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Independent Sets in Corona Graphs

Carmen Ortiz , Mònica Villanueva

Abstract

The corona of two graphs G and H is the graph $G \odot H$ obtained by taking one copy of G; $|V(G)|$ copies of H; and joining each vertex of the *i-th* copy of H to the *i-th* vertex of G. For general graphs, counting independent sets as well as maximal independent sets are $\#\mathcal{P}-complete$ problems. In this work, for a general graph H we determine the number of independent sets in $K_n \odot H$, $K_{1,n} \odot H$, $\overline{K_n} \odot H$ and $W_{1,n} \odot H$. We also establish the number of maximal independent sets of the corona graph $G \odot H$ of two general graphs G and H.

1 Introduction

$G = (V, E)$ is a finite undirected connected graph with no multiple edges or self loops, vertex-set $V(G) = V$ and edge-set $E(G) = E$. The neighbourhood of a vertex v in G is $N[v] = \{u \in V | (u, v) \in E\} \cup \{v\}$. For a positive integer n, the complete graph, the chordless path and the chordless cycle on n vertices are denoted by K_n, P_n and C_n, respectively. $K_{1,n}$ is the star tree with n pendant vertices. The wheel graph $W_{1,n}$ has a single vertex u connected to all vertices of an n-cycle.

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A *clique* in a graph G is a maximal complete subgraph of G. An *independent set* of G is a subset S of V such that no two vertices are adjacent. $S \subseteq V$ is a *maximal independent set* (mis) if it is not properly contained in any other independent set of G. The number of mis of a graph G is $\mu(G)$. An independent set is maximum if it has maximum cardinality. The size of a maximum independent set of G is denoted by $\alpha(G)$.

A molecule can be modeled as a graph with vertices representing atoms and edges representing bonds. A topological index is associated with chemical compounds to predict some properties since there is a very close relation between chemical characteristics of many compounds and the topological structure of its molecular graph. The Merrifield-Simmons index is the number of independent sets of the associated molecular graph and it is correlated with the boiling point of the molecule [9].

The molecular graph of some chemical compounds is obtained as a corona graph. For example, cycloalkanes with a single ring [17]. Various topological indices of different corona graphs have been studied. The Merrifield-Simmons index of a caterpillar graph $P_n \odot K_1$ and a sunlet graph $C_n \odot K_1$ are determined by Reyhani et al. [14]. Wu et al. [16] studied this index for the corona graphs $P_n \odot K_2$ and $C_n \odot K_2$. Hamzed et al. [6] discussed the cases $C_n \odot H$ and $P_n \odot H$.

A Clar structure is a mis of the Clar graph of the corresponding benzenoid hydrocarbons [3]. This paper deals with benzenoid hydrocarbons whose Clar graphs are either paths or cycles.

Valiant [15] showed that the problem of counting the number of mis is $\#\mathcal{P}-$ *complete* for a general graph. Okamoto et al. [10] proved that the problem remains so even for chordal graphs. Li et al. [8] determined the largest number of mis among all n-vertex bipartite graphs with at least one cycle. Hujter and Tuza [7] and Chang and Jou [1] solved the problem for triangle-free graphs. Ortiz and Villanueva determined $\mu(G)$ of a caterpillar graph [11] and also of grid graphs [12].

In this work, we determine the number of independent sets of various corona graphs: $K_n \odot H$, $K_{1,n} \odot H$, $\overline{K_n} \odot H$ and $W_{1,n} \odot H$ where H is an arbitrary graph. We establish the number of mis of $G \odot H$ as a function of the number of independent sets of G and the number of mis of H. In the special case of $C_n \odot K_1$ we show that its number of mis is given by the Fibonacci sequence. We build the independent graph (intersection graph of mis) and the clique graph (intersection graph of cliques) of $C_n \odot K_1$.

2 Preliminaries

The Fibonacci sequence is given by $F_0 = 0$, $F_1 = 1$ and the recurrence formula $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence is defined by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.
Prodinger and Tichy [13] introduced the *Fibonacci number* $f(G)$ of a graph G as the number of independent sets of G, not necessarily maximal, including the empty set. They proved the following results.

Lemma 2.1. *The Fibonacci number of*
a) a complete graph is $f(K_n) = n + 1$ *for* $n \geq 1$,
b) a chordless path is $f(P_n) = F_{n+2}$ *for* $n \geq 1$,
c) a chordless cycle is $f(C_n) = L_n$ *for* $n \geq 3$,
d) a star tree is $f(K_{1,n}) = 2^n + 1$ *for* $n \geq 1$,
e) a wheel graph is $f(W_{1,n}) = f(C_n) + 1$ *for* $n \geq 3$.

Let $f_k(G)$ be the number of independent sets of size k of the graph G. We define $f_0(G) = 1$.

Lemma 2.2. *a)* $f_1(K_n) = n$ *and* $f_k(K_n) = 0$ *with* $n \geq 1$ *and* $k \geq 2$,
b) $f_k(\overline{K_n}) = \binom{n}{k}$ *with* $1 \leq k \leq n$,
c) $f_k(K_{1,n}) = \binom{n}{k}$ *for* $2 \leq k \leq n$ *and* $f_1(K_{1,n}) = n + 1$,
d) $f_1(W_{1,n}) = 1 + f_1(C_n)$ *and* $f_k(W_{1,n}) = f_k(C_n)$ *for* $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Lemma 2.3. *For* $n \geq 2$ *and* $1 \leq k \leq \lceil \frac{n}{2} \rceil$,
$f_k(P_n) = f_k(P_{n-1}) + f_{k-1}(P_{n-2})$ *with* $f_0(P_n) = 1$, $f_1(P_n) = n$,
$f_2(P_n) = \binom{n-1}{2}$, $f_1(P_1) = 1$, $f_1(P_2) = 2$ *and* $f_2(P_2) = 0$.

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_a, ..., v_b, ..., v_n\}$. If $k = 1$, we have n alternatives for choosing one vertex of P_n.
If $k = 2$, let $S = \{v_a, v_b\}$ with $1 \leq a \leq b-2 \leq n-2$, be an independent set of P_n. Since v_a can be chosen in $n-2$ different ways then v_b has $n-(a+1)$ options. Thus, $f_2(P_n) = \sum_{a=1}^{n-2}(n-(a+1)) = \frac{1}{2}(n-1)(n-2)$.

For $k \geq 3$, let S be an independent set of P_{n-1} having k vertices. S is also an independent set of P_n. If we consider an independent set R of P_{n-2} that has $k-1$ vertices, $R \cup \{v_n\}$ is an independent set of P_n. ■

Lemma 2.4. *For $n \geq 3$ and $1 \leq k \leq \lceil \frac{n}{2} \rceil$, $f_k(P_n) = \binom{n-(k-1)}{k}$.*

Proof. Applying induction on n we have that $f_k(P_{n-1}) = \binom{n-1-(k-1)}{k} = \binom{n-k}{k}$ and $f_{k-1}(P_{n-2}) = \binom{n-2-(k-2)}{k-1} = \binom{n-k}{k-1}$. By binomial coefficient property and Lemma 2.3, the result follows.

■

Lemma 2.5. *For $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $f_k(C_n) = f_k(C_{n-1}) + f_{k-1}(C_{n-2})$ with $f_0(C_n) = 1$, $f_1(C_n) = n$ and $f_2(C_n) = \frac{n(n-3)}{2}$.*

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$. If $k = 1$, we have n alternatives for choosing one vertex of C_n.
If $k = 2$, let $S = \{v_a, v_b\}$ be an independent set of C_n with $a + 1 < b$. We have that v_a can be either v_1 or v_2, ..., or v_{n-2}. If $a = 1$ then v_b has $n-3$ options but if $a \neq 1$ then v_b has $n-(a+1)$ options. Thus, $f_2(C_n) = (n-3) + \sum_{a=2}^{n-2}(n-(a+1)) = \frac{n(n-3)}{2}$.

For $k \geq 3$, let S be an independent set of C_{n-1} having k vertices. We have that S is also an independent set of C_n. Now consider an independent set R in C_{n-2} that has $k-1$ vertices. There are three cases:

-If $v_1 \in R$ and $v_{n-2} \notin R$ then $R \cup \{v_{n-1}\}$ is an independent set of C_n.

-If $v_1 \notin R$ and $v_{n-2} \in R$ then $R \cup \{v_n\}$ is an independent set of C_n.

-If $v_1 \notin R$ and $v_{n-2} \notin R$ then $R \cup \{v_n\}$ is an independent set of C_n. In this case $R \cup \{v_{n-1}\}$ is also an independent set of C_{n-1} and of C_n. But it has already been considered in $f_k(C_{n-1})$. ■

Lemma 2.6. *For $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $f_k(C_n) = \binom{n-k}{k} + \binom{n-(k+1)}{k-1}$.*

Proof. Analogous to the proof of Lemma 2.4. ■

Lemma 2.7. *The number of maximal independent sets of*
a) a complete graph is $\mu(K_n) = n$ for $n \geq 1$,
b) [5] a path graph is $\mu(P_n) = \mu(P_{n-2}) + \mu(P_{n-3})$ for $n \geq 4$ with $\mu(P_1) = 1$ and $\mu(P_2) = \mu(P_3) = 2$,
c) [5] a cycle graph is $\mu(C_n) = \mu(C_{n-2}) + \mu(C_{n-3})$ for $n \geq 6$ with $\mu(C_3) = 3$, $\mu(C_4) = 2$ and $\mu(C_5) = 5$,
d) the complement of a complete graph $\mu(\overline{K_n}) = 1$ for $n \geq 1$,
e) a star tree is $\mu(K_{1,n}) = 2$ for $n \geq 1$,
f) a wheel graph is $\mu(W_{1,n}) = \mu(C_n) + 1$.

3 Independent Sets in Corona Graphs

Frucht and Harary [4] defined the corona of two graphs G and H as the graph $G \odot H$ obtained by taking one copy of G; $|V(G)|$ copies of H and joining the *i-th* vertex of G to every vertex in the *i-th* copy of H.

Lemma 3.1 ([2])**.**

a) *For two disjoint graphs G and H: $f(G \cup H) = f(G) \cdot f(H)$.*

b) *For a graph G and a vertex $v \in V(G)$: $f(G) = f(G-v) + f(G-N[v])$.*

Theorem 3.1. *Let H be an arbitrary graph. The number of independent sets of the graph $K_n \odot H$ is:*

$$f(K_n \odot H) = f(H)^{n-1}[f(H) + n].$$

Proof. Let $G_n = K_n \odot H$ and $V(K_n) = \{v_1, v_2, ..., v_n\}$. Applying Lemma 3.1 to G_n, we have that $G_n - v_n = G_{n-1} \cup H^n$ and $G_n - N[v_n] = \cup_{i=1}^{n-1} H^i$ where H^i is the i-th copy of H, with $i = 1, ..., n$.

Thus $f(G_n) = f(H)f(G_{n-1}) + f(H)^{n-1}$ and the result follows. ■

Theorem 3.2. *The number of independent sets of the graph $\overline{K_n} \odot H$ is* $f(\overline{K_n} \odot H) = (f(H)+1)^n$.

Proof. Direct consequence of Lemma 3.1. ■

Theorem 3.3. *The number of independent sets of the graph $K_{1,n} \odot H$ is:*

$$f(K_{1,n} \odot H) = f(H)^n + f(H)(f(H)+1)^n.$$

Proof. Let $G_n = K_{1,n} \odot H$ and $V(K_{1,n}) = \{u, v_1, v_2, ..., v_n\}$. Applying Lemma 3.1 to G_n we have that $G_n - v_n = G_{n-1} \cup H^n$ and $G_n - N[v_n] = H^u \cup [\cup_{i=1}^{n-1}(v_i \odot H^i)]$ where H^i is the i-th copy of H and H^u is the copy of H connected to the universal vertex u.

Thus, $f(G_n) = f(H)f(G_{n-1}) + f(H)(1+f(H))^{n-1}$. This is equivalent to the recurrence equation: $X_n = kX_{n-1} + k(k+1)^{n-1}$ whose solution is $X_n = Ck^{n-1} + k[(k+1)^n - k^n]$ where $k = f(H)$ and $C = k(k+1)$. ■

Theorem 3.4. *The number of independent sets of $G_n = W_{1,n} \odot H$ is:*

$$f(W_{1,n} \odot H) = [f(C_n \odot H)]f(H) + f(H)^n.$$

Proof. Let u be the universal vertex of $W_{1,n}$. By Lemma 3.1 we have that $G_n - u = (C_n \odot H) \cup H$ and $G_n - N[u] = H^1 \cup H^2 ... \cup H^n$. Therefore, the result follows. ■

Hamzed et al. [6] proved that

$$f(C_n \odot H) = f(P_{n-1} \odot H)f(H) + f(P_{n-3} \odot H)f(H)^2 \ .$$

4 Maximal Independent Sets in Corona Graphs

Theorem 4.1. *Let $|V(G)| = n$ and $|V(H)| = m$. The number of mis of the graph $G \odot H$ is:*

$$\mu(G \odot H) = \sum_{k=0}^{\alpha(G)} f_k(G)\mu(H)^{n-k}.$$

Proof. Let S be an independent set of size k in G. For each vertex v_i in G such that $v_i \notin S$ consider a mis T^i of H^i. We have that $T = \bigcup_{\substack{i=1 \\ v_i \notin S}}^{n} T^i$ is an independent set of $G \odot H$ because $T^i \cap T^j = \phi$ for $i \neq j$. Thus, $S \cup T$ is also an independent set in $G \odot H$. Moreover, $S \cup T$ is maximal. In fact, if we add a vertex $v_j \in V(G) \backslash S$ to $S \cup T$ then it is not an independent set since v_j is adjacent to every vertex of $T^j \subset T$. On the other hand, if we add a vertex $x \in H^i$ for some $i \in \{1, 2, ..., n\}$ such that $v_i \notin S$ then T^i is not maximal in H^i, a contradiction.

Since H has $\mu(H)$ mis and S has size k, there are $\mu(H)^{n-k}$ different sets that can be added to S. Furthermore, a set that contains one mis of every copy H^i is also a mis of $G \odot H$. ■

Table 1 summarizes the number of mis for some particular cases.

$G \diagdown H$	K_m	P_m	C_m
K_n	$(n+m)m^{n-1}$	$(n+\mu(P_m))\mu(P_m)^{n-1}$	$(n+\mu(C_m))\mu(C_m)^{n-1}$
P_n	$\sum_{k=0}^{\lceil \frac{n}{2} \rceil} f_k(P_n)m^{n-k}$	$\sum_{k=0}^{\lceil \frac{n}{2} \rceil} f_k(P_n)\mu(P_m)^{n-k}$	$\sum_{k=0}^{\lceil \frac{n}{2} \rceil} f_k(P_n)\mu(C_m)^{n-k}$
C_n	$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f_k(C_n)m^{n-k}$	$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f_k(C_n)\mu(P_m)^{n-k}$	$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f_k(C_n)\mu(C_m)^{n-k}$
$K_{1,n}$	$m^n + \sum_{k=0}^{n} \binom{n}{k} m^{n-k+1}$	$\mu(P_m)^n + \sum_{k=0}^{n} \binom{n}{k} \mu(P_m)^{n-k+1}$	$\mu(C_m)^n + \sum_{k=0}^{n} \binom{n}{k} \mu(C_m)^{n-k+1}$

Table 1: Number of Maximal Independent Sets of Corona Graphs.

In [11] it was proven that for $n \geq 1$: $\mu(P_n \odot \overline{K_m}) = F_{n+2}$ with $\mu(P_1 \odot \overline{K_m}) = 2$ and $\mu(P_2 \odot \overline{K_m}) = 3$.

The number of mis of $P_n \odot K_m$ satisfies
$\mu(P_n \odot K_m) = m[\mu(P_{n-1} \odot K_m) + \mu(P_{n-2} \odot K_m)]$ for $m \geq 2; n \geq 3$.

5 Independent and Clique Graph of $G_n = C_n \odot K_1$

The intersection graph $\mathcal{I}(G)$ of all mis on G is called the *Independent Graph* of G.

Theorem 5.1. *The Independent Graph of $G_n = C_n \odot K_1$ is a complete* *graph with $\mu(G_n)$ vertices minus one edge if n is even and it is a complete graph if n is odd.*

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $R = \{r_1, r_2, ..., r_n\}$ with r_i the vertex of the i-th copy of K_1. Given an independent set S of C_n, we have that $S \cup (R \setminus Adj_R(S))$ is a mis of G_n. If n is even, the only pair of disjoint mis in G_n is $\{v_1, r_2, v_3, ..., r_i, v_{i+1}, r_{i+2}, ..., r_n\}$ and $\{r_1, v_2, r_3, ..., v_i, r_{i+1}, v_{i+2}, ..., v_n\}$. But if n is odd there are no disjoint mis in G_n. In fact, since any mis of G_n contains at most $\frac{n-1}{2}$ vertices of the cycle C_n, it must contain at least $\frac{n+1}{2}$ vertices of R. Thus, any pair of mis has at least one common vertex in R. ■

The *Clique Graph* $\mathcal{K}(G)$ of G is the intersection graph of its cliques.

A chordless sun graph has vertex set $V(C_n) \cup V(R)$ such that $V(C_n) = \{v_1, ..., v_n\}$ induces a chordless cycle C_n and $R = \{r_1, ..., r_n\}$ is an independent set. Each vertex r_i is adjacent to v_i and $v_{(i mod n)+1}$ for $i = 1, ..., n$.

Theorem 5.2. *For $n \geq 4$, the Clique Graph $\mathcal{K}(G_n)$ of the corona graph $G_n = C_n \odot K_1$ is a chordless sun graph $S(C_n)$.*

Proof. Every clique of G_n is isomorphic to K_2. Let L_i be the clique induced by $\{v_i, v_{i+1}\}$ with $v_i, v_{i+1} \in V(C_n)$ for $i = 1, ..., n-1$ and the clique L_n induced by $\{v_n, v_1\}$. Call M_i the clique induced by v_i and r_i. L_1 shares v_1 with M_1 and with L_n. For $i = 2, ..., n-1$, L_i shares v_i with L_{i-1} and M_i. Moreover, L_i shares v_{i+1} with both L_{i+1} and M_{i+1}. Thus, G_n has $2n$ cliques such that their associated vertices in $\mathcal{K}(G_n)$ induce a chordless cycle C_n with n triangles, each one sharing an edge with C_n. Triangle T_i has vertices L_{i-1}, L_i and M_i, $i = 2, ..., n$ and T_1 has vertices L_1, L_n and M_1. ■

6 Conclusions

We have determine the number of independent sets of various corona graphs. It can be deduced that $f_k(C_n) = f_k(P_n) - f_{k-2}(P_{n-1})$. We obtain recursive and combinatorial expressions for the number of independent sets with a given size of a chordless path P_n and a cycle C_n. It would be interesting to study the number of independent sets and to characterize the independent and the clique graph of other corona graphs.

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Carmen Ortiz
Industrial Engineering School
Universidad de Valparaiso
Chile
carvic.ortiz@gmail.com

Mónica Villanueva
Informatic Engineering Department
Universidad de Santiago de Chile
Chile
monica.villanueva@usach.cl