Range-Relaxed Graceful Game on Graphs

Deise L. de Oliveira, Simone Dantas, Atílio G. Luiz

Abstract

Given a graph $G = (V, E)$ and a set of integer labels $\mathcal{L} = \{0, \ldots, k\}$, $k \geq |E(G)|$, a *range-relaxed graceful labeling* is a function $f : V(G) \to \mathcal{L}$ such that: (i) f is injective; (ii) the absolute difference of the labels of the endpoints of an edge $uv \in E(G)$ already labeled is unique. The *range-relaxed graceful game* (RRG game) is a graph labeling game recently introduced by Tuza and it is a variant of a two person graph labeling game. The players, traditionally called Alice and Bob, alternately assign an unused label $f(v) \in \mathcal{L} = \{0, \ldots, k\}$, $k \geq |E(G)|$, to an unlabeled vertex $v \in V(G)$ and the induced edge label is given by $\phi(uv) = |f(u) - f(v)|$. Alice's goal is to end up with an injective vertex labeling of all vertices of G which induces distinct edge labels, and Bob's goal is to prevent it from happening. In 2017, Tuza posed a question about the RRG game asking for which values of k Alice wins the RRG game in a simple graph G, given a set of consecutive non-negative integer labels $\mathcal{L}$. We partially answer Tuza's question presenting an upper bound on the number of consecutive non-negative integer labels necessary for Alice to win the RRG game on a simple graph G, a cycle, and a path graph.

1 Introduction

A simple graph G consists of a nonempty set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of vertices $u, v \in V(G)$, called *edges*. The vertices u and v are called *endpoints* of the edge uv. All graphs considered in this paper are simple and finite.

A *labeling* of a graph is an assignment of *labels* (elements of a set) to the vertex set (vertex labeling) or the edge set (edge labeling) of a graph.

Graph labeling is an area of study in graph theory whose origin goes back to the well-known Four Color Theorem. Indeed, graph coloring can be viewed as a type of graph labeling since it is an assignment of labels (colors) to the elements of a graph subject to certain conditions.

However, since the 1960s, many other graph labeling problems have arisen that use numbers as labels and not colors, whose properties to be satisfied are strictly numerical. The first labelings of this type were investigated by Rosa [6], who introduced four types of graph labelings and proved that if a graph G with m edges has any one of that four labelings, then the complete graph K_{2m+1} can be decomposed into copies of G. The interested reader can consult Gallian's dynamic survey on graph labeling [4], that presents a vast literature on graph labeling problems and open conjectures, as well as algorithmic and complexity results.

Let G be a simple graph, where $|V(G)| = n$ and $|E(G)| = m$, and let $f \colon V(G) \to \{0, \dots, k\}$ be an injective function that induces an injective mapping $\phi \colon E(G) \to \{1, \dots, k\}$ given by $\phi(uv) = |f(u) - f(v)|$, for all $uv \in E(G)$.

By Golomb [5], if $k = m$, then graph G is called *graceful* and f is called a *graceful labeling*. When $k \geq m$, the function f is said to be a *range-relaxed graceful labeling* (RRG labeling). Figure 1 exhibits an RRG labeling of a graph that does not have graceful labeling.

A large number of graphs are classified as graceful. Rosa [6] proved that Eulerian graphs such that $m \equiv 1, 2 \pmod 4$ are not graceful (it is known as parity condition). The gracefulness of classic families of graphs

like paths, cycles, complete bipartite graphs [6], and many other families of graphs [4] is already established.
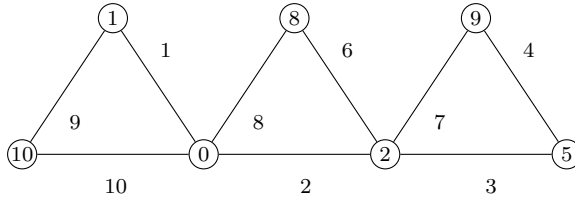

Figure 1: RRG labeling of a graph with no graceful labeling.

In 2017, Tuza [8] surveyed the area of labeling games, proposing the study of new game variants. Combinatorial games are a branch of mathematics that includes no chance games in which players take turns with perfect information until one of them is the winner. It is important to observe that there is absolutely no luck involved and there is no randomness. The range-relaxed graceful game and the graceful game are examples of combinatorial labeling games introduced by Tuza [8].

The *range-relaxed graceful game* is a maker-breaker game played on a graph G and it is defined as follows: Alice (the maker) and Bob (the breaker) alternately assign an unused label $f(v) \in \{0, \ldots, k\}$, $k \geq |E(G)|$, to a previously unlabeled vertex $v \in V(G)$. A vertex of G is *unlabeled* if no label has been assigned to it. If both ends of an edge $uv \in E(G)$ are already labeled, then the *(induced) label* of the edge is defined by $\phi(uv) = |f(u) - f(v)|$. Note that after any move, the next player can only label unlabeled vertices. A move is *legal* if, after it, all edge labels are distinct. Only legal moves are allowed during the game and both players play optimally. Alice's goal is to construct a range-relaxed gracefully labeled graph and Bob's goal is to prevent this. The game ends if an RRG labeling is created (Alice wins) or there is no possible legal move (Bob wins). If $k = |E(G)|$, then the game is called *graceful game*.

Tuza [8] called RRG game as *edge-difference distinguishing* game. However, we call it a *range-relaxed graceful game* in order to match with the range-relaxed graceful labeling nomenclature previously established in the

literature [1, 2, 7].

The graceful game was later studied by Frickes et al. [3] who investigated winning strategies for Alice and Bob in some classic families of graphs such as paths, complete graphs, cycles, complete bipartite graphs, caterpillars, trees, gear graphs, helms, web graphs, prisms, hypercubes, and 2-powers of paths.

In this work, we investigate the RRG game, presenting an upper bound on the number of consecutive non-negative integer labels necessary for Alice to win the RRG game on a simple graph G and contribute to the study of the question posed by Tuza [1]:

Question 1.1 (Tuza [8])**.** Given a simple graph G and a set of consecutive non-negative integer labels $\mathcal{L} = \{0, \ldots, k\}$, for which values of k Alice wins the range-relaxed graceful game?

In the following section, we present definitions that are useful in this work and show our main results on the RRG game.

2 Main results

Before showing our results, first, we present notations and definitions used throughout the text. Let $G = (V(G), E(G))$ be a graph. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$; in such a case, edge $e = uv$ and vertices u and v are called *incident*, and vertices u and v are also called *neighbors*. Note that $uv = vu$, for all $u, v \in V(G)$. The set of neighbors of a vertex $v \in V(G)$ is usually denoted by $N(v)$. The *degree* of a vertex $v \in V(G)$ is the number of edges incident to v and is denoted by $d(v)$. The *maximum degree* of G is the number $\Delta(G) = \max\{d(v) \colon v \in V(G)\}$.

A *path* P_n is a graph whose vertices can be arranged in a linear sequence $(v_1, v_2, \ldots, v_n)$ in such a way that two vertices are adjacent if and only if they are consecutive in the sequence. Similarly, a *cycle* C_n on three or more vertices is a graph whose vertices can be arranged in a cyclic sequence $(v_1, v_2, \ldots, v_n, v_1)$ in such a way that two vertices are adjacent if and only if they are consecutive in the sequence.

Given a graph G, its *game graceful number*, denoted by $grac_g(G)$, is the minimum non-negative integer k such that Alice wins playing the RRG game on G with a set $\mathcal{L} = \{0, \ldots, k\}$. In the context of the RRG game, we say that a vertex label $\ell \in \{0, \ldots, k\}$ is *available* to a vertex v of a graph G if it was not assigned to any other vertex of G and if it can be assigned to v without generating repeated induced edge labels.

Next, we present the main result of this work: Alice always wins the RRG game on a simple connected graph G (independently of who starts the game) using a set of consecutive integer labels $\mathcal{L} = \{0, \ldots, k\}$ for any $k \geq (n-1) + \frac{\Delta(\Delta-1)}{2} + \max\{d(v) \cdot 2(m-d(v)) \colon v \in V(G)\}$. This result partially answers Question 1.1 posed by Tuza [8].

Theorem 2.1. *If G is a simple connected graph on n vertices, m edges and maximum degree Δ, then*

$$grac_g(G) \leq (n-1) + \frac{\Delta(\Delta-1)}{2} + \max\{d(v) \cdot 2(m-d(v)) \colon v \in V(G)\}.$$

Proof. Let G be a graph as described in the hypothesis. Also, let $\mathcal{L} = \{0, \ldots, k\}$ be a set of consecutive integer labels such that $k \geq (n-1) + \frac{\Delta(\Delta-1)}{2} + \max\{d(v) \cdot 2(m-d(v)) \colon v \in V(G)\}$. Alice (or Bob) starts the range-relaxed graceful game on G and our objective is to show a winning strategy for Alice.

For each vertex $w \in V(G)$, define a set $L(w)$ called the *set of available labels for w*. At the beginning of the game, let $L(w) = \mathcal{L}$ for all $w \in V(G)$. At each round of the game, a player (Alice or Bob) chooses an unlabeled vertex and assigns to it an available label α such that $0 \leq \alpha \leq k$ (recall that a label $\alpha \in \{0, \ldots, k\}$ is *available* to a vertex w of G if it was not assigned to any other vertex of G and it can be assigned to w without generating repeated induced edge labels). Right after a player's move, the sets of available labels of the remaining unlabeled vertices are updated so as to maintain the property that these sets only contain available labels for the respective vertices. Thus, when it is Alice's turn, she always choose an unlabeled vertex w and assign to w any label in the set of available labels $L(w)$ if $L(w) \neq \emptyset$. At the end of this proof, we show that $L(w) \neq \emptyset$

for every unlabeled vertex $w \in V(G)$ at any point of the game. Next, it is described how the sets of available labels of unlabeled vertices are updated right after each player's move.

At the j-th move, a player (Alice or Bob) chooses an unlabeled vertex $v_j \in V(G)$ and assigns an available label $f(v_j)$ to v_j. Right after the j-th move, the set of available labels $L(u)$ of each remaining unlabeled vertex $u \in V(G)$ is updated. Only unused vertex labels and vertex labels that cannot generate repeated edge labels in future iterations can remain in each set. The sets of available labels are updated according to the following three steps:

1. for every unlabeled vertex $u \in V(G)$, remove $f(v_j)$ from $L(u)$. Note that, since a range-relaxed graceful labeling is injective, the label $f(v_j)$ cannot be assigned to more than one vertex;

2. for every unlabeled vertex $u \in N(v_j)$ and for every labeled vertex $w \in N(u)$, with $w \neq v_j$, such that $f(w) \equiv f(v_j) \pmod 2$, delete from $L(u)$ the label $(f(w) + f(v_j))/2$. Note that the deletion of label $(f(w) + f(v_j))/2$ from $L(u)$ is necessary since it generates the repeated edge label $|f(w) - f(v_j)/2|$ at edges uv_j and uw;

3. for every unlabeled vertex $u \in V(G)$ and for every labeled vertex $u' \in N(u)$, delete from $L(u)$ every label ℓ such that $|\ell - f(u')| = \phi(e)$, for every edge $e \in E(G)$ that has both endpoints labeled.

Next, we determine the maximum number of labels that are deleted, throughout the game, from each set of available labels. First, note that exactly one label is deleted from each unlabeled vertex at each execution of Step 1. Thus, exactly $n - 1$ labels are deleted at Step 1 after the first $n - 1$ moves.

Now, we count how many labels are deleted after the first $n - 1$ executions of Step 2. All vertices of G have degree at most Δ. Thus, after the first $n - 1$ moves, at most $\binom{\Delta}{2} = \frac{\Delta(\Delta-1)}{2}$ labels are deleted from the set of available labels of any unlabeled vertex of G, including the last unlabeled vertex. The number $\binom{\Delta}{2}$ corresponds to the maximum number of distinct

vertex labels that can be obtained by taking the Δ labels of the neighbors of u in pairs, summing them up and dividing them by two.

Next, we count how many labels are deleted after the first $n-1$ executions of Step 3. Let u be the last unlabeled vertex. Right after $n-1$ moves, all the neighbors of u are labeled and there are exactly $m - d(u)$ edges in G with induced labels. For each neighbor $w \in N(u)$, each induced edge label ℓ' can preclude at most two vertex labels from being assigned to vertex u, namely, the labels $f(w) + \ell'$ and $f(w) - \ell'$. Thus, each neighbor of u contributes for the deletion of at most $2(m - d(u))$ labels from $L(u)$. Since u has $d(u)$ neighbors, the number of labels deleted from $L(u)$ after the first $n-1$ executions of Step 3 is at most $d(u) \cdot 2(m - d(u))$. Therefore, the maximum number of labels that can be excluded on Step 3 is $\max\{d(u) \cdot 2(m - d(u)) \colon u \in V(G)\}$.

From the previous analysis, we conclude that at most $(n-1) + \frac{\Delta(\Delta-1)}{2} + \max\{d(v) \cdot 2(m - d(v)) \colon v \in V(G)\}$ labels are deleted from each set of available labels. Since $|\mathcal{L}|$ is greater than this value, we conclude that there is always an available label at each set $L(u)$ to be assigned to an unlabeled vertex u, and the result follows. ∎

Frickes et al. [3] proved that Alice wins the graceful game ($k = |E(G)|$) on the cycle C_3, but loses on any larger cycle. This implies that $grac_g(C_n) \geq |E(C_n)| + 1 = n + 1$ for every $n \geq 4$. Since every cycle C_n with $n \geq 3$ has maximum degree $\Delta = 2$ and has n edges, the next result follows immediately from Theorem 2.1 and from these observations.

Corollary 2.1. *For any integer $n \geq 4$, $n + 1 \leq grac_g(C_n) \leq 5n - 8$. Moreover, $grac_g(C_3) = 3$.* ∎

From Corollary 2.1, we know that there exists at most $5n - 8$ different labels in the set $\mathcal{L}$ that can be assigned to a vertex $v_i \in V(G)$ in order to obtain a range-relaxed graceful labeling of the graph. Figure 2 exhibits a sequence of moves for the RRG game in a cycle with $n = 5$ vertices.

Frickes et al. [3] also proved that Alice loses the graceful game on any path P_n with $n \geq 4$. This implies that $grac_g(P_n) \geq |E(P_n)| + 1 = n$ for

$n \geq 4$. They also proved that, on P_3, the winner is the player who starts the game; and that Alice wins on P_1 and P_2.

A similar result as Corollary 2.1 can be obtained for paths.

Corollary 2.2. *For any integer $n \geq 3$, $n \leq grac_g(P_n) \leq 5n - 12$. Moreover, $grac_g(P_n) = n - 1$ for $n \in \{1, 2\}$.* ∎

(a) 1st move.   (b) 2nd move.   (c) 3rd move.

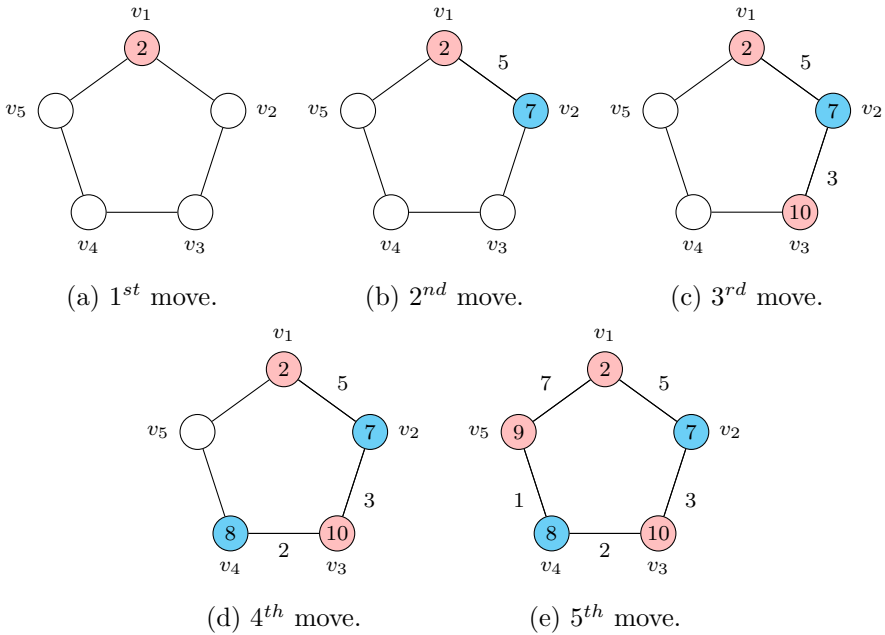(d) 4th move.   (e) 5th move.

Figure 2: A sequence of moves of the RRG game illustrating the strategy described in the proof of Theorem 2.1. A player chooses an unlabeled vertex v_i of the cycle C_5. At the beginning of the game, $L(v_i) = \mathcal{L} = \{0, \ldots, 17\}$, for every $v_i \in V(G)$. After each iteration, the set of available labels is updated for each vertex not labeled. In the last move we guarantee that there exists an available label that can be assigned for the last vertex v_i in the set $L(v_i) \subseteq \mathcal{L}$.

References

[1] C. Barrientos and E. Kroop, *Improved bounds for relaxed graceful trees*, Graphs and Combinatorics **33** (2017), 287–305.

[2] F. V. Bussel, *Relaxed Graceful labelings of trees*, The Eletronic Journal of Combinatorics **9** (2002), 1–12.

[3] L. Frickes, S. Dantas, and A. G. Luiz, *The graceful game*, In: Proceedings of the 17th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, Enschede, Netherlands (2019), 45–48.

[4] J. A. Gallian, *A dynamic survey of graph labeling*, The Eletronic Journal of Combinatorics **DS6** (2019), 1–535.

[5] S. W. Golomb, *Graph Theory and Computing*, chapter How to number a graph, Academic Press, New York (1972), 23–37.

[6] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs, International Symposium, Rome, July 1966 (1967), 349–355.

[7] G. Sethuraman, P. Ragukumar, and P. J. Slater, *Any tree with m edges can be embedded in a graceful tree with less than 4m edges and in a graceful planar graph*, Discrete Mathematics **340** (2017), 96–106.

[8] Z. Tuza, *Graph labeling games*, Eletronic Notes in Discrete Mathematics **60** (2017), 61–68.

Deise L. de Oliveira
Niterói - Brazil
Fluminense Federal University
Federal Institute of Rio de Janeiro
deiseoliveira@id.uff.br

Simone Dantas
Niterói - Brazil
Fluminense Federal University
IME - Campus Gragoatá
sdantas@id.uff.br

Atílio G. Luiz
Quixadá - Brazil
Federal University of Ceará
Campus Quixadá
gomes.atilio@ufc.br