Matemática Contemporânea, Vol. 6, 7–12
http://doi.org/10.21711/231766361994/rmc62

E-IDEALS IN BARIC ALGEBRAS

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1. Introduction

Let F be a field of characteristic zero, A an algebra over F, not necessarily associative but commutative. If $\omega : A \longrightarrow F$ is a nonzero homomorphism, the ordered pair (A, ω) is a (commutative) baric algebra over F, ω its weight function and for each $x \in A, \omega(x)$ is its weight. Elements of weight 0 form an ideal N of codimension 1. The concept of baric algebra has been introduced by I.M.H.Etherington [3]. We define two classes of basic algebras.

I) Let $\gamma_1, ..., \gamma_{n-1}$ be arbitrary elements in the field F such that

$$1 + \gamma_1 + ... + \gamma_{n-1} = 0.$$

The formal expression

$$p(x) = x^n + \gamma_1 \omega(x) x^{n-1} + ... + \gamma_{n-1} \omega(x)^{n-1} x, \tag{1}$$

is called a train polynomial with coefficients $\gamma_1, ..., \gamma_{n-1}$ and degree n. If a baric algebra (A, ω) satisfies identically

$$p(a) = a^n + \gamma_1 \omega(a) a^{n-1} + ... + \gamma_{n-1} \omega(a)^{n-1} a = 0, \ all \ a \in A \tag{2}$$

it is called a train algebra. Here a^k is defined by $a^1 = a$ and $a^k = a^{k-1} a$ for $k \geq 2$. If $p(x)$ has mininal degree among those train polynomials satisfied by A, it is called the train equation of A and n is the rank of A. For these algebras, $N = \{a \mid \omega(a) = 0\}$ is a nil ideal.

*The results presented in this talk are contained in the paper "E-ideals in basic algebras", by myself and R. Costa. This author was supported by financial support from the projet DIUFRO 9204, Temuco, Chile.

II) Recently, Walcher [4] proved some results about baric algebras satisfying

$$(a^2)^2 = \omega(a)^3 a \tag{3}$$

These algebras have a Peirce decomposition $A = Fe \oplus N_{\frac{1}{2}} \oplus N_{-\frac{1}{2}}$ relative to an idempotent e, where $N_{\frac{1}{2}} = \{a \in N \mid ea = \frac{1}{2}a\}, N_{-\frac{1}{2}} = \{a \in N \mid ea = -\frac{1}{2}a\}$. It is not hard to prove that the dimensions of these subspace are independent of e and so the type can be defined as $(1 + dimN_{\frac{1}{2}}, dimN_{-\frac{1}{2}})$. Moreover

$$N_{\frac{1}{2}}^2 \subseteq N_{-\frac{1}{2}}, N_{\frac{1}{2}}N_{-\frac{1}{2}} \subseteq N_{\frac{1}{2}} \; and \; N_{-\frac{1}{2}}^2 \subseteq N_{-\frac{1}{2}} \tag{4}$$

The following set of equations hold for $u \in N_{\frac{1}{2}}$ and $v \in N_{-\frac{1}{2}}$:

$$\begin{array}{ll} i)\; u^3 = 0 & vi)\; u^2(uv) = 0 \\ ii)\; 2u(uv) = u^2v & vii)\; 2(uv)^2 + u^2v^2 = 0 \\ iii)\; 2(uv)v = uv^2 & viii)\; (uv)v^2 = 0 \\ iv)\; v^3 = 0 & ix)\; (v^2)^2 = 0 \\ v)\; (u^2)^2 = 0 & \end{array} \tag{5}$$

For this, impose the identity $(x^2)^2 = \omega(x)^3 x$ to a generic vector $x = \omega(x)e + \lambda u + \mu v$, where $\lambda, \mu \in F$ and equate like powers of λ and μ. Let now $x = \alpha e + u + v$ be an element of A, so $\alpha = \omega(x)$. Then for $k \geq 1$,

$$x^k = \alpha^k e + \alpha^{k-1}(u + a_k v) + \alpha^{k-2}(b_k u^2 + c_k uv + d_k v^2) + \alpha^{k-3}(e_k u^2 v + f_k uv^2) \tag{6}$$

where $a_k, ..., f_k$ are suitable rational numbers. Monomials in u and v of degree at least 4 will disappear, due to relations (5). By imposing that $x^{k+1} = x^k x$, we get the following system of difference equations, with initial values $a_1 = 1, b_1 = c_1 = d_1 = e_1 = f_1 = 0$:

$$\begin{cases} 2a_{k+1} + a_k + 1 = 0 \\ 2b_{k+1} + b_k - 2 = 0 \\ 2c_{k+1} - c_k - 2a_k - 2 = 0 \\ 2d_{k+1} + d_k - 2a_k = 0 \\ 2e_{k+1} + e_k - 2b_k - c_k = 0 \\ 2f_{k+1} - f_k - c_k - 2d_k = 0 \end{cases}$$

It is easy to prove by induction (but we do not do the details here) that:

$$a_k + c_k = 1, a_k + 2b_k = 1, 2d_k + 2e_k + a_k = 1, e_k = f_k \; \forall k \geq 1 \tag{7}$$

Definition 1. The *generalized Etherington's ideal* (in short, E-ideal) of the baric algebra (A, ω), associated to the train polynomial $p(x)$, is the ideal of A generated by all $p(a) = a^n + \gamma_1\omega(a)a^{n-1} + ... + \gamma_{n-1}\omega(a)^{n-1}a, a \in A$. This ideal will be denoted $E_A(1, \gamma_1, ..., \gamma_{n-1})$ or $E_A(p)$

The Etherington's ideal is $E_A(1, -1)(see[3])$.

Observe that $E_A(p) \subseteq ker\ \omega, A/E_A(p)$ satisfies $p(x) = 0$ and $E_A(p)$ is the smallest ideal $I \subseteq N$ such that A/I satisfies $p(x) = 0$. Moreover, (A, ω) is a train algebra when some of its E-ideals is zero.

Proposition 2. *For every baric algebra (A, ω) and every train polynomial $p(x)$, $E_A(p) \subseteq E_A(1, -1)$.*

Proof: Let $E_A(n, k)$ be the E-ideal associated to the train polynomial $x^n - \omega(x)^{n-k}x^k(1 \leq k \leq n-1)$. For any given train polynomial (1), we have for $a \in A : p(a) = a^n + \gamma_1\ \omega(a)a^{n-1} + ... + \gamma_{n-1}\omega(a)^{n-1}a = a^n - (1 + \gamma_2 + ... + \gamma_{n-1})$ $\omega(a)a^{n-1} + ... + \gamma_{n-1}\omega(a)^{n-1}a = (a^n - \omega(a)a^{n-1}) - \gamma_2\omega(a)(a^{n-1} - \omega(a)a^{n-2}) - ... -$ $\gamma_{n-1}\omega(a)(a^{n-1} - \omega(a)^{n-2}a) \in E_A(n, n-1) + E_A(n-1, n-2) + ... + E_A(n-1, 1)$ which implies that

$$E_A(p) \subseteq E_A(n, n-1) + \sum_{k=1}^{n-2} E_A(n-1, k)$$

Similarly, for every $k = 1, 2, ..., n-2$, $E_A(n-1, k) \subseteq E_A(n-1, n-2) + \sum_{r=1}^{n-3} E_A(n-2, r)$. By repeated application, we get finally that $E_A(p) \subseteq \sum_{k=2}^{n} E_A(k, k-1)$. But each of the ideals $E_A(k, k-1)$ is contained in $E_A(1, -1)$, according to

$$a^k - \omega(a)a^{k-1} = (...(a^2 - \omega(a)a)...)a \in E_A(1, -1).\square$$

As a rule, different train polynomials may give rise to the same E-ideals, so we introduce the following equivalencia relation.

Definition 2. Let Ω be a fixed class of baric algebras over the field F. Two train polynomials $p(x)$ and $q(x)$ are *equivalent modulo* Ω when $E_A(p) = E_A(q)$ for every $A \in \Omega$.

2. E-ideals for algebras satisfying $(x^2)^2 = \omega(x)^3 x$

Denote by $A, ..., F$ the sequences $(a_k)_{k\in N}, ..., (f_k)_{k\in N}$ so that (7) means

$$A + C = \mathbf{1},\ A + 2B = \mathbf{1},\ 2D + 2E + A = \mathbf{1},\ E = F \tag{7'}$$

Where $\mathbf{1} = (1, 1...)$. Let $k \geq 1, A_k = (a_k, a_{k-1}, ..., 1) \in F^k$ and similarly $B_k, C_k, D_k, E_k, F_k, \mathbf{1}_k$ so that relations(7') hold also for these vectors of F^k.

A given train polynomial $p(x) = x^n + \gamma_1\omega(x)x^{n-1} + ... + \gamma_{n-1}\omega(x)^{n-1}x$ can be identified with the vector $p = (1, \gamma_1, ..., \gamma_{n-1}) \in F^n$. The set of all train polynomials of degree n is then identified with the linear variety of F^n defined by the equations $x_1 + ... + x_n = 0$ and $x_1 = 1$. Let $<,>$ be the usual bilinear form in F^n.

When replacing each power x^k of x given by (6) into $p(x)$, we get $p(x) =< p, \mathbf{1}_n > \alpha^n e + \alpha^{n-1}[< p, \mathbf{1}_n > u + < A_n, p > v] + \alpha^{n-2}[< B_n, p > u^2 + < C_n, p > uv + < D_n, p > v^2] + \alpha^{n-3} < [E_n, p > u^2 v + < F_n, p > uv^2]$.

As $< p, 1_n >= 1 + \gamma_1 + ... + \gamma_{n-1} = 0$, the first two summands disappear. Also by (7'), we can express A_n, C_n, D_n and F_n as linear combinations of B_n, E_n and $\mathbf{1}_n$, thus getting:

$$p(x) = -2\alpha^{n-1} < B_n, p > v + \alpha^{n-2}[< B_n, p > u^2 + 2$$
$$< B_n, p > uv + < B_n - E_n, p > v^2] + \alpha^{n-3} < E_n, p > (u^2 v + uv^2) \tag{8}$$

It is now possible to prove that there are 3 equivalence classes of train polynomials, described geometrically by the following propositions.

Proposition 2. *If the vectors p and B_n are not orthogonal then $E_A(p) = N_{\frac{1}{2}} N_{-\frac{1}{2}} \oplus N_{-\frac{1}{2}} = E_A(1, -1)$.*

Proof: We already know that $E_A(1, -1) = N_{\frac{1}{2}} N_{-\frac{1}{2}} \oplus N_{-\frac{1}{2}}$ (see [1]) and $E_A(p) \subseteq E_A(1, -1)$. To prove the proposition, it is enought to show that $E_A(p) \supseteq N_{\frac{-1}{2}}$.

If $v \in N_{-\frac{1}{2}}$, then $p(e+v) = -2 < B_n, p > v + < B_n - E_n, p > v^2$ and $p(e-v) = 2 < B_n, p > v + < B_n - E_n, p > v^2$ so $p(e-v) - p(e+v) = 4 < B_n, p > v$ and $v = \frac{1}{4} < B_n, p >^{-1} (p(e-v) - p(e+v)) \in E_A(p)$. □

Proposition 3. *If p and B_n are orthogonal but p and E_n are not, then $E_A(p) = N_{\frac{1}{2}} N^2_{-\frac{1}{2}} \oplus N^2_{-\frac{1}{2}}$.*

Proof: With $< B_n, p >= 0$, the equality (8) reduces to

$$p(x) = \alpha^{n-2} < E_n, p > v^2 + \alpha^{n-3} < E_n, p > (u^2 v + u v^2)$$

$$= \alpha^{n-3} < E_n, p > (u^2 v - \alpha v^2 + u v^2) \in N_{\frac{1}{2}} N^2_{-\frac{1}{2}} \oplus N^2_{-\frac{1}{2}} \tag{9}$$

It is easy to see, using (5), that this subspace is an ideal. This implies that $E_A(p) \subseteq N_{\frac{1}{2}} N^2_{-\frac{1}{2}} \oplus N^2_{-\frac{1}{2}}$. For the converse inclusion it is enough to prove that $N^2_{-\frac{1}{2}} \subseteq E_A(p)$ when p and E_n are not orthogonal. Suposse initially $v^2 \in N^2_{-\frac{1}{2}}$. Then $p(e+v) = - < E_n, p > v^2$ so $v^2 = - < E_n, p >^{-1} p(e+v) \in E_A(p)$. For an additive generator $v_1 v_2$ of $N^2_{-\frac{1}{2}}$, it is enough to remember that $2 v_1 v_2 = (v_1 + v_2)^2 - v_1^2 - v_2^2 \in E_A(p)$. □

Proposition 4. *If p is orthogonal to both B_n, and E_n, then $E_A(p) = 0$*

Proof: It is enough to see from (9) that $p(x) = 0$ identically when p and E_n are also orthogonal. □

Remark. The E-ideals for algebras satisfying $(x^2)^2 = \omega(x)^3 x$ are determined by $x^2 - \omega(x)x$, $x^3 - \frac{1}{2}\omega(x)x^2 - \frac{1}{2}\omega(x)^2 x$ and $x^4 - \frac{3}{4}\omega(x)^2 x^2 - \frac{1}{4}\omega(x)^3 x$ respectively, in minimal degrees. In particular all algebras satisfying $(x^2)^2 = \omega(x)^3 x$ are train of rank 4, as remaked by Walcher, [4, Eq.(11)].

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