Matemática Contemporânea, Vol. 48, 84–93
http://doi.org/10.21711/231766362021/rmc489

On 4-regular square-complementary graphs of large girth

Ariadna Juárez-Valencia, Miguel Pizaña

Abstract

A *square-complementary (squco) graph* is a graph such that its square and its complement are isomorphic. Here, we provide two computer-assisted proofs showing that there are no 4-regular squco graphs of girth greater than 4, thus solving in the negative an open problem in M. Milanič, A.S. Pedersen, D. Pellicer and G. Verret [Discrete Mathematics 327 (2014) 62-75].

1 Introduction

All our graphs are finite and simple. The *square* G^2 of a graph G is the graph with the same vertex set as G and where two vertices are adjacent if and only if they are at distance 1 or 2 in G. The *complement* $\overline{G}$ of G is the graph with the same vertex set as G, and where two vertices are adjacent if and only if they are not adjacent in G. A *square-complementary* graph (*squco* for short), is a graph satisfying $G^2 \cong \overline{G}$, or equivalently, $\overline{G^2} \cong G$. Note that two vertices are adjacent in $\overline{G^2}$ if and only if they are at distance at least 3 in G. The order of a graph G is denoted by $|G|$.

2020 AMS Subject Classification: 05C60 (05C76).
Keywords: graph equations, square-complementary graphs, regular graphs.
Partially supported by SEP-CONACYT, grant A1-S-45528.

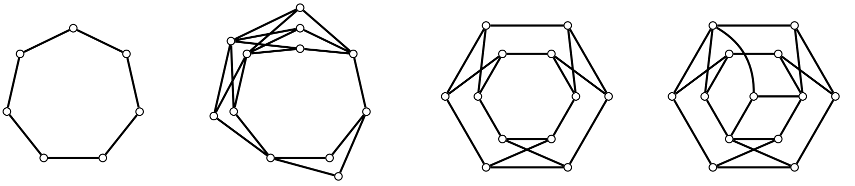

Figure 1: Some squco graphs.

Squco graphs where introduced by Akiyama et al. in [1] and also (as reported in [2]) independently by Schuster in [14], the term “squco” was introduced by Milanič et al. in [12]. In the context of graph equations, several authors have dealt with the topic including [13, 6, 4, 5, 2, 1, 8]. Examples of squco graphs include the trivial graph K_1, the 7-cycle C_7, (see Figure 1(a)) and the Franklin graph (see Figure 1(c)) [1]. There is at least one squco graph for every order $n \geq 7$ [1].

Almost all known squco graphs are either *d-regular* for some d (all the vertices have degree d) or contain some non-trivial d-regular squco induced subgraph (the only known exception is the graph in [12, Figure 3]). Indeed, Baltić et al. [2, Lemma 2.1] and later Milanič et al. [12, Proposition 2.5] presented separate procedures to extend squco graphs and the resulting squco extensions may be non-regular (for example, in Figure 1, (b) and (d) are extensions of (a) and (c) respectively). Also, the construction in [8, Theorem 4.1] always produce bipartite squco graphs containing the Franklin graph, which is 3-regular. Therefore the d-regular case is very prominent here.

A d-regular squco graph G must have at most $d^2 + d + 1$ vertices [2] (see Figure 2): pick any vertex x, it has d neighbors, at most $d(d-1)$ vertices at distance 2, and exactly d vertices at distance at least 3 (since those are the neighbors of x in $\overline{G^2} \cong G$, which must also be d-regular), thus $|G| \leq 1 + d + d(d-1) + d = d^2 + d + 1$. Note also that the described structure must look the same from each vertex, and hence, that the *girth* (length of the smallest cycle) $g(G)$ of G satisfies $g(G) \geq 5$ if and only if $|G| = d^2 + d + 1$.

The only squco graphs on at most 7 vertices are K_1 and C_7 [2, 12], whose girths are $g(K_1) = \infty$ and $g(C_7) = 7$. Except for these two examples, it is known that a squco graph G must satisfy $g(G) \in \{3, 4, 5\}$ [8]. There are known examples of such squco graphs of girth 3 and 4, but it is an open problem to determine whether there is actually a squco graph with $g(G) = 5$, even in the d-regular case, as reported in [12].

We already said that for a d-regular squco graph G, we have $|G| = d^2 + d + 1$ if and only if $g(G) \geq 5$. Note that this can only happen for even d, as no graph can have an odd number of vertices of odd degree. For $d = 0$ and $d = 2$, K_1 and C_7 are the only such graphs. The problem of determining whether there exist a d-regular squco graph of order d^2+d+1, for even $d \geq 4$ was posed in [2], and it was already reported as equivalent to the problem in the previous paragraph in [12].

Here we provide two computer-assisted proofs of Theorem 1.1. The first proof uses Meringer's `genreg` [11]. We also present a fully independent computer-assisted proof based on an exhaustive depth-first search (backtracking), with automorphism reductions, as described in detail in the next section. Both approaches construct all 4-regular graphs on 21 vertices and girth 5 and then check whether any these graphs are squco (none of them are). Meringer's approach has the advantage of being a very well established tool, more general and very fast. The second approach has the advantage of being very easy to understand and reproduce, and the technique may be easily adapted for other similar problems. We also present both approaches because, in our opinion, computer-assisted proofs benefit from redundancy.

Theorem 1.1. *There is no 4-regular squco graph of girth 5, equivalently, there is no 4-regular squco graph on $d^2 + d + 1 = 21$ vertices.*

First Proof. We take the list of all 4-regular graphs of girth 5 on 21 vertices from [3] which where generated using `genreg` [11], and we directly check that all of these 8 graphs are not squco. □

2 Second Proof of Theorem 1.1

We used GAP [9] and YAGS [7] to implement a depth-first search (DFS, aka backtracking) algorithm, with isomorphism reductions, to perform an exhaustive search and determine the inexistence of such graphs. YAGS's backtracking facilities were specially useful. The algorithm took 8.6 minutes to finish on an Intel Core i5, at 3.2GHz. To facilitate the reproducibility of this result, we provide the full code as supplementary material [10] and a concise description here.

The algorithm starts with an initial scaffolding H (see Figure 2), which must be a subgraph of any hypothetical 4-regular graph of girth 5 on 21 vertices. The possible additional edges are all the possible pairs of vertices in the set $\{6, 7, \dots, 21\}$ except those that already form a triangle with the initial scaffolding (like the edge $\{6, 7\}$). There are $\binom{16}{2} - 12 = 108$ such edges. We sort these edges heuristically for performance: first the edges connecting vertices in $\{18, 19, 20, 21\}$, then the edges connecting vertices in $\{18, 19, 20, 21\}$ with vertices in $\{6, 7, \dots, 17\}$ and then, the rest of them. Let us call this list of possible additional edges $U = [u_\ell : \ell \in \{1, 2, \dots, 108\}]$. Since the sought graph is 4-regular, H needs $(12 \cdot 3 + 4 \cdot 4)/2 = 26$ additional edges to become a solution.

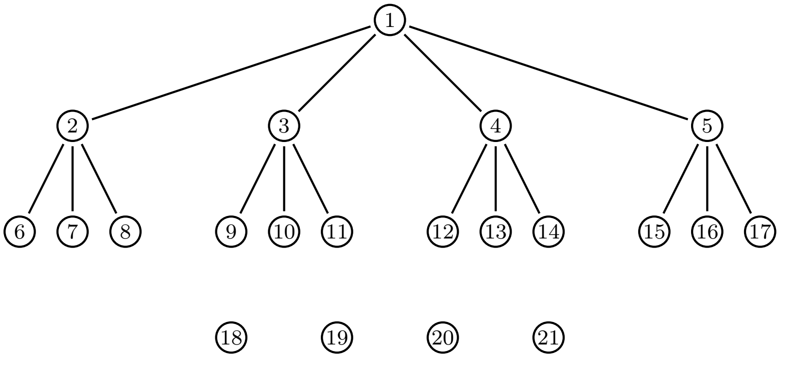

Figure 2: The initial scaffolding H.

The DFS part of the algorithm is standard: At any given time, the algorithm considers a (partial) feasible solution encoded as a list $L = [\ell_1, \ell_2, \ldots, \ell_s]$ (with $1 \leq \ell_1 < \ell_2 < \cdots < \ell_s \leq 108$ and $s \leq 26$) of the indices of a candidate set of edges $S \subset U$, that is, $S = \{u_\ell \in U : \ell \in L\}$. Then, the algorithm generates the graph $G = H + S$ and verifies whether G is still a feasible solution (as described below). If it is feasible, we try adding an additional edge ℓ_{s+1} to get $L = [\ell_1, \ell_2, \ldots, \ell_s, \ell_{s+1}]$; if it is not feasible, we discard the last choice ℓ_s and try the next possibility $\ell'_s = \ell_s + 1$ (whenever $\ell_s < 108$) to get $L = [\ell_1, \ell_2, \ldots, \ell_{s-1}, \ell'_s]$; whenever we are out of options, at the current depth s, we cut out the last index and try the next option at depth $s - 1$. All of this is accomplished by YAGS's backtracking facilities, by means of the YAGS's function `Backtrack` [7].

A (partial) candidate solution $G = H + S$ is considered feasible, only if none of the following conditions hold:

1. A vertex in G already exceeds degree 4.
2. A vertex in G will not be able to achieve degree 4 with the remaining edges (i.e. the edges not yet considered: $\{u_\ell \in U : \ell > \ell_s\}$).
3. The girth of G becomes less than 5.
4. The current candidate edge indices L is equivalent, up to an isomorphism of the initial scaffolding H, to a previously considered case.

Not all of the automorphism group of H is used in the above condition 4, since $|\mathrm{Aut}(H)| = (4!)^2(3!)^4 = 746,496$ and that would make the verification too slow. Instead it was sufficient for us to consider 6 subgroups of $\mathrm{Aut}(H)$, namely the group that permutes freely the vertices in $\{18, 19, 20, 21\}$, the 4 subgroups that respectively permute freely the four bunches of sibling leaves: $\{6, 7, 8\}$, $\{9, 10, 11\}$, $\{12, 13, 14\}$ and $\{15, 16, 17\}$, and the subgroup that freely permutes the vertices $\{2, 3, 4, 5\}$ (the corresponding leaves are permuted accordingly, leaving the relative order of the sibling leaves intact). The number of permutations to consider is then $2 \cdot 4! + 4 \cdot 3! = 72$. These subgroups act on vertices, but they also inherit a natural action on the possible additional edges U (element-wise) and hence on the positions $\{1, 2, \ldots, 108\}$ of these edges in U (such that

$\sigma \cdot \ell = \ell'$ if and only if $\sigma \cdot u_\ell = u_{\ell'}$, for any permutation σ in any of these six groups). It follows that these groups also act naturally on the configurations L (considering L as a set of indices).

Instead of storing all the previously considered cases, we simply try all cases L in lexicographic order and hence, whenever we have a new case L, we compute the orbit of L under the action of each of the previous 6 subgroups and discard the current case, whenever any of these 6 orbits contain a case L' which is lexicographically smaller than L.

The set of candidate edges (and hence $G = H + S$) is accepted when $s = 26$ and the graph G is squco (explicitly tested). Otherwise the algorithm returns 'fail' after all of the search space is explored, which is what actually happens.

3 Open problems

We point out that our algorithm considered $2,210,423$ cases at an average rate of $4,245$ cases per second (8.6 minutes). Removing the condition 4 (the symmetry-checking condition) from the algorithm gives us an estimated of 10^{10} cases to consider, at an average rate of $4,314$ cases per second (8 months). As a reference, we mention that not checking any of the four conditions, gives us $\sum_{i=0}^{26}\binom{108-26+i}{i} \approx 10^{25}$ cases at an average rate of $4,879$ cases per second (58 billion years).

For $d = 6$, the total search space is about 10^{108} (compared to the 10^{25} of the case $d = 4$). Therefore, different techniques would be required for solving the following problem:

Problem 1. [2, 12] Is there a d-regular squco graph on $d^2 + d + 1$ vertices for even $d \geq 6$?

We already said that when d is odd, G can not achieve the order d^2+d+1. But what about $d^2 + d$? Well, it turns out that The Franklin graph (see Figure 1(c)) achieves precisely this for $d = 3$. Besides, it is easy to show that any d-regular squco graph on $d^2 + d$ vertices, must contain an initial scaffolding like that in Figure 2, but with one pair of leaves from different bunches identified (say, 8 and 9 in Figure 2), hence the graph has girth 4. Since this structure must look the same as viewed from any vertex, every vertex must be contained in a unique 4-cycle. This can only happen when $d^2 + d$ is a multiple of 4 and hence, since d is odd, when $d \equiv 3 \pmod 4$. This motivates the following problem:

Problem 2. Is there a d-regular squco graph on $d^2 + d$ vertices for $d \equiv 3 \pmod 4$, $d \geq 7$?

When $d \equiv 1 \pmod 4$, we must have, $|G| \leq d^2 + d - 1$, but this upper bound can not be met, since that would imply an impossible graph having an odd number of vertices of odd degree. Hence for $d \equiv 1 \pmod 4$, we must have $|G| \leq d^2 + d - 2$:

Problem 3. Is there a d-regular squco graph on $d^2 + d - 2$ vertices for $d \equiv 1 \pmod 4$, $d \geq 5$?

Actually, besides the Franklin graph, all known d-regular squco graphs have an even d. So it is even interesting to ask the following:

Problem 4. Is there a d-regular squco graph for odd $d \geq 5$?

Finally, if 21 is not the maximum order of a 4-regular squco graph, which is it?

Problem 5. Which is the maximum order of a 4-regular squco graph?

Acknowledgments We are grateful to the anonymous reviewers for their observations that made this a better paper.

References

[1] J. Akiyama, H. Era and G. Exoo. *Further results on graph equations for line graphs and nth power graphs.* Discrete Math. **34** (1981) 209–218. DOI 10.1016/0012-365X(81)90001-7.

[2] V. Baltić, S. Simić and V. Tintor. *Some remarks on graph equation* $G^2 = \overline{G}$. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **5** (1994) 43–48.

[3] G. Brinkmann, K. Coolsaet, J. Goedgebeur and H. Mélot. *House of graphs: a database of interesting graphs.* Discrete Applied Mathematics **161** (2013) 311–314. available at: `⟨http://hog.grinvin.org⟩`. DOI 10.1016/j.dam.2012.07.018.

[4] M. Capobianco, M. Karasinski and M. Randazzo. On some quadratic graph equations, in Proceedings of the thirteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1982). In *Proceedings of the thirteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1982)*, volume 35, pages 149–156, 1982.

[5] M. Capobianco and S.R. Kim. More results on the graph equation $G^2 = \overline{G}$. In *Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., pages 617–628. Wiley, New York, 1995.

[6] M.F. Capobianco, K. Losi and B. Riley. $G^2 = \overline{G}$ has no nontrivial tree solutions. In *Combinatorial Mathematics: Proceedings of the Third International Conference (New York, 1985)*, volume 555 of *Ann. New York Acad. Sci.*, pages 103–105. New York Acad. Sci., New York, 1989.

[7] C. Cedillo, R. MacKinney-Romero, M.A. Pizaña, I.A. Robles and R. Villarroel-Flores. *YAGS - Yet Another Graph System, Version 0.0.5*, 2019. `⟨http://xamanek.izt.uam.mx/yags/⟩`.

[8] R. Darda, M. Milaniĉ and M. Pizaña. *Searching for square-complementary graphs: Complexity of recognition and further nonexistence results.* Discrete Mathematics **344** (2021) 112369. DOI 10.1016/j.disc.2021.112369.

[9] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10*, 2019. (`http://www.gap-system.org`).

[10] A. Juárez-Valencia and M. Pizaña. *On 4-regular square-complementary graphs of large girth (full code).* Available (2021) at : (`http://xamanek.izt.uam.mx/map/squco/`).

[11] M. Meringer. *Fast generation of regular graphs and construction of cages.* J. Graph Theory **30** (1999) 137–146. (`http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html#CRG`). DOI 10.1002/(SICI)1097-0118(199902)30:2<137::AID-JGT7>3.0.CO;2-G.

[12] M. Milanič, A.S. Pedersen, D. Pellicer and G. Verret. *Graphs whose complement and square are isomorphic.* Discrete Mathematics **327** (2014) 62 – 75. DOI 10.1016/j.disc.2014.03.018.

[13] E. Prisner. *Graph dynamics.* Longman, Harlow, 1995.

[14] S. Schuster. Problem 10. In *Theory and Applications of Graphs (Proc. of the Fourth Inter. Conf. on the Graph Theory and Applications of Graphs), Eds. G. Chartrand, Y. Alavi, D.L. Goldsmith, L. Lesniak-Foster, D.R. Lick)*, page 611. John Wiley Sons, New York - Chichester - Brisbane - Toronto - Singapore, 1981.

Ariadna Juárez-Valencia
Universidad Autónoma Metropolitana – Iztapalapa
Departamento de Matemáticas
jva.950@gmail.com

Miguel Pizaña
Universidad Autónoma Metropolitana – Iztapalapa
Departamento de Ingeniería Eléctrica
mpizana@gmail.com, http://xamanek.izt.uam.mx/map/