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INDECOMPOSABLE BARIC ALGEBRAS*

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Abstract

We introduce the concept of indecomposable baric algebras and prove a Krull-Schmidt theorem for algebras satisfying ascending and descending chain conditions.

1. Baric Algebras

Baric algebras play a central role in the theory of genetic algebras. They were introduced by I.M.H. Etherington [3], aiming for an algebraic treatment of Population Genetics. But the whole class of baric algebras is too large, some conditions (usually with a background in Genetics) must be imposed in order to obtain a workable mathematical object. With this in mind, several classes of baric algebras have been defined: train, Bernstein, special triangular, etc. But there are relevant examples in Genetics which do not give rise to baric algebras, see [5]. As a sample of the recent work in the field of genetic algebra, see [2], [6], [8], [9] and [10].

Let F be a field of characteristic not 2, A an algebra over F, not necessarily associative, commutative or finite dimensional. If $\omega: A \longrightarrow F$ is a nonzero homomorphism, then the ordered pair (A, ω) will be called a baric algebra over F and ω its weight function. For $x \in A$, $\omega(x)$ is called the weight of x. The set $N = \{x \in A \mid w(x) = 0\}$ is a two-sided ideal of A of codimension 1.

A baric homomorphism from (A, ω) to (A', ω') is a homomorphism of F-algebras $\varphi: A \longrightarrow A'$ such that $\omega' \circ \varphi = \omega$. In particular, $\varphi(\ker \omega) \subseteq \ker \omega'$.

*This is an announcement of the results of the paper "Indecomposable baric algebras", by myself and R. Costa, to appear in Linear Algebra and its Applications.

If φ and φ' are baric homomorphisms, the same holds for $\varphi \circ \varphi'$ and for φ^{-1}, when φ is bijective.

Every baric algebra (A, ω) can be decomposed as $A = Fc \oplus N$, where c is any element of A with $\omega(c) = 1$: for $x \in A$, $x = \omega(x)c + (x - \omega(x)c)$ and $x - \omega(x)c \in \ker \omega$. From this, every left ideal of N, say J, such that $cJ \subseteq J$ is also a left ideal of A. Similarly for right ideals, with the condition $Jc \subseteq J$. The converse is also true. Many (but not all) baric algebras relevant in Genetics, have an idempotent e such that $\omega(e) = 1$. In this case, the subspace Fe is a commutative subalgebra of A. We will always assume the existence of an idempotent of weight 1. There is a natural method of obtaining such algebras. If N is any F-algebra, $\lambda: N \longrightarrow N$ and $\rho: N \longrightarrow N$ are F-linear mappings, define on the vector space $F \oplus N$ a multiplication and a weight function by

$$(\alpha, a)(\beta, b) = (\alpha\beta, ab + \alpha\lambda(b) + \beta\rho(a)); \qquad \omega(\alpha, a) = \alpha \tag{1}$$

where α, $\beta \in F$, a, $b \in N$. Clearly ω is a non-zero homomorphism, $e = (1, 0)$ is an idempotent of weight 1, $(1,0)(0,n) = (0, \lambda(n))$, $(0,n)(1,0) = (0, \rho(n))$, for all $n \in N$. We denote this algebra by $[N, \lambda, \rho]$. Every baric algebra (A, ω) with idempotent e of weight 1 is obtained by this method, taking $N = \ker\omega$, $\lambda = L_e$ and $\rho = R_e$, where L and R are left and right multiplication operators. For easy reference, we denote this class by Ω.

Proposition 1 *Let N_1 and N_2 be F-algebras, $\lambda_i, \rho_i: N_i \longrightarrow N_i$ F-linear mappings.*

a) *Suppose $\varphi: [N_1, \lambda_1, \rho_1] \longrightarrow [N_2, \lambda_2, \rho_2]$ is a baric homomorphism. Then there exist a homomorphism $\theta: N_1 \longrightarrow N_2$ and $c \in N_2$ such that*

i) $c^2 + \lambda_2(c) + \rho_2(c) = c$

ii) *The following diagrams are commutative:*

$$\begin{array}{ccc} N_1 & \xrightarrow{\theta} & N_2 \\ {\scriptstyle\lambda_1}\downarrow & & \downarrow{\scriptstyle\lambda_2+L_c} \\ N_1 & \xrightarrow{\theta} & N_2 \end{array} \qquad\qquad \begin{array}{ccc} N_1 & \xrightarrow{\theta} & N_2 \\ {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\rho_2+R_c} \\ N_1 & \xrightarrow{\theta} & N_2 \end{array}$$

b) *Conversely, given a homomorphism $\theta: N_1 \longrightarrow N_2$ and $c \in N_2$ such that i) and ii) hold, then $\varphi : [N_1, \lambda_1, \rho_l] \longrightarrow [N_2, \lambda_2, \rho_2]$ defined by $\varphi(\alpha, a) = (\alpha, \alpha c + \theta(a))$ is a baric homomorphism.*

Corollary *With the above notations, φ is monomorphism (resp. epimorphism) if and only if θ is monomorphism (resp. epimorphism).*

2. The Krull-Schmidt Theorem

Suppose (A, ω) is a baric algebra with idempotent e of weight 1, so $A = Fe \oplus N$ where $N = \ker\omega$. The additive group $(N, +)$ can be endowed with a structure of Abelian M-group, see [7,chap V, Def.1]. The set M is formed by all right and left multiplications R_a and L_a, where a belongs to $A \cup F$ and $\psi : N \times M \to N$ definide by $\psi(n, T) = T(n)$. In this case, the M-subgroups of $(N, +)$ are the two-sided ideals of the algebra A , contained in N. These are exactly the two-sided ideals of N which are invariant under L_e and R_e (in short, invariant). The M-endomorphisms of $(N, +)$ are the mappings φ satisfying $\varphi(n+n') = \varphi(n)+\varphi(n')$, $\varphi(\alpha n) = \alpha\varphi(n)$, $\varphi(na) = \varphi(n)a$ and $\varphi(an) = a\varphi(n)$ where $n, n' \in N$, $\alpha \in F$ and $a \in A$. This set of conditions is equivalent to φ being F-linear, $\varphi(nn') = \varphi(n)n' = n\varphi(n')$ and $\varphi(en) = e\varphi(n)$, $\varphi(ne) = \varphi(n)e$.

According to [7, chap. V, §12] an Abelian M-group N is decomposable if there are two non trivial M-subgroups N_1 and N_2 of N such that $N = N_1 \oplus N_2$. In our context, this concept is translated to the following definition:

Definition 1 *A baric algebra (A, ω) with an idempotent of weight 1 is decomposable if there are non trivial two-sided ideals N_1 and N_2 of A , both contained in $N = \ker\omega$, such that $N = N_1 \oplus N_2$. Otherwise, it is indecomposable.*

Clearly all two dimensional algebras are indecomposable.

Theorem 1 *For any member (A,ω) of Ω, the following conditions are equivalent:*

a) *(A,ω) is decomposable.*

b) *There exists a M-endomorphism θ of the M-group $(N,+)$ such that $\theta^2 = \theta$, $\theta \neq 0$, id_N.*

c) *There exists $\theta: N \longrightarrow N$, a homomorphism of F-algebras, such that $\theta^2 = \theta$, $\theta \neq 0$, id_N, $\theta(N)$ is two-sided ideal of N and for some idempotent e of weight 1 in A, $\theta(en) = e\theta(n)$, $\theta(ne) = \theta(n)e$ for all $n \in N$.*

d) *There exists a baric endomorphism $\varphi: A \longrightarrow A$ such that $\varphi^2 = \varphi$, $\varphi \neq id_A$, $\varphi(N)$ is a nonzero two-sided ideal of N.*

There is a natural method of obtaining decomposable algebras. Suppose (A_1,ω_1) and (A_2,ω_2) belong to Ω, with idempotents e_1 and e_2 resp, so $A_1 = Fe_1 \oplus N_1$ and $A_2 = Fe_2 \oplus N_2$. Consider $N = N_1 \oplus N_2$, endowed with the componentwise multiplication. Let $\lambda, \rho: N \longrightarrow N$ be the linear operators

$$\lambda(n_1,n_2) = (e_1n_1, e_2n_2), \quad \rho(n_1,n_2) = (n_1e_1, n_2e_2)$$

So in $[N,\lambda,\rho]$, according to (1), we have the multiplication:

$$(\alpha,n_1,n_2)(\alpha',n_1',n_2') = (\alpha\alpha', n_1n_1' + \alpha e_1n_1' + \alpha' n_1e_1, n_2n_2' + \alpha e_2n_2' + \alpha' n_2e_2)$$

Clearly N_1 and N_2 are two-sided ideals of $[N,\lambda,\rho]$. If A_1 and A_2 are at least two dimensional, $[N,\lambda,\rho]$ is decomposable. This algebra is called the join of (A_1,ω_1) and (A_2,ω_2) and is denoted $(A_1 \vee A_2, \omega_1 \vee \omega_2)$ or simply $A_1 \vee A_2$. The idempotent $(1,0,0)$ is the join of e_1 and e_2, denoted $e_1 \vee e_2$. It is not difficult to prove that this construction is independent of the idempotents e_1 and e_2. If e_1' and e_2' are also idempotents in A_1 and A_2 then $e_i = e_i' + c_i$, where $c_i \in \ker\omega_i$. Take now $c = (c_1,c_2)$, $\theta = id_{N_1 \oplus N_2}$ and apply Prop.1. Conversely, every decomposable baric algebra can be obtained by this method, taking $A_1 = Fe \oplus N_1$ and $A_2 = Fe \oplus N_2$ if $N = N_1 \oplus N_2$.

The join $A_1 \vee A_2$ can be identified with a subalgebra of the usual direct sum $A_1 \oplus A_2 = Fe_1 \oplus Fe_2 \oplus N_1 \oplus N_2$. The element $e_1 + e_2$ is an idempotent, $F(e_1+e_2) \oplus N_1 \oplus N_2$ is baric (send $e_1 + e_2$ to 1 and the elements of $N_1 \oplus N_2$ to 0). An isomorphism is $(\alpha, u, v) \in A_1 \vee A_2 \mapsto \alpha(e_1+e_2) + u + v \in F(e_1 + e_2) \oplus N_1 \oplus N_2$.

Proposition 2 *Let F be a field, (A_i, ω_i) F-algebras in Ω, $i = 1, 2, 3$. We have the following baric isomorphisms:*

(i) $(F \vee A_1, id_F \vee \omega_1) \cong (A_1, \omega_1)$

(ii) $(A_1 \vee A_2, \omega_1 \vee \omega_2) \cong (A_2 \vee A_1, \omega_2 \vee \omega_1)$

(iii) $((A_1 \vee A_2) \vee A_3, (\omega_1 \vee \omega_2) \vee \omega_3) \cong (A_1 \vee (A_2 \vee A_3), \omega_1 \vee (\omega_2 \vee \omega_3))$

Condition (iii) allows us to define recursively the join $(A_1 \vee \ldots \vee A_n, \omega_1 \vee \ldots \vee \omega_n)$ of n members of Ω. When all A_i are finite dimensional, we have $\dim F(A_1 \vee \ldots \vee A_n) = \left(\sum_{i=1}^{n} \dim F A_i\right) - n + 1$.

Which algebras A in Ω can be obtained by joining a finite number of indecomposable algebras? As expected, some finiteness condition should be imposed on A.

Definition 2 *A baric algebra (A, ω), with $N = \ker\omega$, satisfies d.c.c. if the M-group $(N, +)$ satisfies d.c.c., as stated in [7, p.153].*

This means that strictly descending chains of two-sided ideals of A , contained in N, must be finite.

Proposition 3 *If (A, ω) is a baric algebra in Ω satisfying d.c.c, there exist m indecomposable baric subalgebras (A_i, ω_i) of (A, ω) such that $(A, \omega) = (A_1 \vee \ldots \vee A_m, \omega_1 \vee \ldots \vee \omega_m)$.*

Proposition 4 *Let (A, ω) and (A_i, ω_i) $(i = 1, \ldots, m)$ be members of Ω, $e \in A$ idempotent of weight 1 and suppose there is a baric isomorphism $(A, \omega) \cong (A_1 \vee \ldots \vee A_m, \omega_1 \vee \ldots \vee \omega_m)$. Then there exist two-sided ideals $N_1, \ldots, N_m$ of A, contained in $N = ker\omega$, such that:*

a) $N = N_1 \oplus \ldots \oplus N_m$

b) *The baric subalgebra $Fe \oplus N_i$ is isomorphic to (A_i, ω_i)*

Definition 3 *A baric algebra (A, ω), with $N = ker\omega$, satisfies the ascending chain condition a.c.c. if the M-group $(N, +)$ satisfies a.c.c., as stated in [7, p.154]*

Theorem 2 (Krull-Schmidt) *Suppose $(A, \omega) \in \Omega$ satisfies both d.c.c. and a.c.c. and let $(A_1, \omega_1), \ldots, (A_n, \omega_n), (B_1, \gamma_1) \ldots, (B_m, \gamma_m)$ be indecomposable members of Ω such that*

$$(A, \omega) \cong (A_1 \vee \ldots \vee A_n, \omega_1 \vee \ldots \vee \omega_n)$$

$$(A, \omega) \cong (B_1 \vee \ldots \vee B_m, \gamma_1 \vee \ldots \vee \gamma_m)$$

Then $n = m$ and for some permutation $i \mapsto i'$ of indices, we have $(A_i, \omega_i) \cong (B_{i'}, \gamma_{i'})$ for all $i = 1, \ldots, n$.

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