathbf{2,0})_{\boldsymbol{E}}$, if C' is a clique, and if the partition is equitable. If that is the case for any of the possibilities, then $G \in (\mathbf{2,1})_{\boldsymbol{E}}$. Otherwise, $G \notin (\mathbf{2,1})_{\boldsymbol{E}}$.

There are $\mathcal{O}(|V|)$ ways to choose one vertex from I_1, $\mathcal{O}(|V|)$ ways for I_2 and $\mathcal{O}(|V|^2)$ ways to choose two vertices from C. For every choice, checking whether $I_1' \cup I_2' \in (\mathbf{2,0})_{\boldsymbol{E}}$ can be done in $\mathcal{O}(|V|^2)$ time, from Theorem 3.1. It is trivial to check if $C' \in (\mathbf{0,1})$ in $\mathcal{O}(|V|^2)$. Hence, checking each of the $\mathcal{O}(|V|^4)$ possibilities is $\mathcal{O}(|V|^2)$, which gives a total of $\mathcal{O}(|V|^6)$. It is known a $(\mathbf{2,1})$ partition can be found in $\mathcal{O}(|V|^4)$ time [2, 3, 4], so deciding if $G \in (\mathbf{2,1})_{\boldsymbol{E}}$ can be done in $\mathcal{O}(|V|^6)$ time.

5 Recognition of $(\mathbf{2,2})_{\boldsymbol{E}}$ Graphs

We will use the same technique that was used in the $(\mathbf{2,1})_{\boldsymbol{E}}$ recognition for the $(\mathbf{2,2})_{\boldsymbol{E}}$ recognition. If $G \notin (\mathbf{2,2})$, then $G \notin (\mathbf{2,2})_{\boldsymbol{E}}$. Now suppose that $G \in (\mathbf{2,2})$. Let I_1, I_2, C_1, and C_2 be respectively the independent sets and the cliques that partition $V(G)$. We want to find sets I'_1, I'_2, C'_1, and C'_2, respectively the independent sets and cliques that equitably partition $V(G)$.

Since I_1 is an independent set, then at most two of its vertices can belong to $C'_1 \cup C'_2$, each one in one of the cliques, and the same is true for I_2. Moreover, since C_1 is a clique, then at most two of its vertices can belong to $I'_1 \cup I'_2$, each one in one of the sets, and that also holds for C_2. So, there are $\mathcal{O}(|V|^8)$ possibilities of insertion in $I'_1 \cup I'_2$ and $C'_1 \cup C'_2$. If any of them is valid, that is, if they make an equitable partition of G, then $G \in (\mathbf{2,2})_{\boldsymbol{E}}$. Otherwise, $G \notin (\mathbf{2,2})_{\boldsymbol{E}}$.

For every one of the $\mathcal{O}(|V|^8)$ possibilities, we check if $I'_1 \cup I'_2 \in (\mathbf{2,0})_{\boldsymbol{E}}$, if $C'_1 \cup C'_2 \in (\mathbf{0,2})_{\boldsymbol{E}}$ and if the partition is equitable. Those conditions can be verified in $\mathcal{O}(|V|^2)$, from Theorem 3.1. Thus, given a $(\mathbf{2,2})$ partition of G, we can answer if $G \in (\mathbf{2,2})_{\boldsymbol{E}}$ in $\mathcal{O}(|V|^{10})$ time.

The best known algorithm capable of finding a $(\mathbf{2,2})$ partition of a graph runs in $\mathcal{O}(|V|^{12})$ time [2, 3, 4]. Hence, we are bounded by that complexity, so we can check if $G \in (\mathbf{2,2})_{\boldsymbol{E}}$ in $\mathcal{O}(|V|^{12})$ time.

6 $(\boldsymbol{k,\ell})_{\boldsymbol{E}}$ Recognition for Other Values of k and l

We will now show that $(\boldsymbol{k,\ell})_{\boldsymbol{E}}$ recognition for $\max(k,\ell) \geq 3$ is NP-complete. For that, we will make a reduction from the problem Equitable Coloring. As mentioned before, it is known that deciding whether a given graph G can be equitably colored with k colors is NP-complete for any fixed $k \geq 3$.

We define the *join* of two graphs G and H as the graph obtained by the disjoint union of G and H and the addition of all edges between vertices of

G and H. It suffices to show that $\boldsymbol{(k,\ell)_E}$ is NP-complete for $k \geq 3$, since that $G \in \boldsymbol{(k,\ell)_E} \Leftrightarrow \overline{G} \in \boldsymbol{(\ell,k)_E}$. We prove this result in Theorem 6.1.

Theorem 6.1. *Deciding if* $G = (V,E) \in \boldsymbol{(k,\ell)_E}$ *for* $\max(k,\ell) \geq 3$ *is NP-complete.*

Proof. Since recognizing independent sets and cliques can be easily done in $\mathcal{O}(|V|^2)$, the problem belongs to NP. An instance of the EQUITABLE COLORING receives a graph $G = (V,E)$ and an integer k, and outputs YES if G can be equitably colored with k colors. We may assume that $|V| \geq 2k$, otherwise we can color G in polynomial time by finding a maximum matching in the complement, and that k divides $|V|$, by adding an isolated clique of size at most $k-1$. Let $r = \frac{|V|}{k}$. Notice that $r \geq 2$.

Let C be a complete graph of $\ell \cdot r$ vertices and let G' be the join of G and C, as depicted in Figure 1. We will show that $G' \in \boldsymbol{(k,\ell)_E}$ if and only if G can be equitably colored with k colors. If G can be equitably colored with k colors, then each color class of an equitable coloring of G has exactly r vertices. Hence in G' we can partition the vertices from G into k independent sets of size r, and vertices from C into the ℓ cliques also of size r.

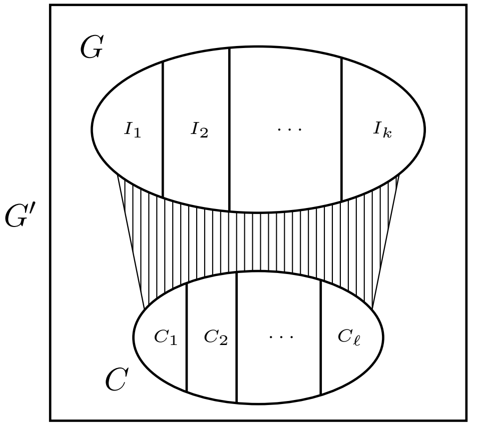

Figure 1: A valid partition of a graph built as in the proof of Theorem 6.1. Vertices in C are universal, I_i's are independent sets and C_j's are cliques.

Conversely, if $G' \in \boldsymbol{(k,\ell)_E}$, then we must make the following observation: if a vertex from C belongs to one of the k independent sets, then no other vertex of G' can belong to that independent set, because the vertices of C are universal in G'. Now, due to the fact that $|V(G)| \geq 2k$, we have that $|V(G')| \geq 2(k+\ell)$, so each independent set must have at least 2 vertices in any equitable partition. Therefore, no vertex of C can be in an independent set in a valid partition. So the original vertices of G form the k independent sets, that is, G can be equitably colored with k colors. ■

7 Conclusion

We showed that deciding whether a graph belongs to $\boldsymbol{(k,\ell)_E}$ is NP-complete if $\max(k,\ell) \geq 3$ and provided polynomial-time algorithms if $\max(k,\ell) \leq 2$. Table 1 summarizes the time complexities of the latter case.

It is interesting to see that, given a $\boldsymbol{(2,2)}$ partition of a graph G, we can find a $\boldsymbol{(2,2)_E}$ partition in $\mathcal{O}(|V(G)|^{10})$ time, however the best known algorithm for finding a $\boldsymbol{(2,2)}$ partition takes $\Theta(|V(G)|^{12})$ time [2, 3, 4]. So a more efficient algorithm for finding a $\boldsymbol{(2,2)}$ partition of G would also imply a more efficient algorithm for $\boldsymbol{(2,2)_E}$ recognition. Since it is not common for algorithms with complexity given by polynomials with such a high degree to be optimal, an interesting open problem is to find faster algorithms for recognizing $\boldsymbol{(2,2)}$ and even $\boldsymbol{(2,1)}$.

k \ ℓ	0	1	2
0	$\mathcal{O}(1)$	$\mathcal{O}(\|V\|^2)$	$\mathcal{O}(\|V\|^2)$
1	$\mathcal{O}(\|V\|^2)$	$\mathcal{O}(\|V\|^3)$	$\mathcal{O}(\|V\|^6)$
2	$\mathcal{O}(\|V\|^2)$	$\mathcal{O}(\|V\|^6)$	$\mathcal{O}(\|V\|^{12})$

Table 1: Time complexities of $\boldsymbol{(k,\ell)_E}$ recognition

Another line of research would be to find $\boldsymbol{(k,\ell)}$ analogues for other classical coloring problems, the same way we did for equitable coloring.

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