Exact Algorithms for Biclique Coloring

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Abstract

A k-biclique-coloring of G is k-coloring of the vertices of G with
no maximal monochromatic biclique. A recent work showed that
determining if such a coloring exists is Σ_2^{P}-complete for $k > 2$. We
present algorithms to determine if a graph has a k-biclique-coloring:
an $\mathcal{O}^*(4^n)$-time algorithm using exact cover, and an $\mathcal{O}^*\big((k+2)^{2d}\big)$-
time FPT algorithm, where d is the neighborhood diversity of G.

1 Introduction

Coloring problems are among the most studied ones in graph theory.
The VERTEX COLORING problem asks if it is possible to assign one of k
possible colors for each vertex of a graph in such a way that adjacent
vertices have different colors or, equivalently, in such a way that no edge
is monochromatic. Variants and generalizations of this problem have been
proposed. Among them, the CLIQUE COLORING problem asks whether or
not we can k-color the vertices of a graph such that no maximal clique
is monochromatic, and was shown to be Σ_2^{P}-**Complete** [9] even we fix $k =$
2, but solvable in $\mathcal{O}^*(4^n)$-time [3]. On the other hand, the BICLIQUE

COLORING problem asks for a k-coloring of the input graph such that no maximal biclique is monochromatic. Similarly to CLIQUE COLORING, BICLIQUE COLORING was shown to be a Σ_2^P-Complete problem in [5], even if the number of colors is fixed to 2, while in [7] it is shown checking if a coloring is a biclique coloring is coNP-Complete. Little is known about the complexity of BICLIQUE COLORING on graph classes. For unichord-free graphs, a polynomial-time algorithm for BICLIQUE COLORING is given by Macêdo Filho et al. [8]. They also show that the smallest number of colors needed to biclique-color unichord-free graphs is either equal to or one unit greater than the size of the largest true twin class. In [7], a polynomial-time algorithm for powers of cycles and powers of paths is presented. Finally, Groshaus et al. [5] deal with H-free graphs, for every H on three vertices, with BICLIQUE COLORING being polynomial for $H \in \{K_3, P_3, \overline{P_3}\}$ and NP-Complete for $\overline{K_3}$-free graphs; moreover they show that the problem is NP-Complete for diamond-free graphs and split graphs, and polynomial for threshold graphs. Both CLIQUE COLORING and BICLIQUE COLORING can be seen as hypergraph coloring problems [1] where we want to color its vertices such that no hyperedge is monochromatic. In this paper we present the first exact algorithms for BICLIQUE COLORING, beginning with a $\mathcal{O}^*(4^n)$-time algorithm, then show a $\mathcal{O}^*\big((k+2)^{2d}\big)$-time FPT algorithm parameterized by the neighborhood diversity d and the number of colors. We conclude with directions for future work.

2   Preliminaries

We denote by $[n]$ the set $\{1, \ldots, n\}$ and by 2^S the powerset of a set S. Given a set V and a subset $S \subseteq V$, let us denote by $\overline{S} = V \setminus S$. An algorithm is FPT if, given a parameter k, its running time is bounded by $f(k)n^{\mathcal{O}(1)}$, for some computable function f. A k-coloring of a graph G is a function $\varphi : V(G) \mapsto [k]$; the coloring of a set $X \subseteq V(G)$ is defined as $\varphi(X) = \bigcup_{v \in X}\{\varphi(v)\}$ and X is *monochromatic* if $|\varphi(X)| = 1$. A k-coloring is a *proper k-biclique-coloring* if and only if no (inclusion-

wise) maximal induced biclique of G is monochromatic. The BICLIQUE COLORING problem is the problem of deciding, given G and k, whether or not G has a proper k-biclique-coloring. Unless noted, all colorings are "proper". Two vertices $u, v \in V(G)$ are *false twins* if and only if $N(u) = N(v)$; they are *true twins* if and only if $N[u] = N[v]$. A *type* is a maximal subset $D \subseteq V(G)$ of true twins or false twins. Being of the same type is an equivalence relation and partitioning G into the equivalence classes can be done in polynomial time [4]. The *neighborhood diversity* of G is the number of types of G.

A *hypergraph* $\mathcal{H} = (V(\mathcal{H}), \mathcal{E})$ has $V(\mathcal{H})$ as its vertex set and $\mathcal{E} \subseteq 2^V$ as its *hyperedge* set. A graph G is said to be a *host* of $\mathcal{H}$ if $V(G) = V(\mathcal{H})$, every hyperedge of $\mathcal{H}$ induces a connected subgraph of G and every edge of G is contained in at least one hyperedge of $\mathcal{H}$. For a hypergraph $\mathcal{H}$, a *transversal* is a set $X \subseteq V(\mathcal{H})$ that, for every hyperedge $\epsilon \in \mathcal{E}(\mathcal{H})$, $X \cap \epsilon \neq \emptyset$. If X is not a transversal we say that it is an *oblique*. The *biclique hypergraph* $\mathcal{H}_\mathcal{B}(G)$ of a graph G is the hypergraph on the same vertex set of G and with hyperedge set equal to the family of maximal bicliques of G, denoted by $\mathcal{B}(G)$. We denote by $\mathbb{T}_\mathcal{B}(G)$ the *family of all transversals* of the biclique hypergraph of G, by $\mathbb{T}_B^*(G)$ the *family of complements of transversals*, and by $\mathbb{O}_B(G)$ the *family of all obliques* $\mathcal{B}(G)$.

We also make use of the following results and problems studied in the literature; we use Lemma 2.1 and Theorem 2.1 to show an $\mathcal{O}^*(4^n)$-time algorithm for BICLIQUE COLORING, and employ Theorem 2.2 to solve BICLIQUE COLORING parameterized by neighborhood diversity and the size of the largest true twin class in $\mathcal{O}^*((b+2)^{2d})$-time.

EXACT COVER

Instance: A set $A = \{a_1, \ldots, a_n\}$, a covering family $\mathcal{F} \subseteq 2^A$ and $k \in \mathbb{N}$.

Question: Is it possible to k-partition A into φ such that $\varphi \subseteq \mathcal{F}$?

SET MULTICOVER

Instance: A set $A = \{a_1, \ldots, a_n\}$, a family $\mathcal{F} \subseteq 2^A$, $k \in \mathbb{N}$ and $c : A \mapsto \mathbb{N}$.

Question: Can we k-cover A with $\varphi \subseteq \mathcal{F}$ so that $|\{\varphi_i \mid a_j \in \varphi_i\}| \geq c(a_j)$ for every $a_j \in A$?

Lemma 2.1 ([3]). *The down closure $\mathcal{F}_\downarrow = \{X \subseteq V \mid \exists Y \in \mathcal{F}, \ X \subseteq Y\}$ of a family $\mathcal{F} \subseteq 2^V$ can be enumerated in $\mathcal{O}^*(|\mathcal{F}_\downarrow|)$-time.*

Theorem 2.1 ([2]). EXACT COVER *is solvable in $\mathcal{O}^*(2^n|\mathcal{F}|)$-time.*

Theorem 2.2 ([6]). SET MULTICOVER *can be solved in time $\mathcal{O}^*((b+2)^n|\mathcal{F}|)$, with b the maximum coverage requirement.*

3 Exact exponential algorithm

Following [3], we first show the relationships between structures of $\mathcal{B}(G)$ and colorings, then use these to build an $\mathcal{O}^*(4^n)$-time algorithm for BI-CLIQUE COLORING by stating it as an EXACT COVER instance.

Lemma 3.1. *A k-partition $\varphi = \{\varphi_1, \ldots, \varphi_k\}$ is a k-biclique-coloring of G if and only if for every i, $\overline{\varphi_i} \in \mathbb{T}_B(G)$.*

Proof. For the forward direction, suppose that there exists some φ_i such that $\overline{\varphi_i} \notin \mathbb{T}_B(G)$. This implies that there exists some $B \in \mathcal{B}(G)$ such that $B \cap \overline{\varphi_i} = \emptyset$ and that $B \subseteq \varphi_i$; that is, $|\varphi(B)| = 1$, which is a contradiction, since φ is a k-biclique-coloring. For the converse, let φ be a k-partition of G with $\overline{\varphi_i} \in \mathbb{T}_B(G)$, but suppose that φ_k is not a k-biclique-coloring. That is, there exists some maximal biclique $B \in \mathcal{B}(G)$ such that $B \subseteq \varphi_i$ for some i, which implies $B \cap \overline{\varphi_i} = \emptyset$ and $\overline{\varphi_i} \notin \mathbb{T}_B(G)$, a contradiction. ∎

Lemma 3.2. *The maximal obliques of $\mathcal{H}_B(G)$ are exactly the complements of the maximal bicliques of G.*

Proof. Let $X \in \mathbb{O}_B$ be a maximal oblique. By definition, there exists some $B \in \mathcal{B}(G)$ such that $X \cap B = \emptyset$, which implies that $X \subseteq \overline{B}$. Note that, if $X \subset \overline{B}$, there is some $v \in \overline{B} \setminus X$, which implies that $(X \cup \{v\}) \cap B = \emptyset$ and that X is not a maximal oblique. Let $B \in \mathcal{B}(G)$. By definition, $\overline{B} \in \mathbb{O}_B$ and must be maximal because $\{B, \overline{B}\}$ is a partition of $V(G)$. ∎

Corollary 3.1. *Given a graph G and a subset $X \subseteq V(G)$, there exists an $\mathcal{O}\left(n^2\right)$-time algorithm to determine if X is a maximal oblique.*

Theorem 3.1. BICLIQUE COLORING *is solvable in* $\mathcal{O}^*(4^n)$-*time.*

Proof. Our goal is to make use of Theorem 2.1 to solve an instance of EXACT COVER, with $A = V(G)$, $\mathcal{F} = \mathbb{T}_B^*(G)$ and k the partition size. Lemma 3.1 guarantees that there is a solution to our instance of BICLIQUE COLORING if and only if the corresponding EXACT COVER instance is solvable. To compute $\mathbb{T}_B^*(G)$, for each $X \in 2^{V(G)}$, we use Lemma 3.2 and Corollary 3.1 to say whether or not X is a maximal oblique of $\mathcal{H}_B(G)$. Next, we compute $\mathbb{O}_B(G)$ from its maximal elements using Lemma 2.1, use the fact that $\mathbb{T}_B(G) = 2^{V(G)} \setminus \mathbb{O}_B(G)$, and complement each transversal to obtain $\mathbb{T}_B^*(G)$. This procedure takes $\mathcal{O}^*(2^n)$-time to construct $\mathbb{T}_B^*(G)$ and $\mathcal{O}^*(4^n)$-time to obtain a coloring using Theorem 2.1. ∎

4   FPT algorithm

Suppose that we are already given a partition $\{D_1, \ldots, D_d\}$ of $V(G)$ in types. If D_i is composed of true twins, we say that it is a *true twin class* and denote D_i as T_i. Otherwise, D_i is a *false twin class* denoted by F_i. Our algorithm builds an induced subgraph H of G that contains representatives of each true and false twin class. Afterwards, we build and solve an instance of SET MULTICOVER derived from H, which we show to be equivalent to the original BICLIQUE COLORING instance of G.

Observation 1. Given G and a true twin class T_i of G, any k-biclique-coloring φ of G has $|\varphi(T_i)| = |T_i|$.

Lemma 4.1. *If F is a false twin class of G, any k-biclique-coloring φ' of G can be changed into a k-biclique-coloring φ where $|\varphi(F)| \leq 2$.*

Proof. If $|\varphi'(F)| \leq 2$, then $\varphi = \varphi'$. Otherwise, there exist $f_1, f_2, f_3 \in F$ with three different colors. Since every maximal biclique B of G that intercepts F has $F \subset B$, we have $|\varphi'(B)| \geq 3$. By making $\varphi(f_1) = \varphi'(f_1)$ and $\varphi(f_3) = \varphi(f_2) = \varphi'(f_2)$, we have $|\varphi(B)| \geq |\varphi(F)| \geq 2$. Repeating this process until $|\varphi(F)| = 2$ does not make any biclique monochromatic. ∎

Figure 1: A graph and its projected graph

Definition 4.1 (Projection and Lifting). *Let T_i and F_j be as previously defined; in the following $t'_i \in T_i$ and $f'^1_j, f'^2_j \in F_j$. We define the following projection rules: $\forall v \in T_i$, $\mathsf{Proj}(v) = \{t'_i\}$; for $f'^1_j \in F_j$, $\mathsf{Proj}(f'^1_j) = \{f'^1_j\}$; $\forall u \in F_j \setminus \{f^1_j\}$, $\mathsf{Proj}(u) = \{f'^2_j\}$ and $\mathsf{Proj}(X) = \bigcup_{u \in X} \mathsf{Proj}(u)$. Lifting rules are defined as $\mathsf{Lift}(t'_i) = \{t_i\}$; $\mathsf{Lift}(f'^1_j) = \{f^1_j\}$; $\mathsf{Lift}(f'^2_j) = F_j \setminus \{f^1_j\}$ and $\mathsf{Lift}(Y) = \bigcup_{u \in Y} \mathsf{Lift}(u)$. Note that $\mathsf{Proj}(\mathsf{Lift}(X)) = X$.*

For the remainder of this section, (G, k) is the input to BICLIQUE COLORING and H is the projected graph of G, i.e. $V(H) = \mathsf{Proj}(V(G))$ and $v'_i v'_j \in E(H)$ if and only if vertices in D_i are adjacent to vertices in D_j. Our SET MULTICOVER instance is the tuple $(V(H), \mathbb{T}^*_B(H), k, c)$ where $c(t'_i) = |T_i|$ for each true twin class T_i and $c(f'^1_j) = c(f'^2_j) = 1$ for each false twin class F_j. Observation 2 follows from the fact that $\mathbb{T}^*_B(H)$ is closed under the subset operation.

Observation 2. If there is a minimum k-multicover ψ of $V(H)$ by $\mathbb{T}^*_B(H)$, then there exists a minimum k-multicover $\psi' = \{\psi_1, \ldots, \psi_k\}$ such that $|\{j \mid u \in \psi_j\}| = c(u)$ for every $u \in V(H)$.

Lemma 4.2. *If $B' \in \mathcal{B}(H)$, then $B = \mathsf{Lift}(B') \in \mathcal{B}(G)$. If $B \in \mathcal{B}(G)$ and B is not contained in any true twin class, then $B' = \mathsf{Proj}(B) \in \mathcal{B}(H)$.*

Proof. For the first statement, note that B is a biclique by the definition of Lift and the fact that B' is a biclique. By the contrapositive, suppose that $B \notin \mathcal{B}(G)$ and that $u \in V(G) \setminus B$ is such that $B \cup \{u\}$ is a (not necessarily maximal) biclique of G. Note that either: (i) if $u \in F_j$ then $F_j \not\subseteq B$ and $\mathsf{Proj}(u) \cap B' = \emptyset$, otherwise we would have $\mathsf{Lift}(\mathsf{Proj}(u)) \subseteq B$, which is a contradiction since $u \in \mathsf{Lift}(\mathsf{Proj}(u))$; or (ii) if $u \in T_i$ then $T_i \cap B = \emptyset$, which implies that $\mathsf{Proj}(u) \not\subseteq B'$. Since $B \cup \{u\}$ is a biclique, $\mathsf{Proj}(u)$ is

adjacent to only one partition of B'. Since $\mathsf{Proj}(u) \not\subseteq B'$, $\mathsf{Proj}(B \cup \{u\}) = \mathsf{Proj}(B) \cup \mathsf{Proj}(u) = B' \cup \mathsf{Proj}(u)$ is a biclique of H and B' is not maximal.

For the second statement, by the definition of Proj, B' is a biclique of H with parts X, Y. Suppose that B' is not maximal, i.e. there is $u' \in V(H)$ such that $B' \cup \{u'\}$ is a biclique of H, and let $u \in \mathsf{Lift}(u')$. By the definition of Lift, it follows that u is adjacent to only one part of B, say $\mathsf{Lift}(X)$. Thus, for each $v \in \mathsf{Lift}(Y)$, $uv \notin E(G)$, otherwise there would be $v' \in \mathsf{Proj}(v)$ with $u'v' \in E(H)$, implying that $B \cup \{u\}$ is a biclique of G. ∎

Lemma 4.3. $X \subseteq V(H)$ *is in* $\mathbb{T}_B^*(H)$ *if and only if* $\mathsf{Lift}(X) \in \mathbb{T}_B^*(G)$.

Proof. Recall that $X \in \mathbb{T}_B^*(H)$ if and only if there is no $Y \in \mathcal{B}(H)$ with $Y \subseteq X$. Note that, for every $B' \in \mathcal{B}(H)$, $B' \not\subseteq X$ implies that $\mathsf{Lift}(B') \not\subseteq \mathsf{Lift}(X)$, since no two vertices of H are lifted to the same vertex of G, and $\mathsf{Lift}(B') \in \mathcal{B}(G)$ due to Lemma 4.2. Moreover, no biclique of G contained in a true twin class can be a subset of $\mathsf{Lift}(X)$. Thus, $\mathsf{Lift}(X)$ contains a maximal biclique B only if $\mathsf{Proj}(B) \subseteq X$ and $\mathsf{Proj}(B) \notin \mathcal{B}(H)$, which is impossible due to Lemma 4.2 and the maximality of B. Taking the contrapositive, $X \notin \mathbb{T}_B^*(H)$ implies that there is some maximal biclique B' of H such that $B' \subseteq X$. This implies that $\mathsf{Lift}(B') \subseteq \mathsf{Lift}(X)$, and, since $\mathsf{Lift}(B') \in \mathcal{B}(G)$ due to Lemma 4.2, it holds that $\mathsf{Lift}(X) \notin \mathbb{T}_B^*(G)$. ∎

Theorem 4.1. ψ *is a k-multicover of H if and only if G is k-biclique-colorable.*

Proof. Recall that a k-partition is a k-biclique-coloring if and only if all elements of the partition belong to $\mathbb{T}_B^*(G)$. By the construction of our set multicover instance, we have that, for each ψ_i, $\psi_i \in \mathbb{T}_B^*(H)$. By making $\varphi = \{\mathsf{Lift}(\psi_1), \ldots, \mathsf{Lift}(\psi_k)\}$, and recalling Observation 2, we have that each vertex $u \in V(H)$ is covered exactly $c(u)$ times; moreover, since true twins appear multiple times and types are equivalence relations, we can attribute to each $v \in T_i$ any of the $|T_i|$ colors available, as long as no two receive the same color. That is, if $T_i = \{v_1, \ldots, v_\ell\}$ and t_i' is covered by $\{\psi_{i_1}, \ldots, \psi_{i_\ell}\}$, we assign to v_j the color i_ℓ. Therefore, φ is a k-partition of

$V(G)$ and, due to Lemma 4.3, every $\mathsf{Lift}(\psi_i)$ is a complement of transversal, implying that φ is a proper k-biclique-coloring of G. For the converse, we first make use of Lemma 4.1 to guarantee that every false twin class is in at most two color classes. In particular, if two colors are required we force f_j^1 to have the smallest color and $F_j \setminus \{f_j^1\}$ to have the other one. Afterwards, for every color class φ_i, we take $\psi_i = \mathsf{Proj}(\varphi_i)$. Note that, each color class has at most one element of each T_i and, for each F_j and any two distinct color classes φ_l, φ_r, $\mathsf{Proj}(\varphi_l) \cap \mathsf{Proj}(\varphi_r) \cap \mathsf{Proj}(F_j) = \emptyset$, since f_j^1 has a different color from $F_j \setminus \{f_j^1\}$. These observations guarantee that $\mathsf{Lift}(\psi_i) = \varphi_i$ and, because of Lemma 4.3, $\psi_i \in \mathbb{T}_B^*(H)$. Finally, $\psi = \{\psi_1, \ldots, \psi_k\}$ is a valid k-multicover of H since the vertices of $V(H)$ are sufficiently covered.   ∎

Note that the size of the largest true twin class is exactly the largest coverage requirement b of our SET MULTICOVER instance. Moreover, since we need at least b colors to biclique color G, it holds that $b \leq k$.

Theorem 4.2. BICLIQUE COLORING *can be solved in* $\mathcal{O}^*\big((k+2)^{2d}\big)$.

Another option would be not to contract true twin classes, keeping all such vertices in the projected graph, which would effectively yield a kernel linear on the product kd, and, even if we use Theorem 3.1, we would still take $\mathcal{O}^*\big(4^{kd}\big)$-time, which is no better than $(k+2)^{2d} \approx 2^{2d \log k}$.

5   Conclusions and Future Work

In this work we presented two exact algorithms for the BICLIQUE COL-ORING problem. We first showed that, much like CLIQUE COLORING, BICLIQUE COLORING can be solved in $\mathcal{O}^*(4^n)$-time using the inclusion-exclusion principle, and that the problem is solvable in FPT time when parameterized by the number of colors and neighborhood diversity of the input graph. While these are only initial results, further investigations into the parameterized complexity of the problem could be of interest. In particular, we leave the parameterization by neighborhood diversity as an

open question, but are also interested in other structural parameterizations, such as treewidth and max leaf number.

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