Matemática Contemporânea, Vol. 9, 23–40
http://doi.org/10.21711/231766361995/rmc93

SOAP BUBBLES IN SPACE FORMS

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1. Introduction

It is known that the surface tension on a soap bubble in R^3 forces the minimum of its area among all surfaces bounding a region of the same volume. This means that soap bubbles are stable solutions for a certain variational problem. It is also well known, from our experience, that soap bubbles are spherical. But this is far from being a precise mathematical formulation and answer to this physical phenomenon.

In 1884, Schwarz proved that, among the compact surfaces embedded in R^3 enclosing the same volume, the sphere has the least area. Probably, he was the first to study the variational problem of minimizing the area preserving the enclosed volume. In his famous notes [HH], H. Hopf pointed out that immersed surfaces with constant mean curvature are critical points for that variational problem, but he directed the attention of the readers of his notes to the determination of all compact immersed surfaces with constant mean curvature in R^3. In 1973 R. Reilly [RR] retook Hopf's idea and has shown that hypersurfaces of R^{n+1} with constant r-mean curvature H_r are critical points for the variation problem of minimizing the functional

$$A = \int_M S_r \, dM \ ,$$

keeping the volume fixed, where $S_r = \binom{n}{r} H_r$. Reilly also computed the second variation formula for this problem. In 1984 J.L. Barbosa and M.P. do Carmo

*Both authors were partially supported by CNPq and FINEP, Brazil

generalized Schwarz's result in the class of immersed compact hypersurfaces in R^{n+1}. They introduced the notion of stability and showed that spheres are the only stable critical points of the variational problem of minimizing the area functional keeping the volume constant.

Recently the notion of stability was extended to the functionals treated by R. Reilly when the ambient space has constant sectional curvatures. Similar results to the ones of Barbosa and do Carmo have been obtained by Barbosa - do Carmo - Eschenburg [BCE], Alencar - do Carmo - Colares [ACC], Alencar - do Carmo - Rosenberg [ACR] and Barbosa - Colares [BC1] and [BC2].

In this work we give an unified presentation of all the relevant results in this subject; an overview of the stability problem for closed hypersurfaces.

2. Preliminaries

Let $\overline{M}^{n+1}(c)$ be an orientable simply connected complete Riemannian manifold with constant sectional curvatures c. Represent by $\langle .,. \rangle$ its Riemannian structure and by $\overline{D}$ the associated connection form. Denote by $d\overline{M}$ a chosen globally defined volume form of $\overline{M}$.

Let M^n be a compact connected orientable Riemannian manifold and $x : M^n \to \overline{M}^{n+1}(c)$ an isometric immersion. Choose a globally defined unit normal vector field N along M and denote by A the second fundamental form of x, associated to N, defined by

$$A = \overline{D}N \; . \tag{1}$$

At each point $p \in M$, the eigenvalues of A, the principal curvatures of the immersion, are represented by $k_1(p), \ldots, k_n(p)$. The elementary symmetric functions S_r of $k_1, \ldots, k_n$ are globally defined on M, by

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r} \; . \tag{2}$$

Symmetric functions provide examples of invariants associated to the immersion and allow the definition of the r-mean curvatures as $H_r = S_r/\binom{n}{r}$. Another set of invariants can be produced by the Newton transformations P_r, defined inductively by

$$\begin{aligned} P_0 &= I \\ P_r &= S_r I - A P_{r-1}. \end{aligned} \quad (3)$$

If $e_1, \ldots, e_n$ are the eigenvectors of A, we represent by A_i the restriction of the transformation A to the subspace normal to e_i and by $S_r(A_i)$ the r-symmetric function associated to A_i. It can be proved that $e_1, \ldots, e_n$ are, also, eigenvectors of P_r. Consequently, A and P_r are simultaneously diagonalizable and

$$P_r(e_i) = S_r(A_i)e_i \ . \quad (4)$$

In [RR] R. Reilly has shown that S_r =constant is the Euler equation of a variational problem. This matter will be central in the present work.

A variation of the immersion x is a differentiable map $X : I \times M^n \to \overline{M}^{n+1}(c)$ such that, for each $t \in I = (-\varepsilon, \varepsilon)$, $X_t(p) = X(t,p)$, $p \in M$, is an immersion, $X_o = x$, and $X_t|_{\partial M} = x|_{\partial M}$, where ∂M denotes the boundary of M^n. The *balance of volume* is the real function $V : I \to R$ defined by

$$V(t) = \int_{[0,t]\times M} X^* dM \ .$$

We decide to call it "balance of volume" because, for a compact closed hypersurface M in the Euclidean space, it measures the balance of volume of the domain enclosed by M, from the time 0 to the time t. A variation is said to be *volume preserving* if $V(t) \equiv 0$.

For a general variation X, we denote by N_t the unit normal vector field of the immersion X_t and, by dM_t, the volume element of the metric induced on M by X_t. It can be proved that

$$\frac{\partial}{\partial t}(dM_t) = (-S_1 f(t) + div(\partial X/\partial t)^T) dM_t \ , \quad (5)$$

where $f(t) = \langle \partial X/\partial t, N_t \rangle$, *div* stands for the divergence operator and $(\partial X/\partial t)^T$ is the tangent component of $\partial X/\partial t$. It is a standard result that

$$\frac{d}{dt}V(t) = \int_M f(t) dM_t \ . \quad (6)$$

3. Stability for $S_1 =$ constant

For an isometric immersion $x : M^n \to R^{n+1}$ of a compact manifold M^n, we will consider here the variational problem of minimizing the area functional

$$\mathcal{A} = \int_M dM \;, \tag{7}$$

among volume preserving variations. This has been treated in [BC]. To determine the corresponding Euler equation, we use Lagrange's multipliers rule, by considering the operator

$$J(t) = \mathcal{A}(t) + \lambda V(t) \;, \tag{8}$$

where λ is a constant to be determined, and compute its derivative.

Proposition 3.1. (First Variation Formula). *For any variation of x,*

$$J'(0) = \int_M (-S_1 + k) f dM \;, \tag{9}$$

where k is a constant.

As a consequence of this proposition, critical points of the above variational problem are the immersions x with

$$S_1 = \text{constant} \;. \tag{10}$$

We can say more, as shown in the following proposition.

Proposition 3.2. ([BC]). *Under the above notation, the following are equivalent statements:*

(i) x has constant mean curvature;

(ii) $\mathcal{A}'(0) = 0$, for all volume preserving variations;

(iii) $J'(0) = 0$, for all variations.

In order to know whether or not x is a local minimum, one has to consider the second variation formula. By the first variation formula we have

$$J'(t) = \mathcal{A}'(t) = \int_M (-S_1(t) + k) f(t) dM_t \ , \tag{11}$$

for the function f satisfying

$$\int_M f(t) dM_t = 0 \ .$$

Since at the point $t = 0$ we have $S_1(0) - k = 0$, we just need to computer $S_1'(0)$. In [RR] it is proved that

$$S_1' = \Delta f + \|A\|^2 f \ ,$$

where $\|A\|^2$ is the square of the norm of the second fundamental form of x and Δ is the Laplacian. Therefore, the derivative of (11) at $t = 0$, is

$$\mathcal{A}''(0) = -\int_M S_1' f dM = -\int_M f(\Delta f + \|A\|^2 f) dM \ .$$

Thus, we have proved the following proposition.

Proposition 3.3. (Second Variation Formula). *Let $x : M^n \to R^{n+1}$ an isometric immersion with S_1 constant. For volume preserving variations, the second derivative of $\mathcal{A}$ at $t = 0$ is given by*

$$\mathcal{A}''(0) = -\int_M f(\Delta f + \|A\|^2 f) dM \ . \tag{12}$$

Definition. *Let $x : M^n \to R^{n+1}$ be an isometric immersion with $S_1 =$ constant. We say that x is stable if $\mathcal{A}'(0) \geq 0$, for any volume preserving variation.*

The first result about stable submanifolds with constant S_1 was proved by Barbosa and do Carmo.

Theorem 3.4. ([BC], 1984). *Let $x : M^n \to R^{n+1}$ be an isometric immersion of an orientable, compact without boundary, Riemannian manifold M^n with $S_1 =$ constant. Then, x is stable if and only if $x(M^n)$ is a hypersphere.*

Sketch of the proof. Assuming stability of the immersion, the proof is done by considering the "test function"

$$f = n + \langle x, N \rangle S_1. \tag{13}$$

Standard computation and stability then yields

$$0 \leq -\int_M f(\Delta f + \|A\|^2 f) dM = \int_M (S_1^2/n - \|A\|^2) dM \leq 0 \, , \tag{14}$$

from where one concludes that the immersion is totally umbilic.

A first generalization of Theorem 3.4 was obtained four years later.

Theorem 3.5. ([BCE], 1988). *Consider $x : M^n \to \overline{M}^{n+1}(c)$, where $\overline{M}^{n+1}(c)$ represents either $S^{n+1}(1)$ or $H^{n+1}(-1)$ and M^n is an orientable, compact without boundary, Riemannian manifold. Assume x has constant S_1. Then, x is stable if and only if $x(M^n)$ is a geodesic sphere.*

Here "stable" means the same as above. The formula of first variation is also the same as (9). The second variation formula is the same as the one in (12) with one more term involving the curvature c of the ambient manifold. This arises from the equation

$$S_1'(t) = \Delta f + \|A\|^2 f + cnf \, ,$$

(see [RR]). In 1988, Heintz in [HE] obtained an unified and simple proof of Theorems 3.4 and 3.5, by using estimates of the first eigenvalue of the Laplacian. A survey on stability for S_1 =constant is given in [CM].

4. Stability for S_2 = constant

First, we seek a functional on M whose critical points are immersions with S_2 = constant. Such functional turns out to be:

$$\mathcal{A}_1 = \int_M S_1 dM \, . \tag{15}$$

This is going to be shown by the use of the first variation formula. We point out that, for S_1 = constant, we used the functional

$$\mathcal{A}_0 = \mathcal{A} = \int_M dM = \int_M S_0 dM \ ,$$

with $S_0 = 1$. Therefore, our choice is a natural generalization of the previous one.

The first difficulty to work with (15) is in the computation of the first variation formula, which is given below.

Proposition 4.1. (First Variation Formula). *Let $x : M^n \to R^{n+1}$ be as above. Set $J_1 = \mathcal{A}_1 + kV$. For any variation of x,*

$$J_1'(0) = \int_M (-S_2 + k) f dM, \tag{16}$$

where k is a constant.

Proof. By ([BC2], Proposition 4.1)

$$S_1'(t) = \Delta(f) + (S_1^2 - 2S_2)f + D_{(\partial X/\partial t)^T} S_1 \ .$$

Using this and (5) we get

$$\begin{aligned} \mathcal{A}_1'(0) &= \int_M [(S_1^2 - 2S_2)f + D_{(\frac{\partial X}{\partial t})^T} S_1 \\ &+ S_1(-S_1 f + \operatorname{div}(\frac{\partial X}{\partial t})^T] dM \\ &= \int_M [-2S_2 f + D_{(\frac{\partial X}{\partial t})^T} S_1 + S_1 \operatorname{div}(\frac{\partial X}{\partial t})^T] dM \\ &= -2\int_M S_2 f dM \ , \end{aligned}$$

where we have used that

$$D_{(\frac{\partial X}{\partial t})^T} S_1 + S_1 \, div(\frac{\partial X}{\partial t})^T = div(S_1 (\frac{\partial X}{\partial t})^T) \ .$$

To finish the proof, just use (6).

This proposition says that critical points of this variational problem are the immersions with S_2 = constant. An analogous of Proposition 3.2 also holds with the changes:

1. S_2 in place of S_1 ;

2. A_1 in place of A ;

3. J_1 in place of J .

In order to decide whether or not the immersion is a local minimum we compute the second variation formula.

Proposition 4.2. (Second Variation Formula). *Let $x : M^n \to R^{n+1}$ be an isometric immersion with $S_2 =$ constant. For volume preserving variations,*

$$\mathcal{A}_1''(0) = -2\int_M [f[L_1(f) + (S_1S_2 - 3S_3)f]dM \ . \tag{17}$$

The proof uses that

$$S_2' = L_1(f) + (S_1S_2 - 3S_3)f + D_{(\partial X/\partial t)^T}S_2 + c(n-1)S_1f \tag{18}$$

(see [BC2], Prop. 4.1), where L_1 is a second order operator given by

$$L_1(f) = div(P_1\nabla f), \tag{19}$$

with $P_1 = S_1I - A$, as in (3), and c is the curvature of the ambient space (which vanishes in R^{n+1}). This operator coincides with the Laplacian (up to a constant factor) if M is umbilic; in fact, in this case

$$P_1 = (n-k)I$$

and so

$$L_1(f) = (n-k)\Delta(f),$$

where k is the eigenvalue of A. This operator has appeared by the first time in [CY].

This variational problem has been extended (in [ACC]) to the case when ambient space is $\overline{M}^{n+1}(c)$. One uses the same functional $\mathcal{A}_1$ and the first variation formula (16), holds. On the other hand, formula (17) for the second variation has one more term involving the curvature c of the ambient space, as follows:

$$\mathcal{A}_1''(0) = -2\int_M f[L_1(f) + (S_1S_2 - 3S_3)f + c(n-1)S_1f]dM \ , \tag{20}$$

as one can see from (18).

Stability for this problem is defined as follows:

Definition. *Let $x : M^n \to \overline{M}^{n+1}(c)$ be an isometric immersion with $S_2 =$ constant. We say that x is 1-stable if $\mathcal{A}_1''(0) \geq 0$, for any volume preserving variation.*

Theorem 4.3. ([ACC], 1993). *Let $x : M^n \to \overline{M}^{n+1}(c)$ be an isometric immersion of an orientable, compact without boundary, Riemannian manifold M^n in $\overline{M}^{n+1}(c)$, $c \geq 0$, with $S_2 =$ constant. When $c > 0$ we further assume that $x(M^n)$ is contained in an open hemisphere of S^{n+1}. Then, x is 1-stable if and only if $x(M^n)$ is a geodesic sphere.*

The proof begins by seeking a "test function" f with

$$\int_M f dM = 0 .$$

For $S_1 =$ constant in R^{n+1}, Barbosa and do Carmo used the integrand of the first Minkowski formula (13). Here, a natural candidate is the integrand of the second Minkowski formula

$$f = \frac{n-1}{2} S_1 + \langle x, N \rangle S_2 .$$

Since $S_2 =$ constant, to compute $L_1 f$ we must only compute $L_1 S_1$ and $L_1 \langle x, N \rangle$. This was done in [ACC]. The proof in R^{n+1} is completed in a similar way as the Barbosa - do Carmo's proof for Theorem 3.4. For $c > 0$, the proof is similar to that of Theorem 3.5.

The case when the ambient manifold has curvature $c < 0$ was left as an open problem. As matter of fact, by that time, nobody knew even a formulation for a variation problem that would give, as critical points, hypersurfaces with $S_r =$ constant, $r > 2$, in the sphere or in the hyperbolic space.

5. Stability for $S_{r+1} =$ constant

Which are the immersed hypersurfaces with $S_{r+1} =$ constant? A first answer is the following theorem.

Theorem 5.1. ([ACR], 1993). *Let $x : M^n \to R^{n+1}$ be an isometric immersion of an orientable, compact without boundary, Riemannian manifold M^n with S_{r+1} = constant. Then, x is r-stable if and only if $x(M^n)$ is a sphere.*

Here r-stable means that the immersion has S_{r+1} = constant and furthermore $\mathcal{A}_r''(0) \geq 0$, with respect to volume preserving variations, where $\mathcal{A}_r$ is the functional

$$\mathcal{A}_r = \int_M S_r dM \ , \tag{21}$$

defined by R. Reilly [RR]. In fact, the formulation of the problem of r-stability in Euclidean spaces became natural after the paper of Barbosa and do Carmo.

The formulas for the first and second variation follow a precise pattern, as we can see in the following comparative table.

In R^{n+1}

$$S_1 = \text{constant} \quad \begin{cases} \mathcal{A} = \int_M S_0 dM \\ \mathcal{A}'(0) = -\int_M S_1 f dM \\ \mathcal{A}''(0) = -\int_M f\{\Delta f - (S_1^2 - 2S_2)f\} dM \end{cases}$$

$$S_2 = \text{constant} \quad \begin{cases} \mathcal{A}_1 = \int_M S_1 dM \\ \mathcal{A}_1'(0) = -2\int_M S_2 f dM \\ \mathcal{A}_1''(0) = -2\int_M f\{L_1 f - (S_1 S_2 - 3S_3)f\} dM \end{cases}$$

$$S_{r+1} = \text{constant} \quad \begin{cases} \mathcal{A}_r = \int_M S_r dM \\ \mathcal{A}_r'(0) = -(r+1)\int_M S_{r+1} f dM \\ \mathcal{A}_r''(0) = -(r+1)\int_M f\{L_r f - (S_1 S_{r+1} - (r+2)S_{r+2} f\} dM \end{cases}$$

In $S^{n+1}(1)$ and $H^{n+1}(-1)$

$$S_1 = \text{constant} \quad \begin{cases} \mathcal{A} = \int_M S_0 dM \\ \mathcal{A}'(0) = -\int_M S_1 f dM \\ \mathcal{A}''(0) = -\int_M f\{\Delta f - (S_1^2 - 2S_2)f + cnf\} dM \end{cases}$$

$$S_2 = \text{constant} \quad \begin{cases} \mathcal{A}_1 = \int_M S_1 dM \\ \mathcal{A}_1'(0) = -2\int_M S_2 f dM \\ \mathcal{A}_1''(0) = -2\int_M f\{L_1 f - (S_1 S_2 - 3S_3)f + c(n-1)S_1 f\} dM \end{cases}$$

$$S_{r+1} = \text{constant} \quad \begin{cases} \mathcal{A}_{r,c} = ? \\ \mathcal{A}_{r,c}'(0) = ? \\ \mathcal{A}_{r,c}''(0) = ? \end{cases}$$

Remark 1. We use that $\|A\|^2 = S_1^2 - 2S_2$ in (12) to get the above expression of $\mathcal{A}''(0)$.

Remark 2. To compute $\mathcal{A}_r'(0)$, we use that

$$S_r' = L_{r-1}(f) + (S_1 S_r - (r+1)S_{r+1}f + c(n-r+1)S_{r-1}f + D_{(\partial X/\partial t)^T}S_r \quad (22)$$

(see [BC2], Prop. 4.1).
This was first proved by Reilly in [RR].

Remark 3. The expression for S_{r+1}' is used to obtain the formula of $\mathcal{A}_r''(0)$, where appears a second order operator L_r (generalization of L_1 given in (19), given by

$$L_r(f) = div(P_r \nabla f) \ ,$$

where P_r is the Newton transformation defined in (3). The operator L_r satisfies:

(i) $L_0(f) = \Delta f$;

(ii) if $S_{r+1} > 0$, then L_r is elliptic ([BC2], Prop. 3.2);

(iii) if $e_1, \ldots, e_n$ is an orthonormal basis of proper vectors of A (hence of P_r), then L_r is elliptic if and only if $\langle P_r e_i, e_i \rangle > 0$, $\forall i$.

To formulate the variational problem, with $S_{r+1} =$ constant, in $S^{n+1}(1)$ and $H^{n+1}(-1)$, we are going the define a functional $\mathcal{A}_{r,c}$, on M, whose critical points are hypersurfaces with $S_{r+1} =$ constant, for volume preserving variations.

We can not simply use the functional

$$\mathcal{A}_r = \int_M S_r dM \ ,$$

given above for the Euclidean space. In fact, its derivation would give us

$$\mathcal{A}_r'(0) = -(r+1) \int_M [S_{r+1}f + c(n-r+1)S_{r-1}f] dM \ .$$

This involves also S_{r-1}, and so $S_{r+1} =$ constant would not imply $\mathcal{A}_r'(0) = 0$.

In [BC2] was proposed the functional

$$\mathcal{A}_{r,c} = \int_M F_r(S_1, S_2, \ldots, S_r) dM \ ,$$

where the functions F_r are defined inductively by

$$\begin{aligned} F_0 &= 1 \\ F_1 &= S_1 \\ F_r &= S_r + \frac{c(n-r+1)}{r-1} F_{r-2}, \quad 2 \leq r \leq n-1. \end{aligned} \tag{23}$$

Observe that, for $c = 0$, $\mathcal{A}_{r,c}$ coincides with $\mathcal{A}_r$. Consider the operator

$$J_{r,c}(t) = \mathcal{A}_{r,c}(t) + \lambda V(t), \quad \lambda = \text{ constant to be determined.} \tag{24}$$

Proposition 5.2. (First Variation Formula). *Let $x : M^n \to \overline{M}^{n+1}(c)$ be an isometric immersion of an orientable, compact, Riemannian manifold M^n. For any variation of x,*

$$J'_{r,c}(0) = \int_M [-(r+1) S_{r+1} + k] f dM \ , \tag{25}$$

where k is a constant.

Proof. For a volume preserving variation we claim that

$$\mathcal{A}'_{r,c}(t) = -(r+1) \int_M S_{r+1} f(t) dM_t \ .$$

Suppose $c = 0$. By (22) and (5) we get

$$\begin{aligned} \mathcal{A}'_r(t) &= \int_M [(S_1 S_r - (r+1) S_{r+1}) f(t) + D_{(\partial X/\partial t)^T} S_r \\ &+ S_r(-S_1 + \operatorname{div}(\partial X/\partial t)^T) f(t)] dM_t \ . \end{aligned}$$

Because

$$D_{(\partial X/\partial t)^T} S_r + S_r div(\partial X/\partial t)^T = div(S_r (\partial X/\partial t)^T) \ , \tag{26}$$

the claim is proved for $c = 0$.

Suppose $c \neq 0$. Then, the terms of $\mathcal{A}'_{r,c}$ which involves c can be written as

$$\sum_{i=1}^{[\frac{r+1}{2}]} c^i d_i \left[\int_M S'_{r-2i} dM_t + \int_M S_{r-2i} \frac{d}{dt}(dM_t) \right] \ , \tag{27}$$

where $[\frac{r+1}{2}]$ = the greatest integer less than $\frac{r+1}{2}$ and d_i is a constant. By (22) and (5) applied to (27) we obtain that the non-vanishing terms have the form

$$c^i d_i \int_M \{[-(r-2i+1)S_{r-2i+1} + c(r-2i-1)S_{r-2i-1} \\ +D_{(\partial X/\partial t)^T} S_{r-2i}]f(t) + S_{r-2i}\mathrm{div}(\partial X/\partial t)^T\} dM_t \,. \qquad (28)$$

But, the first two terms in (28) cancel with terms in

$$\int_M c^{i-1} d_{i-1} S'_{r-2(i-1)} dM_t$$

and

$$\int_M c^{i+1} d_{i+1} S'_{r-2(i+1)} dM_t \,.$$

On the other hand, the last two terms of (28) also vanishes by (26) with S_{r-2i} in place of S_r, proving the claim also for $c \neq 0$.

To finish the proof of the proposition, we just add a term of the form

$$\int_M kf(t) dM_t \,.$$

This proposition says that the critical points of the above variational problem are the immersions x for which

$$S_{r+1} = \text{constant} \,.$$

To decide whether or not x is a local minimum, we restrict ourselves to volume preserving variation and compute $\mathcal{A}''_{r,c}(0)$ at the point x. For such variations, $J'_{r,c} = \mathcal{A}'_{r,c}$.

Proposition 5.3. (Second Variation Formula). *Let $x : M^n \to \overline{M}^{n+1}(x)$ be an isometric immersion with S_{r+1} = constant. Then, for volume preserving variations,*

$$\mathcal{A}''_{r,c}(0) = -(r+1)\int_M f[L_r f + (S_1 S_{r+1} - (r+2)S_{r+2})f + c(n-r)S_r f] dM. \quad (29)$$

The proof is a simple application of formulas (22) and (5).

Definition. *Let $x : M^n \to \overline{M}^{n+1}(c)$ be an isometric immersion with $S_{r+1} =$ constant. We say that x is r-stable if $\mathcal{A}''_{r,c}(0) \geq 0$, for any volume preserving variation of x.*

The theorem below has been proved by Alencar, do Carmo and Rosenberg in [ACR] for $c = 0$ and by Barbosa and Colares in [BC2], for $c \neq 0$. Here, $\overline{M}^{n+1}(c)$ is R^{n+1}, $S^{n+1}(1)$ or $H^{+1}(-1)$.

Theorem 5.4. (Alencar, Barbosa, do Carmo, Colares and Rosenberg, 1993). *Let $x : M^n \to \overline{M}^{n+1}(c)$ be an isometric immersion of an orientable, compact without boundary, Riemannian manifold, with $S_{r+1} =$ constant. If $c > 0$ suppose $x(M^n)$ is contained in an open hemisphere of S^{n+1}. Then, x is r-stable if and only if $x(M^n)$ is a geodesic sphere.*

Sketch of the Proof. *The condition is sufficient.* We use that, for umbilical hypersurfaces,

$$L_r(f) = \binom{n-1}{r} k^r \Delta f \ ,$$

where k is the principal curvature; then, we follow the argument in [BCE].

The condition is necessary. We consider three cases.

CASE I. Suppose $M^{n+1}(c) = R^{n+1}$. We follow the proof in [ACR]. In terms of

$$H_r = S_r \, / \binom{n}{r} \ ,$$

the second variation formula (29) becomes

$$\mathcal{A}''_r(0) = \int_M -(r+1) - fL_r f + [(r+1)c(r+1)H_{r+2} - nc(r)H_1 H_{r+1}]f^2 \ , \quad (30)$$

where $c(r) = (n-r)\binom{n}{r}$.

Choose as "test function" f, the first eigenfunction of L_r: $L_r(f) + \lambda_1 f = 0$, λ_1 being the first eigenvalue of L_r. By hypothesis, $S_{r+1} =$ constant. Since M^n is compact, $S_{r+1} > 0$.

We will use that ([ACR], Corollary 1.2)

$$\lambda_1 \leq c(r) H_{r+1}^{(r+2)/(r+1)}, \quad (31)$$

and equality holds when $x(M^n)$ is a sphere.

Suppose x is r-stable. Then, by (30) and (31), we get

$$\int_M [(r+1)c(r)H_{r+1}^{\frac{r+2}{r+1}} + (r+1)c(r+1)H_{r+2} - nc(r)H_1 H_{r+1}]f^2 \geq 0 \ . \qquad (32)$$

On the other hand, it can be proved that

$$H_1 H_{r+1} \geq H_{r+2} \ .$$

This applied to the integrand of (32) gives

$$\begin{aligned} &[(r+1)c(r)H_{r+1}^{\frac{r+2}{r+1}} + (r+1)c(r+1)H_{r+2} - nc(r)h_1 H_{r+1} \\ &\leq (r+1)c(r)(H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1}) \leq 0, \qquad (33) \\ &\qquad (34) \end{aligned}$$

because

$$H_1 H_{r+1} \geq H_{r+1}^{\frac{r+2}{r+1}} \ ,$$

with equality at the umbilic points ([MR], Lemma 1). Therefore, (32) and (33) implies that we must have equality everywhere, proving that $x(M^n)$ is a sphere.

CASE II. Suppose $x(M^n)$ is containing an open hemisphere of $S^{n+1}(1) \subset R^{n+2}$. Let $U \in R^{n+2}$ be any constant vector and N an unit normal vector field to M^n. Consider the height functions

$$g = \langle N, U \rangle \quad \text{and} \quad f = \langle x, U \rangle \ .$$

Then,

$$L_r g = -(r+1)S_{r+1}f - c(n-r)S_r g$$

and, if $S_{r+1} =$ constant,

$$L_r f = -(S_1 S_{r+1} - (r+2)S_{r+2})f - c(r+1)S_{r+1} g \ .$$

Take $N_0 = \int_M N dM$ which is different from zero, and consider $U_0, U_1, \ldots, U_{n+1}$, $U_0 = \frac{N_0}{|N_0|}$, $f_i = \langle N, U_i \rangle$ and $g_i = \langle x, U_i \rangle$. Then

$$\int_M f_i dM = 0 \ , \qquad i = 1, \ldots, n+1 \ .$$

Suppose x is r-stable. Then, by (29)

$$\begin{aligned} 0 &\leq -\sum_{i=1}^{n+1}\int_M f_i[L_r f_i + (S_1 S_{r+1} - (r+2)S_{r+2})f_i + (n-r)S_r f_i]dM \\ &= \int_M (r+1)\sum_{i=1}^{r+1} S_{r+1} f_i g_i - (n-r)S_r \sum_{i=1}^{n+1} f_i^2 \\ &\leq \int_M (r+1)S_{r+1}(-f_0 g_0) - (n-r)S_r g_0^2 \\ &= \int_m g_0 L_r g_0 = -\int_M \langle P_r \nabla g_0, g_0\rangle \leq 0 \ , \end{aligned}$$

where the last equality comes from the ellipticity of L_r . In particular, we obtain:

$$\nabla g_0 = 0 \quad \Rightarrow \quad \langle x, U_0\rangle = g_0 = \text{constant} \ ,$$

hence $x(M^n)$is a geodesic sphere.

CASE III. Suppose $\overline{M}^{n+1}(c) = H^{n+1}(-1) \subset I\!\!L^{n+2}$. The proof in this case is essentially the same as that of Case II, but we must start with $\overline{x} = \int_M x dM$.

Remark 1. To prove the Euclidean case of Theorem 5.4 we could have used Minkowski formula

$$\int_M (\frac{n-r}{r+1}S_r + \langle x, N\rangle \, S_{r+1})dM = 0 \ ,$$

(see [MR]) to obtain the test function $f = \frac{n-r}{r+1}S_r + S_{r+1}\langle x, N\rangle$. But we would have to compute both $L_r S_r$and $L_r S_{r+1}$. To see how hard it would be, it suffices to consult the computation in [ACC] for $L_1 S_1$.

Remark 2. It is not known an unified proof of Theorem 5.4 which works for R^{n+1}, $S^{n+1}(1)$ and $H^{n+1}(-1)$.

Remark 3. In stability of closed hypersurfaces there is no loss of generality to work with orthogonal variation (i.e., with $(\partial X/\partial t)^T \equiv 0$), by (5), (22) and (26).

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