Matemática Contemporânea, Vol. 48, 94–104
http://doi.org/10.21711/231766362021/rmc4810

Total Coloring in Some Split-Comparability Graphs

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Abstract

A total coloring for a graph G is an assignment of colors to the edges and vertices of G such that any pair of adjacent or incident elements have different colors. The least number of colors for which G has a total coloring is denoted $\chi''(G)$. It is known that split-comparability graphs have $\chi''(G)$ at most $\Delta(G)+2$. In this work we show that certain split-comparability graphs with odd maximum degree have $\chi''(G) = \Delta(G)+1$.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G) \cup E(G)$ and $C = \{1, 2, \ldots, k\}$, let $c : S \to C$ be a mapping such that $c(x) \neq c(y)$ for each adjacent or incident elements

2000 AMS Subject Classification: 05C15.
Total Coloring; Edge Coloring; Split Graphs; Comparability Graphs
This research was supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil (CNPq)

$x, y \in S$. We say that c is a *k-total coloring* when $S = V(G) \cup E(G)$ and a *k-edge coloring* when $S = E(G)$. The least j and the least ℓ for which G has a j-total coloring and a ℓ-edge coloring are denoted by $\chi''(G)$ and $\chi'(G)$, respectively.

The well known *Total Coloring Conjecture* (TCC) [2, 9] asserts that $\chi''(G) \leq \Delta(G) + 2$ for any graph G, where $\Delta(G)$ is the maximum degree of a vertex in G. If $\chi''(G) = \Delta(G)+1$, G is Type 1; otherwise it is Type 2. To decide if G is Type 1 is an NP-Complete problem [8].

A *clique* of a graph G is a set of pairwise adjacent vertices in G. An *independent set* of a graph G is a set of pairwise non-adjacent vertices in G. A graph $G[Q, S]$ is *split* if $V(G)$ can be partitioned into $[Q, S]$ so that Q is a clique and S is an independent set. Chen, Fu and Ko [3] proved that TCC holds for split graphs and also that Type 2 split graphs must necessarily have odd maximum degree.

Theorem 1.1 ([3])**.** *Let G be a split graph. Then, $\chi''(G) \leq \Delta(G) + 2$. In particular, when $\Delta(G)$ is even G is Type 1.*

In this paper, we are concerned with total colorings in split-comparability graphs. A graph G is a *comparability graph* if it admits a transitive orientation of its edges, i. e., an orientation such that if there are edges uv and vw, then there is an edge uw as well. A graph G is *split-comparability* if it is simultaneously a split and a comparability graph. We show in this paper a sufficient condition for a split-comparability graph to be Type 1.

2 Definitions and Previous Results

The *neighborhood* of a vertex v in a graph G, denoted as $N_G(v)$, is the set of vertices adjacent to v. The *closed neighborhood* of a vertex v, denoted $N_G[v]$, is the set $N_G(v) \cup \{v\}$. When there is no ambiguity, the graph notation is omitted from $N_G(v)$ and $N_G[v]$. Two vertices u and v of a graph G are *true twins* if $N[v] = N[u]$. A *k-vertex* v in a graph G is a vertex with $|N(v)| = k$.

Given a graph $G = (V(G), E(G))$ the graph $H = (V(H), E(H))$ is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H is *induced* by a subset $U \subseteq V(G)$ if $V(H) = U$ and $E(H)$ is the set of all edges in $E(G)$ whose endpoints are both in U.

When $|E(G)| > \left\lfloor \frac{|V(G)|}{2} \right\rfloor \Delta(G)$ we say G is *overfull* and if G has a subgraph H with $\Delta(H) = \Delta(G)$, such that H is overfull, then G is *subgraph-overfull*. A graph G is *neighborhood-overfull* if there is a subgraph H induced by the closed neighborhood $N[v]$ of a $\Delta(G)$-vertex v such that H is overfull. Whenever G is overfull, subgraph-overfull, or neighborhood-overfull, then $\chi'(G) = \Delta(G) + 1$.

Theorem 2.1 ([4]). *A split-comparability graph G has $\chi'(G) = \Delta(G)$ iff G is not subgraph-overfull.*

A graph is *chordal* if every induced cycle has at most three vertices. By definition, split graphs are a subclass of chordal graphs. Theorem 2.2 shows the equivalence between subgraph-overfullness and neighborhood-overfullness for every chordal graph and, consequently, for every split graph.

Theorem 2.2 ([5]). *Let G be a chordal graph. Then, G is subgraph-overfull iff G is neighborhood-overfull.*

A *universal vertex* v in a graph G is a vertex with degree $|V(G)| - 1$, i. e., it is a vertex adjacent to every other vertex in $V(G)$. For graphs with a universal vertex, the definition of overfull graphs is equivalent to the following.

Remark 2.1. Let G be a graph with a universal vertex. The graph G is overfull if and only if $|V(G)|$ is odd and $|E(\overline{G})| < \Delta(G)/2$, where $\overline{G}$ is the complement of G.

A *matching* is a set of edges such that no two edges share a common vertex. The size of a maximum matching of a graph G is denoted by $\alpha'(G)$. Based on the parameter $\alpha'(\overline{G})$, Hilton [6] presents a necessary and sufficient condition for a graph G with a universal vertex to be Type 1.

Theorem 2.3 ([6]). *Let G be a graph with a universal vertex. Then, G is Type 1 iff $|E(\overline{G})| + \alpha'(\overline{G}) \geq |V(G)|/2$.*

Almeida [1] found sufficient conditions for a split graph to be Class 1 based on the degree of a vertex in the independent set.

Theorem 2.4 ([1]). *Let $G = [Q, S]$ be a split graph. If there is a vertex $v \in S$ such that $\lceil |Q|/2 \rceil \leq deg(v) \leq \Delta(G)/2$, then G is Class 1.*

Ortiz and Villanueva [7] characterized the split-comparability graphs as follows.

Theorem 2.5 ([7]). *A split graph $G = [Q, S]$ is a comparability graph iff Q has a partition $[Q_l, Q_t, Q_r]$ and its vertices can be ordered Q_l, Q_t, Q_r so that for any vertex $s \in S$:*

i. $N(s) \cap Q_t = \emptyset$;

ii. $v_{k-1} \in (N(s) \cap Q_l)$ *if* $v_k \in (N(s) \cap Q_l)$; *and*

iii. $v_{k+1} \in (N(s) \cap Q_r)$ *if* $v_k \in (N(s) \cap Q_r)$.

Consider a split-comparability graph $G = [Q, S]$. Let $[Q_l, Q_t, Q_r]$ be a partition of Q as defined in Theorem 2.5. Without loss of generality, we can assume from now on that $|Q_l| \geq |Q_r|$. The subset of vertices of S that are not adjacent to vertices in Q_r are denoted as S_l, and those not adjacent to Q_l are denoted as S_r. Let $S_t = S \setminus S_l \cup S_r$. Note that $[Q_l, Q_t, Q_r, S_l, S_t, S_r]$ is a partition of $V(G)$. Henceforth, we denote a split-comparability graph by $G = [Q_l, Q_t, Q_r, S_l, S_t, S_r]$.

A split-comparability graph with either $S_r = \emptyset$ or $S_l = \emptyset$ has a universal vertex. Therefore, we have the following remark from Theorem 2.3.

Remark 2.2. Let $G = [Q_l, Q_t, Q_r, S_l, S_t, S_r]$ be a split-comparability graph. If $S_l = \emptyset$ or $S_r = \emptyset$, then G is Type 1 iff $|E(\overline{G})| + \alpha'(\overline{G}) \geq |V(G)|/2$.

3 Our Contribution

This section presents new results on total coloring of split-comparability graphs. We establish two sufficient conditions for a split-comparability graph to be Type 1. When both Q_l and Q_r have at most $|Q|/2$ vertices we show that G is always Type 1 (Theorem 3.1). In the remaining case, assuming $|Q_l| > |Q|/2$, we specify a condition between the sizes of Q, Q_l and S_l that ensures G is Type 1 (Theorem 3.2). Although we know that not all split-comparability graphs will satisfy this condition, we understand that most of them will satisfy it and there are only few graphs for which we have not determined the total chromatic number.

Theorem 3.1. *Let $G = [Q_l, Q_t, Q_r, S_l, S_t, S_r]$ be a split-comparability graph. If $S_l \neq \emptyset$, $S_r \neq \emptyset$, $|Q_l| \leq \frac{|Q|}{2}$ and $|Q_r| \leq \frac{|Q|}{2}$, then G is Type 1.*

Proof. We may assume that $\Delta(G)$ is odd, otherwise, by Theorem 1.1, G is Type 1. Label the vertices in Q as $v_0, v_1, \ldots, v_{|Q|-1}$, so that this ordering respects the characterization of Theorem 2.5.

Add a vertex v_f adjacent to all vertices in $Q \cup S_t \cup S_l$ and let G_f be such graph. Note that $G_f = [Q_l^f, Q_t, Q_r, S_l, S_t, S_r]$ is also a split-comparability graph, that $Q_l^f = Q_l \cup \{v_f\}$, and that v_f is a true twin of v_0. Also, $\Delta(G_f) = \Delta(G) + 1$ and, thus, $\Delta(G_f)$ is even. Let Q^f be $Q \cup \{v_f\}$. Since $|Q_r| < \frac{|Q^f|}{2}$ and $|Q_l^f| \leq \frac{|Q^f|}{2}$, it is possible to add edges to G_f between vertices in Q_t and S_l to obtain a supergraph $G'_f = [Q'_l, Q'_t, Q'_r, S_l, S_t, S_r]$ of G_f so that $|Q'_l| = \lceil |Q^f|/2 \rceil$ and $\Delta(G'_f) = \Delta(G_f)$. By Theorem 2.4, G'_f is Class 1 and, thus, G_f is also Class 1. Consider an edge coloring $\pi : E(G_f) \to \{1, 2, \ldots, \Delta(G_f)\}$. It is possible to obtain a total coloring $\beta : V(G) \cup E(G) \to \{1, 2, \ldots, \Delta(G_f)\}$ from π, as follows. For every edge $uv \in E(G)$, let $\beta(uv) = \pi(uv)$. For every $w \in N(v_f)$, let $\beta(w) = \pi(wv_f)$. The only elements of G not yet colored are the vertices of S_r. Since $|Q_r| \leq |Q|/2$, each vertex in S_r has degree at most $|Q|/2$, that is, it is adjacent and incident with at most $|Q|$ elements of G. Since $|S_l| \geq 1$ then $\Delta(G) > |Q|$ and $\Delta(G_f)$ (the number of colors used in β) is at least $|Q|+1$. For each vertex $u \in S_r$ there is a color $c \in \{1, 2, \ldots, \Delta(G_f)\}$, not used in any of its neighbors and none of the edges incident with it. Let $\beta(u) = c$. Since $\Delta(G_f) = \Delta(G) + 1$, the graph G is Type 1. ■

By the characterization of split-comparability graphs given by Theorem 2.5, $Q_r \cap Q_l = \emptyset$. Then one of the sets, Q_r or Q_l, has at most $|Q|/2$ vertices. Recall that we are considering $|Q_l| \geq |Q_r|$ without loss of generality. The next result extends Theorem 3.1 by considering the case when $|Q_l| > |Q|/2$.

Theorem 3.2. *A split-comparability graph* $G = [Q_l, Q_t, Q_r, S_l, S_t, S_r]$ *is Type* 1 *if* $S_l \neq \emptyset$, $S_r \neq \emptyset$, *and*

$$|Q| \geq \left(\frac{|S_l|}{|S_l| - 0.5} \right) |Q_l|.$$

Proof. We may assume that $\Delta(G)$ is odd, otherwise, by Theorem 1.1, G is Type 1. Since $N(S_r) \cap N(S_l) = \emptyset$, one of the sets Q_l or Q_r necessarily has size at most $|Q|/2$. If both the sets Q_l and Q_r have size at most $|Q|/2$, then G is Type 1 by Theorem 3.1. So, it is sufficient to consider the cases where $|Q_l| > |Q|/2$. If $|S_l| = 1$, then $|Q_l| \leq |Q|/2$ by the hypothesis. Therefore, by Theorem 3.1, G is Type 1. Thus, consider $|S_l| \geq 2$.

Label the vertices in Q as $v_0, v_1, \ldots, v_{|Q|-1}$, so that this ordering respects the characterization of Theorem 2.5. Add a vertex v_f adjacent to all vertices in $Q \cup S_t \cup S_l$ and let G_f be such graph. Note that $G_f = [Q'_l, Q_t, Q_r, S_l, S_t, S_r]$ is also a split-comparability graph, that $Q'_l = Q_l \cup \{v_f\}$, and that v_f is a true twin of v_0. Also, $\Delta(G_f) = \Delta(G) + 1$ and, thus, $\Delta(G_f)$ is even. Let Q' be $Q \cup \{v_f\}$. First we show that G_f is not subgraph-overfull. Every $\Delta(G_f)$-vertex is either a true twin of v_0 or of $v_{|Q|-1}$. Hence, by Theorem 2.2 and Remark 2.1, to prove that G_f is not subgraph-overfull, it suffices to show that the subgraphs $G_0 = G_f[N[v_0]]$ and $G_q = G_f[N[v_{|Q|-1}]]$ are not overfull, that is, $|E(\overline{G_0})| \geq \Delta(G_f)/2$ and $|E(\overline{G_q})| \geq \Delta(G_f)/2$. The number of edges in $\overline{G_0}$ is

$$|E(\overline{G_0})| \geq |S_l|(|Q| - |Q_l|) + |S_t||Q_t| + (|S_l| + |S_t|)(|S_l| + |S_t| - 1)/2. \quad (1)$$

By the hypothesis, $|Q| \geq \left(\frac{|S_l|}{|S_l|-0.5}\right)|Q_l|$ which implies

$$|Q| - |Q_l| \geq \frac{|Q|}{2|S_l|} \tag{2}$$

Replacing (2) in (1), the inequality relation is preserved:

$$\begin{aligned} |E(\overline{G_0})| &\geq |Q|/2 + |S_t||Q_t| + (|S_l| + |S_t|)(|S_l| + |S_t| - 1)/2 && (3)\\ &= (|Q| + 2|S_t||Q_t| + |S_l|^2 + 2|S_l||S_t| - |S_l| + |S_t|^2 - |S_t|)/2. && (4)\end{aligned}$$

Since $|S_l| \geq 2$, we have $|S_l||S_t| \geq 2|S_t|$ and $|S_l|^2 \geq 2|S_l|$. From these observations and (4) we obtain

$$\begin{aligned} |E(\overline{G_0})| &\geq (|Q| + 2|S_t||Q_t| + |S_l| + 3|S_t| + |S_t|^2)/2\\ &\geq (|Q| + |S_l| + |S_t|)/2\\ &= \Delta(G_f)/2. \end{aligned}$$

Therefore, G_0 is not overfull. Now we show that G_q is not overfull similarly; the only difference is that vertices in S_r are not adjacent to v_f. The number of edges in $\overline{G_q}$ is

$$|E(\overline{G_q})| \geq |S_r| + |S_r|(|Q_l| + |Q_t|) + |S_t||Q_t| + (|S_r| + |S_t|)(|S_r| + |S_t| - 1)/2. \tag{5}$$

Manipulating (5) we obtain

$$\begin{aligned} |E(\overline{G_q})| \geq |S_r| + |S_r||Q_l| + |S_r||Q_t| + |S_t||Q_t| + |S_r|^2/2 + |S_r||S_t| +\\ |S_t|^2/2 - |S_t|/2 - |S_r|/2 \end{aligned} \tag{6}$$

Since $|S_r| \geq 1$, we deduce that $|S_r||S_t| - |S_t|/2 \geq |S_t|/2$ and $|S_r||Q_l| \geq |Q_l|$. By hypothesis, $|Q_l| > |Q|/2$, so $|S_r||Q_l| > |Q|/2$. So, replacing these terms in (6), we obtain

$$\begin{aligned} |E(\overline{G_q})| &\geq |S_r|/2 + |Q|/2 + |S_r||Q_t| + |S_t||Q_t| + |S_r|^2/2 + |S_t|^2/2 + |S_t|/2\\ &\geq (|Q| + |S_r| + |S_t|)/2\\ &= \Delta(G_f)/2. \end{aligned}$$

Since G_f is not subgraph-overfull, it has an edge coloring with $\Delta(G_f)$ colors, by Theorem 2.1. Let $\pi : E(G_f) \to \{1, 2, \ldots, \Delta(G_f)\}$ be one such edge coloring and $\beta : V(G) \cup E(G) \to \{1, 2, \ldots, \Delta(G_f)\}$ a total coloring obtained from π as follows. For every edge $uv \in E(G)$, let $\beta(uv) = \pi(uv)$. For every $w \in N(v_f)$, let $\beta(w) = \pi(wv_f)$. Figure 1 shows an example of G_f with an edge coloring and the assignment of colors for the neighbors of v_f. The only elements of G not yet colored are the vertices of S_r. Since $|Q_r| \leq |Q|/2$, each vertex in S_r has degree at most $|Q|/2$, that is, it is adjacent and incident with at most $|Q|$ elements of G. Since $|S_l| \geq 1$ then $\Delta(G) > |Q|$ and $\Delta(G_f)$ (the number of colors used in β) is at least $|Q|+1$. For each vertex $u \in S_r$ there is a color $c \in \{1, 2, \ldots, \Delta(G_f)\}$, not used in any of its neighbors and none of the edges incident with it. Let $\beta(u) = c$. Since $\Delta(G_f) = \Delta(G) + 1$, the graph G is Type 1.

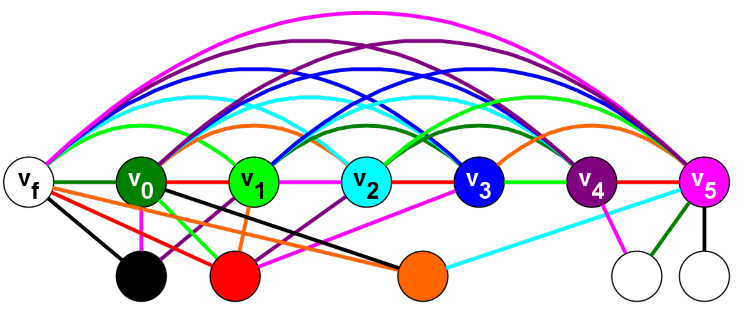

Figure 1: 9-total coloring for G

■

4 Conclusion

In this paper we show sufficient conditions for split-comparability graphs to be Type 1. Theorem 3.2 specifies a sufficient condition in terms of a relation between the sizes of Q, Q_l and S_l. Note that, if we interpret the expression $|S_l|/(|S_l| - 0.5)$ as a function for $|S_l| \geq 1$, it is a monotonically decreasing one, with value 2 when $|S_l| = 1$ and tending to 1 as $|S_l|$ tends to infinity. So, the sufficient condition will tend to be satisfied when $|S_l|$ is sufficiently large, as $|Q| \geq |Q_l|$.

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