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Geography and geometry of the moduli spaces of semi-stable rank 2 sheaves on projective space

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Abstract

In this article, we will give a review of recent results on the geography and geometry of the Gieseker-Maruyama moduli scheme $M = M(c_1, c_2)$ of rank 2 semi-stable coherent sheaves with first Chern class $c_1 = 0$ or -1, second Chern class c_2, and third Chern class 0 on the projective space $\mathbb{P}^3$. We enumerate all currently known irreducible components of M for small values of c_2. We then present constructions of new series of components of M for arbitrary c_2. The problem of connectedness of M and also the problem of rationality of some series of components of M are discussed.

In this article, we will give a review of recent results on the geography and geometry of the Gieseker-Maruyama moduli scheme $M(e,n)$ of rank 2 semi-stable coherent sheaves with first Chern class $e = 0$ or -1, second Chern class n, and third Chern class 0 on the projective space $\mathbb{P}^3$ over $\mathbf{k} = \bar{\mathbf{k}}$, char$\mathbf{k} = 0$. It is known, see, e. g. [20], [21] that $M(e,n)$ is

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projective over $\mathbf{k}$. Here $n \geq 1$ if $e = 0$ and, respectively, $n = 2m,\ m \geq 1$, if $e = -1$.

Let us introduce some basic notation.
• $d(e, n)$ the number of irreducible components of $M(e, n)$
• $B(e, n)^*$ the open subset of $M(e, n)$ consisting of locally free sheaves; $B(e, n)$ the closure of $B(e, n)^*$ in $M(e, n)$.
• $b(e, n)$ the number of irreducible components of $B(e, n)$.
• $S(e, n) = \overline{M(e, n) \smallsetminus B(e, n)}$ the union of components of non-locally free sheaves in $M(e, n)$.
• $s(e, n)$ the number of irreducible components of $S(e, n)$.

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1 Old results on $M(e, n)$ for small values of n

The complete description of $M(0, n)$, resp., $B(0, n)$ is known only for $n \leq 2$, resp., $n \leq 4$; and the complete description of $M(-1, 2m)$, resp., $B(-1, 2m)$ was known only for $m = 1$, resp., $m \leq 2$.

(i) **Old results on $M(0, n)$ and $B(0, n)$ for small n.**

In 1978 R.Hartshorne [10] showed that, for $n = 1$ and 2, $B(0, n) = I_n$ is irreducible of the expected dimension $8n-3$, where I_n is the instanton component (to be defined below). It is easy to see that $M(0, 1) = B(0, 1)$. In 1993 J. Le Potier [19] proved that $M(0, 2)$ contains, together with $B(0, 2)$, two more components, of dimensions 17 and 21, respectively. Next, $B(0, 3)$

consists of two irreducible components of the expected dimension 21, one of which is the instanton component I_3 (1981, G. Ellingsrud and S. Strømme [8]), and $B(0,4)$ consists of two irreducible components of the expected dimension 29, one of which is the instanton component I_4 (1981, W. Barth [4], and 1983, M. Chang [6]).

(ii) **Old (and some new) results on $M(-1,2m)$ and $B(-1,2m)$ for small m.**

In 1981 R. Hartshorne and I. Sols [11] showed that $B(-1,2)$ is irreducible of the expected dimension 11, and in 1985 C. Banica and N. Manolache [3] found that $B(-1,4)$ consists of two components, one of the (expected) dimension 27, and another of the dimension 28. In 1981 J. Meseguer, I. Sols and S. Strømme showed that the space $M(-1,2)$ contains, besides $B(-1,2)$, at least 2 families X_1 and X_2 of sheaves of dimensions 15 and 19, respectively.

Remark. In 2012 M. Zavodchikov [28], [29] proved that X_1 and X_2 are irreducible components of $M(-1,2)$, and that $M(-1,2)$ contains, besides $B(-1,2)$, X_1 and X_2, only one more component X_3 of dimension 11. In 2017 C. Almeida, M. Jardim and A. Tikhomirov [2] gave an independent proof of this result.

2 Old results on $B(e,n)$ for general n

(i) $\mathbf{e=0}$. It is known (M. Atiyah, N. Hitchin, V. Drinfeld, Yu. Manin, R. Hartshorne, W. Barth, 70ies) that, for any $n \geq 1$, $B(0,n)$ contains an irreducible component

$$I_n \text{ of the expected dimension } 8n-3;$$

I_n is a component of the closure $\tilde{I_n}$ in $B(0,n)$ of the open subset I_n^* constituted by the so-called *mathematical instanton vector bundles* (these are those bundles $[E] \in B(0,n)$ which satisfy the *instanton condition*

$h^1(E(-2)) = 0)$. Historically, $\{I_n\}_{n\geq 1}$ was the first known infinite series of irreducible components of $B(0,n)$ having the expected dimension $\dim I_n = 8n-3$. It was shown in 2012-13 by A. Tikhomirov [23], [24] that $\tilde{I_n}$ is irreducible, i. e. $\tilde{I_n} = I_n$, and in 2014 M. Jardim and M. Verbitsky proved that I_n is smooth along I_n^*.

(ii) $\mathbf{e} = -\mathbf{1}$. For any $m \geq 1$, $B(-1,2m)$ contains an irreducible component

$$H(-1,2m) \text{ of the expected dimension } 16m-5$$

(1978, Hartshorne [10]). We call $H(-1,2m)$ the *Hartshorne's component.*

(iii) The other infinite series of families of vector bundles of dimension $3k^2 + 10k + 8$ from $B(0, 2k+1)$ was constructed in 1978 by W. Barth and K. Hulek [5], and in 1981 G. Ellingsrud and S. Strømme showed that these families are open subsets of irreducible componens distinct from the instanton components I_{2k+1}.

(iv) Later in 1985-87 V. Vedernikov [26], [27] constructed two infinite series of families of bundles from $B(0,n)$, and one infinite family of bundles from $B(-1,2m)$. A more general series of families of rank 2 bundles depending on triples of integers a, b, c with $b \geq a \geq 0,\ c > a+b$, was found in 1984 by A. Prabhakar Rao [22]. Namely, these are vector bundles which are the cohomology sheaves of monads of the form

$$0 \to \mathcal{O}(-c+e) \to \begin{array}{c} \mathcal{O}(a) \oplus \mathcal{O}(-a+e) \\ \oplus \\ \mathcal{O}(b) \oplus \mathcal{O}(-b+e) \end{array} \to \mathcal{O}(c) \to 0,$$

They are called the *generalized null correlation bundles.*

(v) Soon after that, L. Ein in 1988 independently studied these bundles and proved that they constitute open subsets of irreducible components $N(e,a,b,c)$ of $B(e,n)$,
$n = c^2 - a^2 - b^2 - e(c-a-b)$, for both $e = 0$ and $e = -1$. We will call these components $N(e,a,b,c)$ the *Ein components.* It is not clear whether Ein

components exist for *any* large enough n. However, Ein showed that the number of these components of $B(0,n)$, resp., of $B(-1,2m)$ is unbounded as n, resp., m grows infinitely. Hence,

$$\limsup_{n\to\infty} b(0,n)=\infty, \quad \limsup_{m\to\infty} b(-1,2m)=\infty.$$

This shows that the moduli space $B(e,n)$ differs drastically from the corresponding moduli space $B_{\mathbb{P}^2}(e,n)$ of semistable rank 2 vector bundles on $\mathbb{P}^2$ which is irreducible for any admissible e,n.

3 Recent results on $M(0,3)$, $M(0,4)$ and $B(0,5)$

We now proceed to the first very recent result on $M(0,3)$ obtained in 2017-18 by M. Jardim, D. Markushevich and A. Tikhomirov [15], and by A. Ivanov and A. Tikhomirov [12], [13].

Theorem 1. $M(0,3)=B(0,3)\cup S(0,3)$, *where B(0,3) consists of two components mentioned above in section 1(i), and* $S(0,3)$ *contains at least the following 8 irreducible components:*
(a) four components of dimesions, respectively, 25, 29, 33 and 37, a general sheaf of any of which has 0-dimensional singular locus,
(b) one component of dimension 21, a general sheaf of which has purely 1-dimensional singular locus,
(c) three components of dimensions, resp., 22, 24 and 26, such that a general sheaf of any of them has singular locus of mixed dimension.
(d) The union of the above 10 components of $M(0,3)$ *is connected.*

Here we say that the sheaf E has a *purely 1-dimensional singular locus* (respectively, has a *singular locus of mixed dimension*) if $E^{\vee\vee}/E$ is a purely 1-dimensional sheaf (respectively, is a 1-dimensional sheaf with a 0-dimensional subsheaf).

The next recent result from 2018, obtained by M. Jardim, M. Maican and

A. Tikhomirov [14], concerns the description of the so called instanton sheaves in $M(0,n)$ for small values of n. By an *instanton sheaf* we mean a sheaf $[E] \in M(0,n)$ (not necessary locally free) satisfying the instanton conditions $h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0$. Clearly, any mathematical instanton bundle from I_n^* is an instanton sheaf. An important feature of non-locally free instanton sheaves is that they have a purely 1-dimensional singular locus [9]. We call an irreducible component of $M(0,n)$ an *instanton component* if a general sheaf in this component is an instanton sheaf. The main result of [14] is the following theorem.

Theorem 2. *(i) The only instanton component of* $M(0,2)$ *is* $B(0,2) = I_2$.
(ii) $M(0,3)$ *has two instanton components, both of dimension 21. They are* I_3 *and the component mentioned in Theorem 1(b) above.*
(iii) $M(0,4)$ *has four instanton components, three of dimension 29, including* I_4*, and one of dimension 32.*

The third result from 2017 due to C. Almeida, M. Jardim, A. Tikhomirov and S. Tikhomirov [1] gives a complete description of $B(0,5)$.

Theorem 3. $B(0,5)$ *has three irreducible components:*
1) the 37-dimensional instanton component $\bar{I}_5$*;*
2) the 40-dimensional Ein component $N(0,0,2,3)$*;*
3) the 37-dimensional component described as the closure in $M(0,5)$ *of the set* $\{[E] \in M(0,5) \mid E$ *is a cohomology bundle of a monad of the type* $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0\}$.

4 New infinite series of irreducible components of $B(e,n)$

We now proceed to the description of the three recently obtained new infinite series of irreducible components of $B(e,n)$ (*components of locally* *free sheaves*). It was obtained in 2017 by A. Kytmanov, A. Tikhomirov and S. Tikhomirov in [18].

Theorem 4. *For $a = 2$ and $a \geq 4$, the rank 2 bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to 6 \cdot \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

fill out a dense subset of an irreducible rational component of $B(0, a^2 + 1)$ of dimension

$$4 \cdot \binom{a+3}{3} - a - 1.$$

The other two new series were constructed by A. Tikhomirov, S. Tikhomirov and D. Vassiliev [25] in 2018.

Theorem 5. *Let $m \geq 0$, $\varepsilon \in \{0, 1\}$ and $a \geq 5$ be integers such that either $a \leq 11$, $m \leq a - 5$, or $a \geq 12$, $m \leq a$. There exists an irreducible component $B_{0,n}$ of $B(0, n)$, where $n = 2m+\varepsilon+a^2$, of dimension $\dim B_{0,n} = 4\binom{a+3}{3} + (2m + \varepsilon)(10 - a) - 11$. A general bundle in this component is the cohomology bundle of the monad*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to E_4 \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

*where the rank 4 bundle E_4 is a versal symplectic deformation of the bundle $E_1 \oplus E_2$, and E_1 and E_2 are instanton bundles: $[E_1] \in I_m$, $[E_2] \in I_{m+\varepsilon}$. The set $\Sigma_0 = \{B_{0,n}\}_{n\geq 1}$ of these components is an infinite series distinct from the series of instanton components $\{I_n\}_{n\geq 1}$ and from the series of Ein components $\{N(0, a, b, c) \mid$
a, b, c admissible$\}$. Furthermore, at least for each $n \geq 146$ there exists an irreducible component of $B(0, n)$ belonging to the series Σ_0.*

Theorem 6. *(i) For $n = 4m+2\varepsilon+a(a+1)$, where $m \geq 1$, $\varepsilon \in \{0, 1\}$ and $a \geq 2(m+\varepsilon)+3$, there exists an irreducible component $B_{-1,n} \subset B(-1, n)$ of dimension $4\binom{a+3}{3} + 2\binom{a+3}{2} - (2m + \varepsilon)(2a - 19) - 17$. This component is characterized by the property that it contains the cohomology bundle of* *the monad*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a-1) \to E_1 \oplus E_2 \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

where E_1 and E_2 are bundles of the Hartshorne's series: $[E_1] \in H(-1, 2m)$, $[E_2] \in H(-1, 2(m+\varepsilon))$. The set $\Sigma_{-1} = \{B_{-1,n}\}_{n \geq 1}$ of these components is an infinite series distinct from the Hartshorne's series $\{H(-1, 2n)\}_{n \geq 1}$ and from the series of Ein components $\{N(0, a, b, c) |\ a, b, c$ admissible$\}$.
(ii) Let $\mathcal{N}$ be the set of all values of n for which $B_{-1,n} \neq \emptyset$, i. e. $\mathcal{N} = \{n \in 2\mathbb{Z}_+ \mid n = 4m + 2\varepsilon + a(a+1),$ where $m \in \mathbb{Z}_+,\ \varepsilon \in \{0, 1\}$, $a \geq 2(m+\varepsilon)+3\}$. Then

$$\lim_{r \to \infty} \frac{\mathcal{N} \cap \{2, 4, ..., 2r\}}{r} = 1.$$

5 Three new infinite series of irreducible components of $S(e, n)$

We next turn to the description of the three new infinite series of irreducible components of $S(e, n) = \overline{M(e, n) \setminus B(e, n)}$ of *non-locally free sheaves*. (It were unknown before whether such infinite series of components actually exist.) The first two of them were constructed in 2017 by M. Jardim, D. Markushevich and A. Tikhomirov [15], and the third was obtained recently by C. Almeida, M. Jardim and A. Tikhomirov in [2].

We begin with the description of the first of these series of components, general sheaves E of which have 0-dimensional singularities:

$$\dim \mathrm{Sing} E = 0.$$

For this, introduce one more piece of notation:

- $R(0, n, 2l)$ the moduli space of rank 2 reflexive sheaves on $\mathbb{P}^3$ with $c_1 = 0,\ c_2 = n,\ c_3 = 2l$.

Proposition A.
For every nonsingular irreducible component $\mathcal{F}$ of $R(0,n,2m)$ of the expected dimension $8n-3$, there exists an irreducible component $T(n,m)$ of dimension $8n-3+4m$ in $M(0,n)$ whose general point $[E]$ has $\dim \mathrm{Sing} E = 0$ and satisfies $[E^{\vee\vee}] \in \mathcal{F}$ and $\ell(E^{\vee\vee}/E) = m$: $\quad 0 \to E \to E^{\vee\vee} \to E^{\vee\vee}/E \to 0$.

Proposition B.
For each triple (a,b,c) of nonnegative integers such that $3a+2b+c$ is nonzero and even, the rank 2 reflexive sheaves defined by the exact triple

$$0 \to a\mathcal{O}_{\mathbb{P}^3}(-3) \oplus b\mathcal{O}_{\mathbb{P}^3}(-2) \oplus c\mathcal{O}_{\mathbb{P}^3}(-1) \to (a+b+c+2)\mathcal{O}_{\mathbb{P}^3} \to F(k) \to 0,$$

where $k := (3a+2b+c)/2$, so that $c_1(F)=0$, fill out an irreducible, nonsingular, component $S(a,b,c)$ of $R(0,n,m)$, of the expected dimension $8n-3$, where n and m are given by the expressions:

$$\begin{aligned} n =& \frac{1}{4}(3a+2b+c)^2 + \frac{3}{2}(3a+2b+c) - (b+c), \\ m =& m(a,b,c) = 27\binom{a+2}{3} + 8\binom{b+2}{3} + \binom{c+2}{3} \\ &+ 3(3a+2b+5)ab + \frac{3}{2}(3a+c+4)ac + (2b+3c+3)bc + 6abc. \end{aligned}$$

Propositions A and B yield

Theorem 7. *Let $s_0(0,n)$ denote the number of irreducible components of $S(0,n)$ whose generic points correspond to sheaves with 0-dimensional singularities. Then*

$$\limsup_{n\to\infty} s_0(0,n) = \infty.$$

A general sheaf E of any of these components is obtained as a kernel of an epimorphism φ in the exact triple:

$$0 \to E \to F \xrightarrow{\varphi} Q \to 0,$$

where F is a reflexive sheaf from the component $S(a,b,c)$ of $R(0,n,m)$ described above in Proposition B, Q is an artinian sheaf of length m, and m,n are related to a,b,c as in Proposition B.

We now turn to the second series of moduli components of sheaves E from $S(0,n)$ with 1-dimensional singularities:

$$\dim \mathrm{Sing} E = 1.$$

Introduce a few more notation:
• H_{d_1,d_2} the open subset of the Hilbert scheme parametrizing smooth complete intersection curves $C = Y_1 \cap Y_2$ in $\mathbb{P}^3$, where Y_1, Y_2 are surfaces of degree d_1, d_2,
• L an invertible sheaf of degree $g(C)-1$ on the curve $C \in H_{d_1,d_2}$,
• $W(d_1,d_2,m)$ the set of data (C,L,F,φ), where C and L are as above, F is an *instanton bundle* of charge m: $[F] \in I_m$, and $\varphi : F \to L \otimes \mathcal{O}_{\mathbb{P}^3}(2)$ is an epimorphism. $W(d_1,d_2,m)$ is an algebraic variety.

For any point $w = (C,L,F,\varphi) \in W(d_1,d_2,m)$ define a sheaf $E(w)$ by the triple

$$0 \to E(w) \to F \xrightarrow{\varphi} L \otimes \mathcal{O}_{\mathbb{P}^3}(2) \to 0.$$

Theorem 8. *(i) For any $m \geq 0$ and $(d_1,d_2) \neq (1,1)$, $(d_1,d_2) \neq (1,2)$, there is a well-defined morphism*

$$f:\ W(d_1,d_2,m) \to M(0, d_1d_2+m), \quad w \mapsto [E(w)],$$

which is an open embedding. The closure $\mathcal{C}(d_1,d_2,m)$ of $f(W(d_1,d_2,m))$ in $M(0,d_1d_2+m)$ is an irreducible component of $S(0,d_1d_2+m)$, of dimension

$$\dim \mathcal{C}(d_1,d_2,m) = \binom{d_1-1}{3} + \binom{d_2-1}{3} - \binom{d_2-d_1-1}{3}$$
$$+\frac{1}{2}d_1d_2(d_1+d_2-4) + 8(m+d_1d_2) - 3.$$

(ii) Let $s_1(0,n)$ denote the number of irreducible components of $S(0,n)$ whose generic points correspond to sheaves with 1-dimensional singularities. Then

$$\limsup_{n\to\infty} s_1(0,n) = \infty.$$

The last result of this section concerns the third series of moduli components obtained in [2] via elementary transformations of reflexive sheaves along unions of points and a projective line. The result is this:

Theorem 9. *Let $s_{01}(-1,n)$ be the number of irreducible components of M(1, n, 0) the generic point of which corresponds to a non-locally free sheaf E with singular locus* Sing$E$ *of mixed dimension, such that the 1-dimensional part of* Sing$E$ *is supported on a projective line. Then $s_{01}(-1,n)$ is unbounded as n goes to infinity:*

$$\limsup_{n\to\infty} s_{01}(-1,n) = \infty.$$

6 Recent results on the geometry of $M(e,n)$

(i) **Boundary divisors in** I_n.

We now turn to the geometry of the moduli spaces of $M(e,n)$. First, we discuss the recent result of M.Jardim, D.Markushevich and A. Tikhomirov in [16], 2018, on geometry of the *instanton components* I_n, $n \geq 1$. It concerns their boundary $\partial I_n = I_n \smallsetminus I_n^*$.

Theorem 10. *For each $n \geq 2$ and each $m = 1,\ldots,n-1$, consider the boundary ∂I_n of the instanton moduli component I_n and the set*

$$D(m,n)^* = \{[E] \in \partial I_n \mid E = \ker(F \twoheadrightarrow \mathcal{O}_C((2m-1)\mathrm{pt}), \quad \textit{where}$$

$$[F] \in I^*_{n-m} \textit{ and } C \textit{ is a smooth rational curve of degree } m \textit{ in } \mathbb{P}^3\}.$$

Then the closure $D(m,n)$ of $D(m,n)^*$ in I_n is anirreducible divisor in I_n.

This theorem leads to the next result on partial connectedness of $M(0,n)$ proved in [15].

(ii) **Partial connectedness of** $M(0,n)$.

Theorem 11. *For any $n \geq 1$, the partial union of irreducible components of $M(0,n)$,*

$$I_n \cup \Big(\bigcup_{\substack{d_1, d_2, m : \\ d_1 d_2 + m = n}} \mathcal{C}(d_1, d_2, m) \Big)$$

is connected.

Conjecture. *For any $n \geq 1$,* the space $M(0,n)$ is *connected.*

(iii) **Rationality of Ein components** $N(e,a,b,c)$.

The last result to mention concerns the rationality of certain components of $M(e,n)$. It was obtained in 2017 by A. Kytmanov, A. Tikhomirov and S. Tikhomirov in [18].

Theorem 12. *For $e \in \{0,-1\}$ and integers a,b,c with $b \geq a \geq 0$, $c > a+b$, consider the Ein components $N(e,a,b,c)$ of the space $M(e,n)$, $n = c^2 - a^2 - b^2 - e(c-a-b)$. Then, in the case $c > 2a+b-e$, $b > a$, $(e,a) \neq (0,0)$, $N(e,a,b,c)$ is rational and it always contains an open subset which is a fine moduli space. In the remaining cases $N(e,a,b,c)$ is (at least) stably rational.*

The idea of the proof is to relate generalized null correlation bundles introduced in Section 2(ii) to certain rank 2 reflexive sheaves via elementary transforms along surfaces of degree $c-b$. These reflexive sheaves satisfy the property that, being appropriately twisted, they have sections with zero loci that are complete intersection curves. Then the rationality of the family of such sheaves is stated, and this leads to the rationality (respectively, stable rationality) of the components $N(e,a,b,c)$.

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