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A note on Helly-B_1-EPG graphs

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Abstract

*Edge intersection graphs of paths on a grid (*EPG *graphs)* are graphs whose vertices can be represented as nontrivial paths on a grid such that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid. When the paths have at most one change of direction (bend) these graphs are called B_1-EPG *graphs*. In this paper, we delimit some subclasses of B_1-EPG graphs that admit a Helly-B_1-EPG representation. It is known that B_1-EPG and Helly-B_1-EPG are hereditary classes, so they can be characterized by forbidden structures. In both cases, finding the whole list of minimal forbidden induced subgraphs are challenging open problems. Taking a step towards solving those problems, we describe a few structures at least one of which will necessarily be present in any B_1-EPG graph that does not admit a Helly representation. In addition, we show that the well known families of Block graphs, Cactus and Line of Bipartite graphs are totally contained in the class Helly-B_1-EPG.

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1 Introduction

Let $\mathcal{P}$ be a collection of non trivial simple paths on a rectangular grid $\mathcal{G}$. The *edge intersection graph* of $\mathcal{P}$ (denoted by EPG($\mathcal{P}$)) is the graph whose vertices correspond to the paths of $\mathcal{P}$ and two vertices are adjacent in EPG($\mathcal{P}$) if and only if the corresponding paths in $\mathcal{P}$ share at least one edge in $\mathcal{G}$. A graph is called an *edge intersection graph of paths on a grid (EPG graph)* if $G = \text{EPG}(\mathcal{P})$ for some $\mathcal{P}$ and $\mathcal{G}$, and $\langle \mathcal{P}, \mathcal{G} \rangle$ is an EPG *representation of* G. In [6], it was proved that every graph is EPG, and started the study of the subclasses defined by bounding the number of times any path used in the representation can bend. Graphs admitting a representation where paths have at most k changes of direction (*bends*) were called B_k-EPG. In particular, when the paths have at most 1 bend we have the B_1-*EPG graphs* or a *single bend EPG graphs*. Sometimes we can refer to a specific path of the representation and its number of bends. In this particular case, we denote by *k-bend* the path or the set of paths that have at most k bends.

A collection of sets satisfies the *Helly property* when every pair-wise intersecting sub-collection has at least one common element. When this property is satisfied by the set of edges of the paths used in a EPG representation, we get a Helly-EPG representation.

In [3] were studied the Helly-B_1-EPG graphs and it was proved that not every B_1-EPG graph admits a Helly-B_1-EPG representation. We are interested in determining the subgraphs that make B_1-EPG graphs that do not admit a Helly representation. In the present work, we describe some structures that will be present in any such subgraph, and, in addition, we present new Helly-B_1-EPG subclasses.

2 Preliminaries

The *vertex set* and the *edge set* of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G

induced by S. If $\mathcal{F}$ is any family of graphs, we say that G is *$\mathcal{F}$-free* if G has no induced subgraph isomorphic to a member of $\mathcal{F}$.

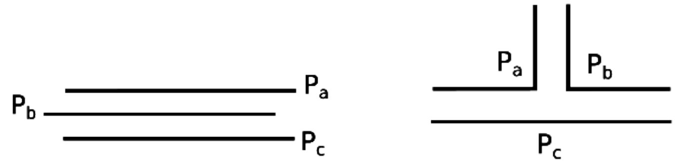
A *cycle*, denoted by C_n, is a sequence of distinct vertices $v_1, \ldots, v_n, v_1$ where $v_i \neq v_j$ for $i \neq j$ and $(v_i, v_{i+1}) \in E(G)$, such that $n \geq 3$. A *chord* is an edge that is between two non-consecutive vertices in a sequence of vertices of a cycle. An *induced cycle* or *chordless cycle* is a cycle that has no chord, in this paper an induced cycle will simply be called a *cycle*.

A *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non adjacent vertices. A *path* (in the grid) is defined as a finite sequence of consecutive edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3), \ldots, e_i = (v_i, v_{i+1}), \ldots, e_m = (v_m, v_{m+1})$, where $v_i \neq v_j$ for $i \neq j$. A *segment* is a path without bends. The notation $[x_i, x_j] \times \{y\}$ (resp. $[y_p, y_q] \times \{x\}$), where $i < j$ (resp. $p < q$), is used to denote the *horizontal segment* (resp. *vertical segment*) between columns x_i and x_j (resp. between rows y_p and y_q), on row y (resp. column x). In B_1-EPG representations we use $\lrcorner$-path notation and $\llcorner$-path notation to denote the path that has horizontal segment and with bend at right to up, and at left to up respectively.

Given an EPG representation of a graph G, we identify each vertex v of G with the corresponding path P_v of the grid used in the representation. We say that a vertex of G *covers or contains some edge of the grid* if the corresponding path does and, that *a set of paths of the representation induces a subgraph of* G if the corresponding set of vertices does.

In a B_1-EPG representation, a clique K is said to be an *edge-clique* if all the vertices of K share a common edge of the grid (see Figure 1(a)). Note that the collection of paths of K is Helly (considering edge intersection). A *claw of the grid* is a set of three edges of the grid incident into the same vertex of the grid, which is called the *center of the claw*. The two edges of the claw that have the same direction form the *base of the claw*. If K is not an edge-clique, then there exists a claw of the grid (and only one) such that the vertices of K are those containing exactly two of the three edges of the claw; such a clique is called *claw-clique* [6] (see Figure 1(b)).

Notice that if three vertices induce a claw-clique, then exactly two of



(a) Representation of a clique as edge-clique.

(b) Representation of a clique as claw-clique.

Figure 1: Examples of clique representations.

them turn at the center of the corresponding claw of the grid, and the third one contains the base of the claw. Furthermore, any other vertex adjacent to the three must contain two of the edges of that claw, then the following lemma holds.

Lemma 2.1. *If three vertices are together in more than one maximal clique of a graph G, then in any B_1-EPG representation of G the three vertices do not form a claw-clique.*

3 Subclasses of Helly-B_1-EPG Graphs

In this section, we describe a set of graphs that define Helly-B_1-EPG families. In particular, we present some features of non-trivial families of graphs properly contained in Helly-B_1 EPG, namely Bipartite, Block, Cactus and Line of Bipartite graphs.

In [1] Asinowski et al. proved the following lemma for C_4-free graphs.

Lemma 3.1. *[1] Let G be a B_1-EPG graph. If G is C_4-free, then there exists a B_1-EPG representation of G such that every maximal claw-clique K is represented on a claw of the grid such that vertices of K are the unique covering the edges of their base.*

We have obtained the following similar result for diamond-free graphs. A *diamond* is a graph G with vertex set $V(G) = \{a, b, c, d\}$ and edge set $E(G) = \{ab, ac, bc, bd, cd\}$.

Lemma 3.2. *Let G be a B_1-EPG graph. If G is diamond-free, then in any B_1-EPG representation of G, every maximal claw-clique K is represented on a claw of the grid such that vertices of K are the unique covering an edge of K.*

Proof. Since every vertex of K covers two of the three edges of the claw, if $v \notin K$ but covers edges of the claw, then it covers exactly one edge and then we have a diamond, a contradiction. ■

Let $S_3, S_{3'}, S_{3''}$ and C_4 be the graphs depicted in Figure 2.

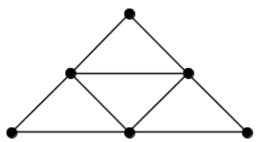
Theorem 3.1. *Let G be a B_1-EPG graph. If G is $\{S_3, S_{3'}, S_{3''}, C_4\}$-free then G is a Helly-B_1-EPG graph.*

Proof. If G is not a Helly-B_1-EPG graph, then in each B_1-EPG representation of G, there is at least one clique that is represented as claw-clique. Consider the B_1-EPG representation of G that satifies Lemma 3.1 and let K be a maximal clique which is represented as a claw-clique. Assume, without lose of generality, K is on a claw of the grid with base $[x_0, x_2] \times \{y_0\}$ and center $C = (x_1, y_0)$. Denote by $\mathcal{P}_K$ the set of paths corresponding to the vertices of K. By Lemma 3.1, the grid segment $[x_0, x_2] \times \{y_0\}$ is covered only by vertices of K. For every $\lrcorner$-path (resp. $\llcorner$-path) belonging to $\mathcal{P}_K$, we do the following: if the path does not intersect any path $P_t \notin \mathcal{P}_K$ on column x_1, then we delete its vertical segment and add to the path the segment $[x_1, x_2] \times \{y_0\}$ (resp. $[x_0, x_1] \times \{y_0\}$). If after this transformation there is no more $\lrcorner$-paths (resp. $\llcorner$-paths) in $\mathcal{P}_K$, then we are done since we have obtained an edge-clique. So we may assume that every $\lrcorner$-path and every $\llcorner$-path in $\mathcal{P}_K$ intersects some path $P_t \notin \mathcal{P}_K$ on column x_1 (notice that we can assume is the same path P_t for all the vertices).

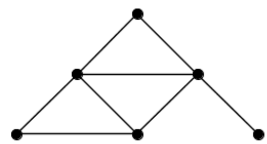
Now, if none of the $\lrcorner$-paths belonging to $\mathcal{P}_K$ intersect a path not in $\mathcal{P}_K$ on the line y_0, then we can replace the horizontal part of those paths by the segment $[x_1, x_2] \times \{y_0\}$, getting an edge representation of the clique K. Thus, we can assume there exists at least one $\lrcorner$-path $P_v \in \mathcal{P}_K$ intersecting some path $P_{t'} \notin \mathcal{P}_K$ on line y_0. Analogously, there exists at least one $\llcorner$-path $P_{v'} \in \mathcal{P}_K$ intersecting some path $P_{t''} \notin K$ on line y_0. Notice that vertex t' cannot be adjacent to any of the vertices t, v' or t''; and, in addition, vertex t'' cannot be adjacent to t, or v.

Finally, since K is claw-clique, there is a path $P_u \in \mathcal{P}_K$ covering the base of the claw. Depending on the possible adjacencies between u and t' or t'', one of the graphs S_3, $S_{3'}$ or $S_{3''}$ is obtained. ∎

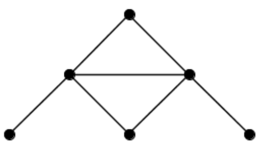
Notice that any bull-free graph is $\{S_3, S_{3'}, S_{3''}\}$-free, so our previous result implies Lemma 5 of [1].



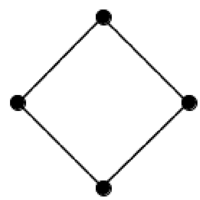
(a) Graph S_3.



(b) Graph $S_{3'}$.



(c) Graph $S_{3''}$.



(d) Graph C_4.

Figure 2: Graphs on the statement of Theorem 3.

Next theorem has as consequence the identification of several graph classes where the existence of a B_1-EPG representation ensures the existence of a Helly-B_1-EPG representation.

Theorem 3.2. *If G is a B_1-EPG and diamond-free graph then G is a Helly-B_1-EPG graph.*

Proof. Following the same argument given in the proof of Theorem we have that $G[v, v', u, t]$ induces a diamond, a contradiction. ∎

A graph G is said to be *Bipartite* if its set of vertices can be partitioned into two distinct independent sets. There are Bipartite graphs that are not B_1-EPG, for instance $K_{2,5}$ and $K_{3,3}$ (see [5]). Clearly, since bipartite graphs are triangle-free, any B_1-EPG representation of a bipartite graph is also a Helly-B_1-EPG representation. It is trivial that Bipartite graphs are diamond-free, so we have the following result which is obtained as a corollary of the previous theorem.

Corollary 3.1. *If G is a Bipartite B_1-EPG graph then G is a Helly-B_1-EPG graph.*

A *Block graph* is a type of graph in which every biconnected component (block) is a clique. Block graphs are known to be exactly the Chordal diamond-free graphs, so by Theorem 19 of [1], all Block graphs are B_1-EPG.

Corollary 3.2. *Block graphs are Helly-B_1-EPG.*

A *Cactus* graph is a connected graph in which any two cycles have at most one vertex in common. Equivalently, it is a connected graph in which every edge belongs to at most one cycle, or (for nontrivial Cactus) in which every block (maximal connected subgraph without a cut-vertex) is an edge or a cycle. It is easy to see, by their definition, that Cactus graphs are diamond-free. In [4], it is proved that every Cactus graph is a B_1-EPG graph.

Corollary 3.3. *Cactus graphs are Helly-B_1-EPG.*

Given a graph G, its *Line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint (i.e. "are incident") in G. A graph G is a *Line graph of a Bipartite graph* (or simply *Line of Bipartite*) if and only if it contains no claw, no odd cycle (with more than three vertices), and no diamond as an induced subgraph [8].

In [9] was proved that every Line graph has an EPG representation with at most two bends. In [7] it was proved that Line of Bipartite graphs are B_1-EPG graphs. In the following corollary we proved that when restricted to the Line of Bipartite we can obtain a Helly and 1-bended representation.

Corollary 3.4. *Line of Bipartite graphs are Helly-B_1-EPG.*

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