Matemática Contemporânea, Vol. 47, 271–300
http://doi.org/10.21711/231766362020/rmc4711

Enumeration of hypersurfaces with prescribed non-isolated singular subschemes

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Abstract

Let $\mathbb{W}$ be an irreducible subvariety of a Hilbert scheme $\mathrm{Hilb}_{P_W(t)}(\mathbb{P}^n)$. We show under mild hypothesis that there are polynomial formulas for the degrees of the loci of hypersurfaces in $\mathbb{P}^n$ with singular subschemes containing some member of the family $\mathbb{W}$. The formulas are made explicit in a number of cases.

1 Introduction

The enumeration of singular hypersurfaces has a rich history. We refer the reader to Kleiman [20], [21] for a guide to the classical sources. Recent work has centered on generalizations of Göttsche's pioneering article [12]. Polynomial formulas have been shown to exist for the counting of any type of specified *isolated* singularities for hypersurfaces in higher dimensions, cf. Rennemo [27]; alas, his method is nonconstructive and doesn't lead to formulas. A different approach for the *existence* of universal polynomials

2010 AMS Subject Classification: 14N10 (Primary); 14C20, 14H20 (Secondary).

Key Words and Phrases: Enumerative Geometry, Bott's residues formula, singularities, Hilbert schemes.

This research was supported by UFVJM, UFMG and CNPq (grant 307495/2014-0).

enumerating singular subvarieties is offered by Tzeng [31], via cobordism theory of bundles and divisors. A few special cases of explicit polynomial formulas, still for isolated singularities in higher dimensions, can be found in [32].

The purpose of this work is to investigate the loci of hypersurfaces with possibly *nonisolated* singularities. More precisely, given a closed, irreducible subvariety of a Hilbert scheme, $\mathbb{W} \subset \mathrm{Hilb}_{P_W(t)}(\mathbb{P}^n)$, we define a *generalized discriminant subvariety* $\Sigma(\mathbb{W}, d) \subset \mathbb{P}^{N_d} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d)))$, the points of which correspond to the hypersurfaces of degree d in $\mathbb{P}^n$ singular along some (variable) member $W \in \mathbb{W}$. Assuming that a general member $W \in \mathbb{W}$ is smooth and of pure dimension $\leq n-2$, we show that the degree of $\Sigma(\mathbb{W}, d)$ is expressed by a polynomial $p^{\mathbb{W}}(d)$ for all $d \gg 0$. Our argument uses Grothendieck-Riemann-Roch. The degree of $p^{\mathbb{W}}(d)$ is shown to be bounded *a priori* by $n \dim \mathbb{W}$. The polynomial is made explicit for a few families $\mathbb{W}$, distinguished by an adequate description in the literature. Notably, we study the cases

$$\begin{array}{lcl} \bullet\, \mathbb{W}_{(k,n)} & := & \{\, \mathbb{P}^k - \text{ linear in } \mathbb{P}^n \,\}, \\ \bullet\, \mathbb{W}_m & := & \{ \text{ plane curves of degree } m \text{ in } \mathbb{P}^3 \}, \\ \bullet\, \mathbb{W}_{twc} & := & \{ \text{ twisted cubic curves in } \mathbb{P}^3 \ \}, \\ \bullet\, \mathbb{W}_{rc} & := & \{ \text{ ruled cubic surfaces in } \mathbb{P}^4 \ \}, \\ \bullet\, \mathbb{W}_{seg} & := & \{ \text{ Segre cubic 3-folds in } \mathbb{P}^5 \ \} \text{ and} \\ \bullet\, \mathbb{W}_{eqc} & := & \{ \text{ elliptic quartic curves in } \mathbb{P}^3 \ \}. \end{array} \tag{1}$$

In all examples we actually find that the degree of our polynomial is

$$\deg p^{\mathbb{W}}(d) = (k+1) \dim \mathbb{W}, \text{ where } k = \dim W (\leq n-2),\ W \in \mathbb{W}. \tag{2}$$

We conjecture this is always the case. Our main result is the following

Theorem 1. *Notation and hypotheses as above, set* $\mathcal{F}_d := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. *There exists a desingularization* $\widetilde{\mathbb{W}} \to \mathbb{W}$ *such that*

(**i**) *for* $d \gg 0$ *there exists a vector subbunble* $\mathcal{E}_d \subset \widetilde{\mathbb{W}} \times \mathcal{F}_d$ *whose fiber over a general* $W \in \widetilde{\mathbb{W}}$ *is the subspace* $H^0(\mathbb{P}^n, (\mathcal{I}_W)^2(d)) \subset \mathcal{F}_d$ *formed by homogeneous polynomials of degree* d *with gradient null along* W*;*
(**ii**) *the map* $\mathbb{P}(\mathcal{E}_d) \longrightarrow \mathbb{P}^{N_d} = \mathbb{P}(\mathcal{F}_d)$ *induced by projection is generically injective and its image,* $\Sigma(\mathbb{W}, d) \subset \mathbb{P}(\mathcal{F}_d)$*, has degree*

$$\deg \Sigma(\mathbb{W}, d) = \int \mathit{Segre}(w, \mathcal{E}_d) \cap [\widetilde{\mathbb{W}}],$$

where $w = \dim \widetilde{\mathbb{W}} = \dim \mathbb{W}$.
(**iii**) $\deg(\Sigma(\mathbb{W}, d))$ *is a polynomial in* d *of degree* $\leq nw$ *for all* $d \gg 0$.

Let us summarize the contents. §2 contains the proof of the theorem. The first step is to associate to a family $\mathbb{W}$ as above a family $\mathbb{W}'$ of thickenings, cf. Def. 4. The general member $W' \in \mathbb{W}'$ has ideal $\mathcal{I}_{W'} = (\mathcal{I}_W)^2$ for $W \in \mathbb{W}$ a smooth member. A hypersurface of degree $d >> 0$ is singular along W if and only if its equation F lies in $H^0(\mathbb{P}^n, \mathcal{I}_{W'}(d)) \subset \mathcal{F}_d$ (Lemma 5). The set of pairs (W', F) such that $F \supset W'$ is a vector subbundle $\mathcal{E}_d$ of $\mathbb{W}' \times \mathcal{F}_d$. Our generalized discriminant $\Sigma(\mathbb{W}, d) \subset \mathbb{P}^{N_d}$ is the image of the projectivization $\mathbb{P}(\mathcal{E}_d)$, cf. (10). Standard techniques of intersection theory enable us to express $\deg \Sigma(\mathbb{W}, d)$ as a top Chern class of the quotient bundle $\mathcal{D}_d := \mathcal{F}_d / \mathcal{E}_d$, cf. (12),(13). The latter bundle is a direct image, (7). Now GRR applies (14) to ensure that the desired top Chern class is a polynomial in d of degree $\leq n \dim \mathbb{W}$.

Polynomial formulas for the families envisaged in (1) are derived via Bott's localization at fixed points (15), as we learn from Ellingsrud and Strømme [9] and Meurer [24]. The fixed points of $\mathbb{W}_{twc}$ are available in op.cit. Additional work is required since our parameter space $\mathbb{W}'_{twc}$ is in fact a blowup of $\mathbb{W}_{twc}$, cf. Prop. 14, Remark 15. Ditto for the families $\mathbb{W}_{rc}$ (§3.3.2), $\mathbb{W}_{seg}$ (§3.3.3) and $\mathbb{W}_{eqc}$ (§3.4). We work over $\mathbb{C}$.

2 There are a vector bundle and a polynomial formula

Let $\mathbb{W}$ be a closed, irreducible subvariety of a Hilbert scheme $\mathrm{Hilb}_{P_{\mathbb{W}}(t)}(\mathbb{P}^n)$. We assume the general member $W \in \mathbb{W}$ is smooth and of pure dimension $\leq n-2$. Let $W' \subset \mathbb{P}^n$ be the subscheme with ideal sheaf $\mathcal{I}_{W'} = (\mathcal{I}_W)^2$.

Lemma 2. *Notation as just above, set $\mathcal{N} = \mathcal{I}_W/(\mathcal{I}_W)^2$. We have the formula for the Hilbert polynomials*

$$P_{W'}(d) := \chi\left(\mathcal{O}_{W'}(d)\right) = \chi\left(\mathcal{O}_W(d)\right) + \chi\left(\mathcal{N}(d)\right). \tag{3}$$

Proof. The assertion follows from the exact sequence

$$0 \to \mathcal{N} \longrightarrow \mathcal{O}_{\mathbb{P}^n}/(\mathcal{I}_W)^2 = \mathcal{O}_{W'} \longrightarrow \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_W = \mathcal{O}_W \to 0.$$

□

Lemma 3. *Notation as just above, we have a generically injective rational map*

$$\begin{array}{ccc} \mathbb{W} & \dashrightarrow & \mathrm{Hilb}_{P_{W'}(t)}(\mathbb{P}^n) \\ W & \mapsto & W',\ \mathcal{I}_{W'} = (\mathcal{I}_W)^2, \end{array} \tag{4}$$

which is a morphism on the open subset of $\mathbb{W}$ consisting of smooth members.

Proof. By Hirzebruch-Riemann-Roch [11, Cor.15.2.1,p. 288] and Conservation of Number [11, 10.21,p. 180], the r.h.s. in (3) is independent of the particular (smooth) W. By the universal property of Hilb [15], (see [8] for a wonderful introduction or [29] for the state of the art) the map exists over the open subset where flatness is ensured. Finally, if $Z, W \in \mathbb{W}$ are smooth members such that $(\mathcal{I}_Z)^2 = (\mathcal{I}_W)^2$ it follows that $\mathcal{I}_Z = \mathcal{I}_W$. □

Definition 4. *Denote by $\mathbb{W}'$ the closure of the image of the map* (4)*.*

Note the occurrence of a new Hilbert polynomial, $P_{\mathbb{W}'}(t)$. For instance, if we take $\mathbb{W}$ as the family of lines in $\mathbb{P}^3$, we have $P_{\mathbb{W}}(t) = t+1$ whereas presently $P_{\mathbb{W}'}(t) = 3t+1$. The latter is the Hilbert polynomial of the subscheme defined by the ideal $\langle x_0^2, x_0x_1, x_1^2\rangle = \langle x_0, x_1\rangle^2$. Our starting point is the elementary fact that a surface in $\mathbb{P}^3$ is singular along the line $\langle x_0, x_1\rangle$ if and only if its defining homogeneous polynomial lies in $\langle x_0, x_1\rangle^2$. Quite generally, to ask a hypersurface F of degree d to be singular along a general member $W \in \mathbb{W}$ is equivalent to requiring F to be an element of $H^0((\mathcal{I}_W)^2(d))$. This approach was probably inaugurated by Harris and Pandharipande [17] and followed by Göttsche and Rennemo for isolated singularities.

We write $\mathrm{Sing}(F)$ for the singular locus of F. The next lemma is a main step towards the proof of Theorem.1**(ii)**.

Lemma 5. *Suppose $\mathcal{J}_d := (\mathcal{I}_W)^2(d)$ globally generated. Let F be a general element of $H^0(\mathcal{J}_d)$. Then Sing$(F) = W$ set-theoretically.*

Proof. The hypothesis that $\mathcal{J}_d$ be globally generated implies by Bertini (cf. [18, 10.9.2]) that $\mathrm{Sing}(F) \subseteq W$. The inclusion $W \subseteq \mathrm{Sing}(F)$ is evident: if F lies in $H^0(\mathcal{J}_d)$ then its gradient is zero all along W. □

Next we borrow from [1] the technical construction of the correspondence

$$\widetilde{\Sigma}(\mathbb{W}', d) := \{(Z, F) \in \mathbb{W}' \times \mathbb{P}^{N_d} \mid Z \subset F\}. \tag{5}$$

Lemma 6. *Notation as in* Definition 4*, consider the projection maps*

$$\begin{tikzcd} & \mathbb{W}' \times \mathbb{P}^{N_d} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{W}' & & \mathbb{P}^{N_d}. \end{tikzcd} \tag{6}$$

Then for all $d \gg 0$, the correspondence $\widetilde{\Sigma}(\mathbb{W}', d)$ (see (5)) *is a projective bundle over $\mathbb{W}'$ via the first projection p_1.*

Proof. Let $\widetilde{Z} \subset \mathbb{W}' \times \mathbb{P}^n$ be the universal subscheme and similarly $\widetilde{F} \subset \mathbb{P}^{N_d} \times \mathbb{P}^n$ the universal hypersurface of degree d. Let us denote $\widehat{Z}, \widehat{F}$ their pullbacks to $\mathbb{W}' \times \mathbb{P}^{N_d} \times \mathbb{P}^n$. We have the following diagram of sheaves

$$\begin{tikzcd} \mathcal{O}_Y \arrow[r] \arrow[dr, "\rho"'] & \mathcal{O}_Y(\widehat{F}) \arrow[r] \arrow[d] & \mathcal{O}_{\widehat{F}}(\widehat{F}) \\ & \mathcal{O}_{\widehat{Z}}(\widehat{F}) & \end{tikzcd}$$

over $Y := \mathbb{W}' \times \mathbb{P}^{N_d} \times \mathbb{P}^n$. By construction, the oblique arrow ρ vanishes at a point $(Z, F, x) \in Y$ if and only if $x \in F \cap Z$. So the inclusion $Z \subset F$ holds when the previous condition occurs for all $x \in Z$. Thus, $\widetilde{\Sigma}(\mathbb{W}', d)$ is equal to the scheme of zeros of ρ along the fibers of the projection $p_{12} : \widehat{Z} \to \mathbb{W}' \times \mathbb{P}^{N_d}$. Recalling Altman & Kleiman [1, (2.1) p. 14], this is equal to the scheme of zeros of the adjoint section of the direct image vector bundle $p_{12*}(\mathcal{O}_{\widehat{Z}}(\widehat{F}))$. Look at the projection maps

$$\begin{tikzcd} \mathbb{W}' \times \mathbb{P}^n \arrow[r, "q_2"] \arrow[d, "q_1"'] & \mathbb{P}^n \\ \mathbb{W}'. & \end{tikzcd}$$

Since $\mathcal{O}(\widetilde{F}) = \mathcal{O}_{\mathbb{P}^{N_d}}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(d)$, by the projection formula we have produced a section of $\mathcal{O}_{\mathbb{P}^{N_d}}(1) \otimes \mathcal{D}_d$, where $\mathcal{D}_d = q_{1*}(\mathcal{O}_{\widetilde{Z}}(d))$. By Castelnuovo-Mumford and base change theory, there is an integer d_0 (= regularity) such that $\mathcal{D}_d$ is a vector bundle of rank $P_{\mathbb{W}'}(d)$ for all $d \geq d_0$, where $P_{\mathbb{W}'}(t)$ denotes the Hilbert polynomial of the members of $\mathbb{W}'$. In fact, $\mathcal{D}_d$ fits into the exact sequence of vector bundles over $\mathbb{W}'$:

$$\begin{tikzcd} 0 \arrow[r] & q_{1*}(\mathcal{I}_{\widetilde{Z}} \otimes \mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r] \arrow[d, equal] & q_{1*}(q_2{}^*\mathcal{O}_{\mathbb{P}^n}(d)) \arrow[r] \arrow[d, equal] & q_{1*}(\mathcal{O}_{\widetilde{Z}}(d)) \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{E}_d \arrow[r, tail] & \mathcal{F}_d \arrow[r, two heads] & \mathcal{D}_d\,. & \end{tikzcd} \tag{7}$$

Taking the projectivization and pulling back to $\mathbb{W}' \times \mathbb{P}^{N_d}$, we get (omitting pullbacks):

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^{N_d}}(-1) \arrow[d, tail] \arrow[dr, "\overline{\rho}"] & \\ \mathcal{E}_d \arrow[r, tail] & \mathcal{F}_d \arrow[r, two heads] & \mathcal{D}_d\,. \end{tikzcd} \tag{8}$$

By construction, $\overline{\rho}$ vanishes precisely over $\widetilde{\Sigma}(\mathbb{W}', d)$. And this tells us that

$$\widetilde{\Sigma}(\mathbb{W}', d) = \mathbb{P}(\mathcal{E}_d). \tag{9}$$

□

Lemma 7. *Notation as in* (7),(9) *we have that* $\widetilde{\Sigma}(\mathbb{W}', d)$ *represents the top Chern class of* $\mathcal{O}_{\mathbb{P}^{N_d}}(1) \otimes \mathcal{D}_d$.

Proof. As $\operatorname{codim}_{\mathbb{W}' \times \mathbb{P}^{N_d}} \widetilde{\Sigma}(\mathbb{W}', d) = P_{\mathbb{W}'}(d)$, which coincides with the rank of $\mathcal{D}_d$ due to (7), the assertion follows from Fulton [11, 3.2.16, p. 61]. □

Definition 8. *We call the* $\mathbb{W}$*-discriminant, denoted by* $\Sigma(\mathbb{W}, d)$*, the subvariety of* $\mathbb{P}^{N_d}$ *corresponding to the hypersurfaces which contain some member of* $\mathbb{W}'$.

For a general point $W' \in \mathbb{W}'$ and a hypersurface F, asking that $F \supset W'$ as schemes is equivalent to requiring that the hypersurface be singular along the reduced scheme $W = W'_{red} \in \mathbb{W}$. With the notation of (6), we have

$$\Sigma(\mathbb{W}, d) = p_2(\widetilde{\Sigma}(\mathbb{W}', d)). \tag{10}$$

The usual discriminant hypersurface corresponds to the choice $\mathbb{W} = \mathbb{P}^n$.

Lemma 9. *Notation as above, the map*

$$\begin{array}{rcl} \mathbb{W}' \times \mathbb{P}^{N_d} \supset \widetilde{\Sigma}(\mathbb{W}', d) & \xrightarrow{p_2} & \Sigma(\mathbb{W}, d) \subset \mathbb{P}^{N_d} \\ (Z, F) & \mapsto & F \end{array} \tag{11}$$

is generically injective for all $d >> 0$.

Proof. We must show that for a general $F \in \Sigma(\mathbb{W}, d)$, the fiber $p_2^{-1}(F) \subset \widetilde{\Sigma}(\mathbb{W}', d)$ consists of a single element. In view of Lemma 3, there is an open subset $\mathbb{W}'_0 \subset \mathbb{W}'$ formed by subschemes W' with ideal of the form $\mathcal{I}_{W'} = (\mathcal{I}_W)^2$ with $W \in \mathbb{W}$ smooth. Now it suffices to show that the restriction of p_2 over $\mathbb{W}'_0$ is injective. Let F be a general hypersurface of degree d containing $W' \in \mathbb{W}'_0$. This means that F is general in $H^0(\mathcal{I}_{W'}(d)) = H^0((\mathcal{I}_W)^2(d))$. By Lemma 5 we have that $\mathrm{Sing}(F) = W$ (as sets). Let $Z' \in \mathbb{W}'_0$ be such that $Z' \subset F$. So we have (W', F) and $(Z', F) \in p_2^{-1}(F)$. By construction of $\mathbb{W}'_0$, we have $\mathcal{I}_{Z'} = (\mathcal{I}_Z)^2$ for some smooth $Z = Z'_{red} \in \mathbb{W}$. So $Z \subseteq \mathrm{Sing}(F) = W$. Since the Hilbert polynomials of Z, W are one and the same, therefore $Z = W$ and so $Z' = W'$. This shows that the map in (11) is generically injective as asserted. $\square$

Lemma 10. *Notation as above, we have*

$$\deg \Sigma(\mathbb{W}, d) = \int \mathit{Segre}(w, \mathcal{E}_d) \cap [\mathbb{W}'], \tag{12}$$

where $w := \dim \mathbb{W}' = \dim \mathbb{W}$.

Proof. We have the equality of cycle classes

$$(p_2)_\star[\widetilde{\Sigma}(\mathbb{W}', d)] = [\Sigma(\mathbb{W}, d)].$$

This follows from [11, §1.4, p. 11] since $\widetilde{\Sigma}(\mathbb{W}, d) \xrightarrow{p_2} \Sigma(\mathbb{W}, d)$ is birational, as shown in Lemma 9. Set $\delta := \dim \Sigma(\mathbb{W}, d)$. We have $\delta = w + \epsilon$, with $\epsilon := \mathrm{rk}\mathcal{E}_d - 1$. Set $H = c_1\mathcal{O}_{\mathbb{P}^{N_d}}(1)$, the hyperplane class. By projection formula we may write

$$\begin{aligned} \deg \Sigma(\mathbb{W}, d) &= \int H^\delta \cap [\Sigma(\mathbb{W}, d)] = \int p_2^\star H^\delta \cap [\widetilde{\Sigma}(\mathbb{W}', d)] \\ &= \int (p_1)_\star \left(p_2^\star H^{w+\epsilon} \cap [\widetilde{\Sigma}(\mathbb{W}', d)]\right) = \int \mathrm{Segre}(w, \mathcal{E}_d) \cap [\mathbb{W}'], \end{aligned}$$

using Fulton [11, §3.1, p. 47, Prop.4.4,p. 83 and Ex. 8.3.14, p. 143]. $\square$

Proposition 11. *The degree of the* $\mathbb{W}$*-discriminant,* $\Sigma(\mathbb{W}, d)$*, is a polynomial in* d *of degree* $\leq n \dim(\mathbb{W})$ *for all* $d \gg 0$.

Proof. Let $\widetilde{\mathbb{W}} \to \mathbb{W}'$ be a desingularization (cf. [19]). Pulling back $\mathcal{E}_d, \mathcal{D}_d$ in (7) to $\widetilde{\mathbb{W}}$, we may as well simplify notation and assume $\widetilde{\mathbb{W}} = \mathbb{W}'$ smooth. We now argue as in [6] and [33]. Recall $\mathcal{D}_d$ is a direct image of a sheaf over $\mathbb{W}' \times \mathbb{P}^n$ (cf. 7). The same diagram of sheaves tells us

$$\mathrm{Segre}(w, \mathcal{E}_d) = c_w(\mathcal{D}_d). \tag{13}$$

Now we can apply Grothendieck-Riemann-Roch (cf. [11, Thm.15.2, p. 286]) to express the Chern character of $\mathcal{D}_d$ as

$$ch(\mathcal{D}_d) = ch((q_1)_!(\mathcal{O}_{\widetilde{Z}}(d))) = (q_1)_* \left(ch(\mathcal{O}_{\widetilde{Z}}) \cdot ch(\mathcal{O}_{\mathbb{P}^n}(d)) \cdot \mathrm{todd}(\mathbb{P}^n) \right). \tag{14}$$

Note that the right hand side is a polynomial in d of degree $\leq n$. On the other hand, the Chern class c_w is a weighted polynomial of degree w on the coefficients of the Chern character ([11, 3.2.3,p. 56]). This implies that $c_w(\mathcal{D}_d)$ is a polynomial in d of degree $\leq nw$.

□

Remark 12. In order to get a polynomial formula, it suffices to calculate the degree of $\Sigma(\mathbb{W}, d)$ for $n \dim \mathbb{W} + 1$ values of d. In all cases treated in this work, we find that the degree of the polynomial $p^{\mathbb{W}}(d)$ actually is $(k+1) \times \dim(\mathbb{W})$, where k denotes the dimension of a member of $\mathbb{W}$. The validity for $\mathbb{W}$ arbitrary remains conjectural. [31] and [27] handle the case $k = 0$.

To compute explicitly the integral in (12), we will apply Bott's residues formula in the equivariant flavor of [9] (see also [23], [24]),

$$\int \mathrm{Segre}(w, \mathcal{E}_d) \cap [\mathbb{W}'] = \sum_F \frac{c_w^{\mathbb{T}}(-\mathcal{E}_d) \cap [F]_{\mathbb{T}}}{c_{top}^{\mathbb{T}}(\mathcal{N}_{F|\mathbb{W}'})}, \tag{15}$$

where the sum runs through all fixed components F of a convenient action of the torus $\mathbb{T} := \mathbb{C}^*$ on $\mathbb{W}'$. The $\mathcal{N}_{F|\mathbb{W}'}$ appearing in the denominator denotes the normal bundle of a fixed component F in $\mathbb{W}'$. In all cases treated in this work the set of fixed points is finite. Thus the denominator in (15) is the $\mathbb{T}$-equivariant top Chern class, $c_{top}^{\mathbb{T}}(\mathcal{T}_F\mathbb{W}')$, where $\mathcal{T}_F\mathbb{W}'$ denotes the tangent space at a fixed point F in $\mathbb{W}'$.

Remark 13. Notation as in (1), for $\mathbb{W} = \mathbb{W}_{(k,n)}$ (as well as $\mathbb{W} = \mathbb{W}_m$), the family $\mathbb{W}'$ which parameterizes subschemes of $\mathbb{P}^n$ defined by $(I_W)^2$ with $W \in \mathbb{W}$ is flat. In fact, we have $\mathbb{W} = \mathbb{W}'$: the map (4) is an isomorphism. However, in the other cases dealt with in this work we have only the generic flatness guaranteed over the locus of smooth $W \in \mathbb{W}$. In fact, the Hilbert polynomial for $(I_W)^2$ may jump at special points. A blowup will be required in order to achieve flatness, following Raynaud [26].

3 Enumerative results

"*For many problems it would be miraculous and totally unexpected if somebody were to find a precise formula for the solution; most of the time one must settle for a rough estimate instead.*"[1]

A detailed exposition of the fixed points and the computations of their contributions on Bott's formula (15), including scripts for Macaulay2 [14], Maple [22] and Singular [7] and for the resolution of indeterminacies in the cases of sections 3.3 and 3.4 can be found in Sellin [28].

3.1 Hypersurfaces singular along a linear $\mathbb{P}^k \subset \mathbb{P}^n$

Here, the parameter space $\mathbb{W}_{(k,n)} := \mathbb{G}(k+1, n+1)$, the grassmannian of $k+1$ dimensional vector subspaces of $\mathbb{C}^{n+1}$. Our goal is to determine the degree of the family of hypersurfaces of degree d singular along some $\mathbb{P}^k \subset \mathbb{P}^n$.

For the reader's benefit we will show the calculations for $\deg \Sigma(\mathbb{W}_{(1,3)}, d)$.

Consider the torus $\mathbb{T} = \mathbb{C}^*$ acting diagonally on $\mathcal{F}_1 = (\mathbb{C}^4)^\vee$ via

$$t \circ x_i := t^{w_i} x_i,$$

with appropriate weights, say:

$$w_0 = 4, w_1 = 11, w_2 = 17, w_3 = 32; \tag{16}$$

The requirement is that denominators appearing in (15), which turn out to be polynomials in the weights $w_0, ..., w_3$, do not vanish.

[1] Tim Gowers, *Mathematics: A Very Short Introduction*

We get a natural induced action on $\mathbb{W}_{(1,3)} = \mathbb{G}(2,4)$. The tautological vector bundles

$$\mathcal{S} \rightarrowtail \mathcal{F}_1 \twoheadrightarrow \mathcal{Q}$$

on $\mathbb{W}_{(1,3)}$ are $\mathbb{T}$-equivariant. The fiber of $\mathcal{S}$ over a line $l \in \mathbb{G}(2,4)$ is the two dimensional subspace of $\mathcal{F}_1$ of linear forms vanishing on l. Presently we have six fixed points corresponding to the coordinate axes

$$\langle x_0, x_1 \rangle, \langle x_0, x_2 \rangle, \ldots, \langle x_2, x_3 \rangle.$$

Referring to (15), we have

$$\deg \Sigma(\mathbb{W}_{(1,3)}, d) = \sum_F \frac{c_4^{\mathbb{T}}(-\mathcal{E}_d) \cap [F]_{\mathbb{T}}}{c_4^{\mathbb{T}}(\mathcal{T}\mathbb{W}_{(1,3)})} \tag{17}$$

summing over the six fixed points. The denominator in (17), let's say for $F = \langle x_0, x_1 \rangle$, is obtained as follows. First we find the fiber of the tangent

$$\mathcal{T}_F \mathbb{W}_{(1,3)} = \mathrm{Hom}(\mathcal{S}_F, \mathcal{Q}_F) = \langle x_0, x_1 \rangle^\vee \otimes \langle x_2, x_3 \rangle = \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_2}{x_1} + \frac{x_3}{x_1},$$

where $\frac{x_i}{x_j}$ denotes the $\mathbb{T}$-space with weight $w_i - w_j$. In this way, we obtain $c_4^{\mathbb{T}}(\mathcal{T}\mathbb{W}_{(1,3)}) \cap [F]_{\mathbb{T}} = (w_2 - w_0)(w_3 - w_0)(w_2 - w_1)(w_3 - w_1)$. With the choice of weights in (16), this gives us the value 45864. Similarly, the numerator requires the weight decomposition of the fiber $(\mathcal{E}_d)_F$. To fix the ideas, take $d = 3$. Now that fiber consists of the cubic forms $f \in H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ with gradient null along the line F. The weight decomposition is given by

$$(\mathcal{E}_3)_F = x_0^3 + x_0^2 x_1 + x_0^2 x_2 + x_0^2 x_3 + x_0 x_1^2 + x_0 x_1 x_2 + x_0 x_1 x_3 + x_1^3 + x_1^2 x_2 + x_1^2 x_3.$$

Since we actually need the Segre class, Chern$(-\mathcal{E}_d)$=Chern$(\mathcal{D}_d)$ cf. (7), we find the complementary decomposition

$$(\mathcal{D}_d)_F = x_0 x_2^2 + x_1 x_2^2 + x_2^3 + x_0 x_2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_0 x_3^2 + x_1 x_3^2 + x_2 x_3^2 + x_3^3.$$

Here $x_i^\alpha x_j^\beta x_k^\gamma$ denotes the $\mathbb{T}$-space with weight $\alpha w_i + \beta w_j + \gamma w_k$. The corresponding numerical contribution is 3217978137. The fixed point

$F = \langle x_0, x_1 \rangle$ contributes the fraction 3217978137/45864. The total contribution of the six fixed points is

$$\frac{3217978137}{45864} - \frac{2152229961}{17640} + \frac{774359841}{28665} + \frac{1227942219}{28665} - \frac{392711889}{17640} + \frac{218302833}{45864} = 504.$$

This is the degree of the subvariety of $|\mathcal{O}^3_{\mathbb{P}}(3)| = \mathbb{P}^{19}$ consisting of the Whitney umbrellas: surfaces of degree 3 in $\mathbb{P}^3$ which are singular along some line (cf. [5]).

Recalling Remark 12, we need the degrees of $\Sigma(\mathbb{W}_{(1,3)}, d)$ for $3 \times 4 + 1$ values of d. Interpolating, we get

$$\deg \Sigma(\mathbb{W}_{(1,3)}, d) = \frac{1}{32}\binom{d}{2}(27d^6 - 117d^5 + 269d^4 - 375d^3 + 312d^2 - 132d + 48). \quad (18)$$

We list below the results for $(k, n) \in \{(2,4), (2,5), (3,5)\}$:

$$\begin{gathered}\deg \Sigma(\mathbb{W}_{(2,4)}, d) = \frac{1}{2^7 \cdot 3^3}\tbinom{d+2}{4}(9d^{14} - 18d^{13} - 63d^{12} \\ +396d^{11} - 405d^{10} - 1530d^9 + 5328d^8 - 4176d^7 - 9414d^6 \\ +27208d^5 - 24347d^4 - 4696d^3 + 36572d^2 - 32544d + 14400).\end{gathered} \quad (19)$$

$$\begin{gathered}\deg \Sigma(\mathbb{W}_{(2,5)}, d) = \frac{1}{(2)^3 3^{11} 5^3}\tbinom{d+2}{4}(12800d^{23} - 25600d^{22} \\ -224000d^{21} + 966400d^{20} + 520800d^{19} \\ -10632000d^{18} + 18128000d^{17} + 35186000d^{16} - 170677265d^{15} \\ +145358830d^{14} + 449576760d^{13} - 1292773830d^{12} + 778144037d^{11} \\ +2164141556d^{10} - 5208921230d^9 + 3728975455d^8 + 3332483181d^7 \\ -10452711042d^6 + 10781927010d^5 - 2523245175d^4 - 7609562253d^3 \\ +11511503406d^2 - 8323547040d + 3637418400).\end{gathered} \quad (20)$$

$$\begin{gathered}\deg \Sigma(\mathbb{W}_{(3,5)}, d) = \frac{1}{2^{27} \cdot 3^3 \cdot 5^4}\tbinom{d+2}{4}\left(1125d^{28} + 15750d^{27}\right. \\ +86625d^{26} + 168750d^{25} - 187875d^{24} - 38250d^{23} \\ +8824725d^{22} + 23473350d^{21} - 32467725d^{20} - 128183670d^{19}\end{gathered} \quad (21)$$

$$+426415635d^{18} + 1377078570d^{17} - 2137554049d^{16}$$
$$-7117020302d^{15} + 15925316455d^{14} + 37514746370d^{13}$$
$$-82840806388d^{12} - 125157483544d^{11} + 422227932240d^{10}$$
$$+287672117600d^{9} - 1529648949952d^{8} + 207120164224d^{7}$$
$$+4517312266240d^{6} - 3047085731840d^{5} - 6253154779136d^{4}$$
$$+11893749153792d^{3} + 2911913902080d^{2}$$
$$-8455245004800d + 2378170368000\big) .$$

Recalling $\dim \mathbb{W}_{k,n} = (k+1)(n-k)$, we remark that the degrees of the above polynomials are in agreement with the expectation (2), to wit, $(k+1)\dim \mathbb{W}$.

3.2 Surfaces singular along plane curves

The family of plane curves of degree $m > 1$ in $\mathbb{P}^3$ is parameterized by a $\mathbb{P}^{N_m}$-bundle over $\check{\mathbb{P}}^3$

$$\mathbb{W}_m \longrightarrow \check{\mathbb{P}}^3,$$

where $N_m = \binom{m+2}{2} - 1$. We have calculated $\deg \Sigma(\mathbb{W}_m, d)$ for $m = 2, 3$:

$$\begin{aligned} \deg \Sigma(\mathbb{W}_2, d) = \frac{1}{2^{13} \cdot 3^2 \cdot 5 \cdot 7}(d-2)(150903d^{15} - 3809754d^{14} \\ +44834472d^{13} - 317080224d^{12} + 1422290970d^{11} - 3579080844d^{10} \\ -455933988d^{9} + 47928493544d^{8} - 237841700217d^{7} + 712127741206d^{6} \\ -1498533401372d^{5} + 2287674925704d^{4} - 2504345972608d^{3} \\ +1873638158208d^{2} - 859900216320d + 182801203200). \end{aligned} \tag{22}$$

$$
\begin{aligned}
\deg \Sigma(\mathbb{W}_3, d) = \frac{1}{2^{20}\cdot 3^4\cdot 5\cdot 7\cdot 11} \big(13286025d^{24} - 1038081420d^{23} \\
+39146062158d^{22} - 946074434976d^{21} + 16407919974303d^{20} \\
-216603408547548d^{19} + 2251372103607528d^{18} \\
-18776305509313968d^{17} + 126579622223230407d^{16} \\
-686155959955971780d^{15} + 2911999863446866566d^{14} \\
-8886007643094113376d^{13} + 12799827743693355329d^{12} \\
+50456388588134712812d^{11} - 483658040042985949724d^{10} \\
+2229927488252098274992d^{9} - 7358275057877141245584d^{8} \\
+18804143410678335462720d^{7} - 38007885859704936084800d^{6} \\
+60658830486712279959808d^{5} - 75133955486596446561280d^{4} + \\
69793667761693681135616d^{3} - 45744106516543857328128d^{2} \\
+18819557445986636267520d - 3636764182567924531200\big)\,.
\end{aligned}
\tag{23}
$$

The reader interested in obtaining $\deg \Sigma(\mathbb{W}_m, d)$ for other m, simply plug in the desired value in the script in [28, Appendix E, p.92]. Notice the degrees of the above polynomials in (22) and (23) are $(k+1)\dim \mathbb{W} = 2\left(2 + \binom{d+2}{2}\right)$.

3.3 Hypersurfaces singular along base loci of nets of quadrics of determinantal type

In this section we discuss the case of hypersurfaces in $\mathbb{P}^n$ $(n = 3, 4, 5)$ singular along base loci of nets of quadrics of determinantal type. By this we mean the nets generated by 2×2-minors of a 3×2 matrix of linear forms. Specifically, we consider the families

$$
\begin{cases}
\mathbb{W}_{twc} & = & \{ \text{ twisted cubics in } \mathbb{P}^3 \}, \\
\mathbb{W}_{rc} & = & \{ \text{ ruled cubics in } \mathbb{P}^4 \} \ \text{ and} \\
\mathbb{W}_{seg} & = & \{ \text{ Segre 3-folds in } \mathbb{P}^5 \}.
\end{cases}
$$

3.3.1 Surfaces singular along twisted cubics

A twisted cubic is a rational, smooth curve of degree 3 in $\mathbb{P}^3$. Any such is projectively equivalent to the scheme of zeros of the 2×2 minors of the matrix $\left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{smallmatrix}\right)$. Its Hilbert polynomial is $3t+1$. Piene & Schlessinger [25] showed that the component $\mathbb{W}_{twc} \subset \mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ is a smooth projective variety of dimension 12. Subsequently, Ellingsrud, Piene & Strømme [10] proved that the subvariety of the Grassmannian

$$\mathbb{X} \subset \mathbb{G}(3, \mathcal{F}_2) \tag{24}$$

formed by nets of determinantal type is smooth. Moreover the component $\mathbb{W}_{twc}$ is the blowup of $\mathbb{X}$ along the subvariety $\mathbb{G}_\omega$ of nets projectively equivalent to the net

$$\omega := (x_0^2, x_0x_1, x_0x_2).$$

A typical element on the fiber of the exceptional divisor over ω corresponds to an ideal of the form $\mathcal{I}_{\omega,f} := \langle x_0^2, x_0x_1, x_0x_2, f\rangle$, where $x_0 = f(x_1, x_2, x_3) = 0$ is a plane cubic singular at the point $x_0 = x_1 = x_2 = 0$. The square $(\mathcal{I}_{\omega,f})^2$ of any such ideal has Hilbert polynomial $9t - 7$, same as for the square of the ideal of the standard twisted cubic, $\langle x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_1^2 - x_0x_2\rangle$.

Unlike the cases $\mathbb{W}_{(k,n)}$ and $\mathbb{W}_m$, the family formed by the subschemes of $\mathbb{P}^3$ defined by $(\mathcal{I}_W)^2, W \in \mathbb{W}_{twc}$ is not flat. In fact, the element

$$\mathbf{o} := \langle x_0, x_1\rangle^2 = \langle x_0^2, x_0x_1, x_1^2\rangle \tag{25}$$

is a member of the good component $\mathbb{W}_{twc}$, but its square has "bad" Hilbert polynomial, namely $P_{\mathbb{W}_{twc}}(t) = 10t - 10$, instead of $9t - 7$.

This is remedied by blowing up $\mathbb{W}_{twc}$ along the orbit $\mathbb{G}_{\mathbf{o}}$. Since $\mathbb{G}_{\mathbf{o}} \cap \mathbb{G}_\omega = \emptyset$, it follows that $\mathbb{G}_{\mathbf{o}}$ lifts isomorphically to an orbit in $\mathbb{W}_{twc}$, still denoted by $\mathbb{G}_{\mathbf{o}}$. Let $\mathbb{W}'_{twc}$ denote the blowup of $\mathbb{W}_{twc}$ along $\mathbb{G}_{\mathbf{o}}$. In fact, $\mathbb{X}$ and $\mathbb{W}_{twc}$ are isomorphic over any neighborhood of $\mathbb{G}_{\mathbf{o}}$ disjoint from $\mathbb{G}_\omega$. The restriction $\mathbb{W}'_{twc|\mathbb{X}\setminus\mathbb{G}_\omega}$ is isomorphic to the restriction $\mathbb{X}'_{|\mathbb{X}\setminus\mathbb{G}_\omega}$ of the blowup $\mathbb{X}'$ of $\mathbb{X}$ along $\mathbb{G}_{\mathbf{o}}$.

Let $\mathcal{C}$ be the tautological subbundle of rank 3 over the grassmannian of nets of quadrics $\mathbb{G}(3, \mathcal{F}_2)$. Write $\mathrm{S}_2(\mathcal{C})$ the symmetric power.

Proposition 14. *Let $\mu : \mathrm{S}_2(\mathcal{C})_{|\mathbb{W}_{twc}} \to \mathcal{F}_4$ be the natural map induced by multiplication. Consider the blowing up diagram of $\mathbb{W}_{twc}$ along $\mathbb{G}_{\mathbf{o}}$*

$$\begin{tikzcd} \mathbb{W}'_{twc} \arrow[d] & \supset & \mathbb{E}' \arrow[d] \\ \mathbb{W}_{twc} & \supset & \mathbb{G}_{\mathbf{o}}. \end{tikzcd} \tag{26}$$

Then **(i)** *$\mathbb{G}_{\mathbf{o}}$ is the scheme of zeros of $\overset{6}{\wedge}\mu$;*
(ii) *$\mathbb{W}'_{twc}$ embeds in $\mathbb{W}_{twc} \times \mathbb{G}(6, \mathcal{F}_4)$ as the closure of the graph of the rational map $\mathbb{W}_{twc} \dashrightarrow \mathbb{G}(6, \mathcal{F}_4)$ induced by μ.*
(iii) *The fiber of the exceptional divisor $\mathbb{E}'$ over $\mathbf{o}$ is the projectivization of the quotient space of quartic forms,*

$$(\langle x_0, x_1\rangle^3)_4 / (\langle x_0, x_1\rangle^4)_4.$$

Proof. The argument is based on local calculations as shown in [28, Appendix F.1]. We just highlight the main steps. Denote by $\mathfrak{Z}$ the scheme of zeros in question; it is invariant under the natural $\mathbb{PGL}_4$ induced action. Recall $\mathbb{X}$ (24) has precisely two closed orbits, represented by the nets

$$\mathbf{o} = (x_0^2, x_0x_1, x_1^2) \text{ and } \omega = (x_0^2, x_0x_1, x_0x_2).$$

Clearly $\mathbf{o} \in \mathfrak{Z} \not\ni \omega$. Consider the list the 10 quadratic monomials,

$$m_1 := x_0^2, m_2 := x_0x_1, m_3 := x_1^2, m_4 := x_0x_2, \dots, m_{10} := x_3^2.$$

Use the affine coordinates $a_{ij}, 1 \le i \le 3, 1 \le j \le 7$ for the open subset $\mathbb{G}^0 \subset \mathbb{G}(3, 10)$ so that the quadrics

$$\begin{cases} q_1 := & x_0^2 & & & + \sum a_{1j} m_{3+j} \\ q_2 := & & x_0x_1 & & + \sum a_{2j} m_{3+j} \\ q_3 := & & & x_1^2 & + \sum a_{3j} m_{3+j} \end{cases}$$

yield a trivialization for the restriction $\mathcal{C}_{|\mathbb{G}^0}$. Over $\mathbb{G}^0$ the multiplication map $\mathcal{C} \otimes \mathcal{F}_1 \to \mathcal{F}_3$ is of generic rank 12; the rank drops to 10 exactly along $\mathbb{X}^0 := \mathbb{X} \cap \mathbb{G}^0 = \mathbb{W}_{twc} \cap \mathbb{G}^0 =: \mathbb{W}^0_{twc}$; the 2nd equality stems from the fact that $\mathbb{G}^0$ is a neighborhood away from the orbit of ω. This yields explicit equations for $\mathbb{X}^0 \subset \mathbb{G}^0$. These equations allow us to express 9 of the coordinates in terms of the 12 remaining ones; these in turn provide affine coordinates for $\mathbb{X}^0$. Working out a matrix representation for $\mu : \mathcal{S}_2\mathcal{C}_{|\mathbb{X}^0} \to \mathcal{F}_4$ we find that the ideal of $6{\times}6$ minors, which defines $\mathfrak{Z}$, is equal to the ideal of $(\mathbb{G}_{\mathbf{o}})^0 := \mathbb{G}_{\mathbf{o}} \cap \mathbb{X}^0 \subset \mathbb{X}^0$. Since $\mathfrak{Z} \supseteq \mathbb{G}_{\mathbf{o}}$ are closed invariant subschemes which agree in a neighborhood of their unique closed orbit, they must be equal. Blowing it up, we get $\mathbb{X}'$ (resp. $\mathbb{W}'_{twc}$) embedded in $\mathbb{X}{\times}\mathbb{G}(6, \mathcal{F}_4)$ (resp. $\mathbb{W}_{twc}{\times}\mathbb{G}(6, \mathcal{F}_4)$) as the closure of the graph of the rational map induced by μ. Likewise, we find that the fiber of $\mathbb{E}'$ over $\mathbf{o}$ is as stated in **(iii)**. $\square$

Remark 15. The previous result implies that the fixed points in $\mathbb{W}'_{twc}$ are obtained from those well known for $\mathbb{W}_{twc}$ (cf. [9]), except for the six ones belonging to $\mathbb{G}_{\mathbf{o}}$. For each of these, say $\mathbf{o} = \langle x_0, x_1\rangle^2$, we form the ideals $\langle x_0, x_1\rangle^4 + \langle Q\rangle, Q \in \{x_0^3x_2, x_0^3x_3, x_0^2x_1x_2, x_0^2x_1x_3, x_0x_1^2x_2, x_0x_1^2x_3, x_1^3x_2, x_1^3x_3\}$. These eight monomials span the exceptional fiber $(\langle x_0, x_1\rangle^3)_4/(\langle x_0, x_1\rangle^4)_4$.

For details about the explicit contribution of each fixed point the reader is again kindly referred to [28, Appendix F.3]. The polynomial that gives us the degree of $\Sigma(\mathbb{W}_{twc}, d)$ is displayed in (27). Note that its degree is equal to $2 \times \dim(\mathbb{W}_{twc}) = 2 \times 12$ in agreement with (2).

$$
\begin{gathered}
\deg \Sigma(\mathbb{W}_{twc}, d) = \tfrac{1095687}{50462720}d^{24} - \tfrac{19230291}{18022400}d^{23} + \tfrac{24114591}{985600}d^{22} \\
-\tfrac{3932462817}{11468800}d^{21} + \tfrac{73665592101}{22937600}d^{20} - \tfrac{23321377833}{1146880}d^{19} + \tfrac{4087404048523}{51609600}d^{18} \\
-\tfrac{205245946577}{2457600}d^{17} - \tfrac{79029321809671}{68812800}d^{16} + \tfrac{2854774357217311}{309657600}d^{15} \\
-\tfrac{6688891988137}{143360}d^{14} + \tfrac{895445339622112187}{3406233600}d^{13} - \tfrac{4177328126526143027}{2270822400}d^{12} \\
+\tfrac{11340295250223 01939}{94617600}d^{11} - \tfrac{29052565860084958379}{464486400}d^{10} + \tfrac{1100107099486708819}{4300800}d^{9} \\
-\tfrac{319500979951588311119}{38707200}d^{8} + \tfrac{365421773568911927}{172800}d^{7} - \tfrac{8318629615873057099}{1935360}d^{6} \\
+\tfrac{615395937691427021}{89600}d^{5} - \tfrac{33777705898251350 8747}{39916800}d^{4} + \tfrac{5167781409451915223}{665280}d^{3} \\
-\tfrac{693707469384158233}{138600}d^{2} + \tfrac{466431399017887}{231}d - 383398629664.
\end{gathered} \tag{27}
$$

Figure 1 shows with the help of Surfer [30] an example of a quartic surface singular along a twisted cubic.

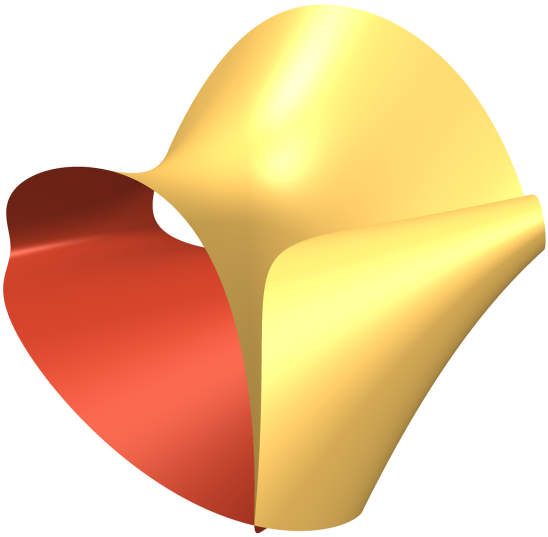


Figure 1: $-8y^4 + 16xy^2z + 8y^3z - 8x^2z^2 - 8xyz^2 - 8y^2z^2 + 2xz^3 + 2yz^3 + 4z^4 - 8xy^2 + 2y^3 + 8x^2z + 10xyz - 2y^2z - 2xz^2 - 8yz^2 - 6x^2 + 2xy + 4y^2 = 0$

3.3.2 Hypersurfaces singular along a ruled cubic surface in $\mathbb{P}^4$

A ruled cubic surface in $\mathbb{P}^4$ is the base locus of a net of quadrics of determinantal type (cf. Beauville [4, Prop. IV.7, p. 44)]; it's projectively equivalent to the subvariety W defined by the ideal $\mathcal{I}_W$ of the $2{\times}2$ minors of the matrix $\left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \end{smallmatrix}\right)$. It has Hilbert polynomial $P_{rc}(t) := (3/2)t^2 + (5/2)t + 1$. Denote by $\mathbb{W}_{rc}$ the corresponding component in $\operatorname{Hilb}\mathbb{P}^4$. The family $\mathbb{W}_{rc}$ has dimension 18. The Hilbert polynomial of the subscheme W'_{rc} defined by $\mathcal{I}_W^2$ is $P_{W'}(t) = (9/2)t^2 - (5/2)t + 2$. The family formed by subschemes of $\mathbb{P}^4$ defined by $\mathcal{I}_W^2$ for some $W \in \mathbb{W}_{rc}$ is not flat. The culprits are again in the orbit of the net $\mathbf{o} = \langle x_0^2, x_0x_1, x_1^2\rangle$, a legitimate member of $\mathbb{W}_{rc}$. Its square has Hilbert polynomial $5t^2 - 5t + 5$ which is different from the expected. Blowing up as before produces a flat family $\mathbb{W}'_{rc}$. Computational details are available in [28, Appendix G.1]. The polynomial that gives the degree of $\Sigma(\mathbb{W}_{rc}, d)$ is described below:

$$\begin{aligned}
\deg \Sigma(\mathbb{W}_{rc}, d) = {} & \frac{1089331}{28207459709485056 00}d^{54} - \frac{4609327}{138135296519700480}d^{53} \\
& + \frac{17053361977}{12432176686773043200}d^{52} - \frac{44006738257}{1243217668677304320}d^{51} + \frac{43540862009}{68559797904998400}d^{50} \\
& - \frac{6776065867607}{822717574859980800}d^{49} + \frac{25203282464989}{329087029943992320}d^{48} - \frac{95461703632727}{205679393714995200}d^{47} \\
& + \frac{3121945759267787}{3290870299439923200}d^{46} + \frac{13975371538743871}{987261089831976960}d^{45} - \frac{1762263793046822003}{9872610898319769600}d^{44} \\
& + \frac{1571373547792223293}{1645435149719961600}d^{43} - \frac{18657333817850689}{21095322432307200}d^{42} - \frac{21162893089184824063}{822717574859980800}d^{41} \\
& + \frac{88172373953883719 83}{42070785078067200}d^{40} - \frac{7285835577039579827299}{7404458173739827200}d^{39} + \frac{1843996517311543646 0101}{2278294822689177600}d^{38} \\
& - \frac{30625726302752154570146789}{251751577907154124800}d^{37} + \frac{2866716053467831514887098 19}{201401262325723299840}d^{36} \\
& - \frac{5957731889573498708183240461}{503503155814308249600}d^{35} + \frac{9462193853605591943184924 23}{130780040471248896 00}d^{34} \\
& - \frac{28843644632003758667785804741}{888534980848779264 00}d^{33} + \frac{35866123088730708454143166 31}{370222908686991360 0}d^{32} \\
& - \frac{277299005780422921177276000 3}{31733392173170688 00}d^{31} - \frac{173239617944054456458227898277}{17770699616975585280}d^{30} \\
& + \frac{310736093407096826889145530073 3}{44426749042438963200}d^{29} - \frac{1302777164405876523072798778669}{4936305449159884800}d^{28}
\end{aligned} \tag{28}$$

$$
\begin{aligned}
&+\frac{2175543494720246680252051667789}{3748506950455787520}d^{27}-\frac{15324266643945858395023213928441}{8885349808487792640 0}d^{26}\\
&-\frac{583723723691983350730395768869707}{133280247127316889600}d^{25}+\frac{295008612506350533900867771909281}{14808916347479654400}d^{24}\\
&-\frac{728822985180459844929716963812 49}{1514548262810419200}d^{23}+\frac{3179423312365559691881647284007591}{59235665389918617600}d^{22}\\
&+\frac{6507491575863414894237209094247570 3}{799681482763901337600}d^{21}-\frac{17650658027740832446748837419090939}{335668770542872166 40}d^{20}\\
&+\frac{43155219287681067897344483362302109}{354025656431935488 00}d^{19}-\frac{30042531700267289895379997718912521}{22377918036191477760}d^{18}\\
&-\frac{749894075579299475576086383836784223}{906305680465754849280}d^{17}+\frac{11528841141262909789036518858178 21}{176296623184281600}d^{16}\\
&-\frac{679247544279215190070362388445065693}{49980092672743833600}d^{15}+\frac{272442096451803568353268951826019 77}{185111454343495680 0}d^{14}\\
&-\frac{14180655522525890878698424573977769}{833001544545730560 0}d^{13}-\frac{17786673868531949329900173945074227}{694167953788108800}d^{12}\\
&+\frac{8140256480874854682039834827204717}{148750275811737600}d^{11}-\frac{15847193428252892198587722393037621}{231389317929369600}d^{10}\\
&+\frac{5120308596714613277827568192502967 1}{851933397830860800}d^{9}-\frac{415833099791358148948760413114949}{10846374277939200}d^{8}\\
&+\frac{1909222806402780987957309330907 99}{10846374277939200}d^{7}-\frac{478337690398387542649533056 41}{8608233553920}d^{6}\\
&+\frac{27647372439801630130761097904 63}{2560949482291200}d^{5}-\frac{155335836443886932189226007 7}{17784371404800}d^{4}\\
&-\frac{198129972820025979593798 3}{242514155520}d^{3}+\frac{157438783435621606 67}{7623616}d^{2}\\
&-\frac{6555215918550187 25}{7351344}d+4625512425.
\end{aligned}
$$

Note that the degree in (28) is $54=(2+1)\times 18$, cf. (2).

3.3.3 Hypersurfaces Singular along a Segre 3-fold in $\mathbb{P}^5$

The Segre variety $\mathbb{S}:=\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5$ has Hilbert polynomial $(1/2)t^3+2t^2+(5/2)t+1$. It moves in a family $\mathbb{W}_{seg}$ of dimension 24. It is well known (cf. Harris [16, p. 99]) that the homogeneous ideal is spanned by a net of quadrics of determinantal type. Identifying $\mathbb{P}^5=\mathbb{P}(\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^3))$, $\mathbb{S}$ corresponds to the locus of rank one matrices up to scalar. As in the previous 2 cases, the family formed by the subschemes of $\mathbb{P}^5$ defined by $\mathcal{I}_W^2$ for some $W\in\mathbb{W}_{seg}$ lacks flatness precisely along the nets coming from the Veronese-like embedding $\mathbb{G}(2,\mathcal{F}_1)\cong\mathbb{G}_{\mathbf{o}}\subset\mathbb{G}(3,\mathcal{F}_2),\langle L_0,L_1\rangle\mapsto$

$\langle L_0^2, L_0L_1, L_1^2\rangle$. Write $\widehat{\mathbb{X}}$ for the blowing up of $\mathbb{X}$ (= nets of quadrics of determinantal type) along $\mathbb{G}_{\mathbf{o}}$. It embeds in $\mathbb{X} \times \mathbb{G}(6, \mathcal{F}_4)$ and the exceptional divisor $\widehat{\mathbb{E}}$ affords the same description as in Proposition 14**(iii)**. Scripts are available in [28, Appendix H].

Although we have all the information needed to calculate $\deg \Sigma(\mathbb{W}_{seg}, d)$ via Bott's residues formula, computations become prohibitive beyond $d = 28$, last entry in Table 1. So we were not able to perform interpolation, which would require pushing d up to $(3+1) \times 24$ (conjecturally).

d	degree
4	4985292672535
5	38085453623924002125608
6	75285508677103874434199729447346
7	6919928722801305898152558631141006297978
8	4218195443246668648480236632794603635056367373
9	30538531184782134440883223805188165885850765266730973
10	422434095172656585934258782287990966927007220991809111509
11	158437528281133532734337703310993668084277908103801228619349318
12	208003535305995749964153455992416379146245711635831375143591990 7641
13	11549735996636189943619254985547139290129087463355134074887299468381440
14	3129677022760327047365764485946385978830331925722648969765576693528286 1144
15	4621825113885445589602828803080710783620639726202691905802598900486034586 5068
16	4057317802501705324816345579199525313833324883021974968190152468051492069464 7875
17	2269640346038978228291812022009661269306699090248673546303769574845835501210206 5130
18	8560094850432050145388608162764331545974912158826771912534187363304630242378140 685505
19	2280218446179281906894436399299532691147069188695294809825377606946754403932028306 244123
20	4459131223707827852686255332456492502745327413011186069785255174835826716801543373 45798650
21	6613604483089078555276316651308847567556264721723232296060553315394391918129952874 3949231995
22	7648060182749239379957328222725038044389468341441118678038359708033154622298562461 760431031987
23	7061221238074707907837554402772427735105063360352750265318179595405111259947381992 69690002855831
24	5312604939340426644092894683412771448675554774725996107833251481810418578806617512 9628674092418346
25	3315561352388199144671538442416320830215174679718026171794913021436371907104234446 224006732128647329
26	1743348574713956673477317282393221129642312106032823567015915985140842044082911710 80634141000772703155
27	7829482987143513944990986949407455476367377747552625701320278751035638067417139596 3140409480403448 57400
28	3039913648205425110026984143365532813960751207492523362139713198718711642625487792 8115364707290713667 1375

Table 1: $\deg \Sigma(\mathbb{W}_{seg}, d)$

3.4 Surfaces singular along elliptic quartic curves

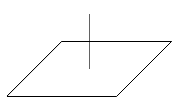
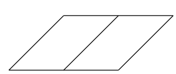
An elliptic quartic curve in $\mathbb{P}^3$ is the complete intersection of a (unique) pencil of quadric surfaces. Avritzer & Vainsencher [34], [3] obtained an explicit description of the component $\mathbb{W}_{eqc}$ of elliptic quartics of the Hilbert scheme $\mathrm{Hilb}_{4t}(\mathbb{P}^3)$. This has been used in [9] for enumerating curves in certain Calabi-Yau 3-folds, and in [6] for studying Noether-Lefschetz loci of systems of surfaces in $\mathbb{P}^3$. G. Gotzmann [13] has shown that $\mathrm{Hilb}_{4t}(\mathbb{P}^3)$ consists of two irreducible components; the second one parameterizes unions of a plane quartic curve and a zero dimensional subcheme of $\mathbb{P}^3$ of length 2.

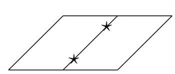
Put $\mathbb{X} = \mathbb{G}(2, \mathcal{F}_2)$, the grassmannian of pencils of quadrics in $\mathbb{P}^3$. We summarize in the diagram below the construction of $\mathbb{W}_{eqc}$.

$$\begin{tikzcd}
\mathbb{G}(19,\mathcal{F}_4) \supset \mathbb{W}_{eqc} & = & \widehat{\mathbb{X}} \arrow[d] & & & & \widehat{\mathbb{E}} \arrow[llll, hook'] \arrow[d] \\
\mathbb{G}(8,\mathcal{F}_3)\times\mathbb{X} & \supset & \widetilde{\mathbb{X}} \arrow[d] & \supset & \widetilde{\mathbb{E}} \arrow[d] & \supset & \widetilde{\mathbb{Y}} \arrow[d] \\
\mathbb{G}(2,\mathcal{F}_2) & = & \mathbb{X} & \supset & Z & \supset & \mathbb{Y}
\end{tikzcd} \tag{29}$$

where

$$\begin{cases}
Z \cong \check{\mathbb{P}}^3 \times \mathbb{G}(2, \mathcal{F}_1) \text{ consists of pencils with a fixed plane;} \\
\mathbb{Y} \cong \{(p, l) \in \check{\mathbb{P}}^3 \times \mathbb{G}(2, \mathcal{F}_1 | p \supset l\} = \text{ closed orbit of } Z; \\
\widetilde{\mathbb{Y}} \to \mathbb{Y} = \mathbb{P}^2\text{-bundle of divisors of degree 2 over a variable line } l \subset p; \\
\widetilde{\mathbb{X}} = \text{ blowup of } \mathbb{X} \text{ along } Z; \\
\widehat{\mathbb{X}} = \text{ blowup of } \widetilde{\mathbb{X}} \text{ along } \widetilde{\mathbb{Y}}.
\end{cases}$$

$Z = \Bigg\{$  $\Bigg\} \supset \mathbb{Y} = \Bigg\{$  $\Bigg\} \longleftarrow \widetilde{\mathbb{Y}} =$

$\Bigg\{$  $\Bigg\}$

Let

$$\mathcal{A} \subset \mathcal{F}_2 \times \mathbb{X} \tag{30}$$

be the tautological subbundle of rank 2 on our grassmannian of pencils of quadrics. There is a natural map of vector bundles on $\mathbb{X}$ induced by multiplication,

$$\mu_3 : \mathcal{A} \otimes \mathcal{F}_1 \longrightarrow \mathcal{F}_3 \times \mathbb{X},$$

with generic rank 8. The rank drops precisely over Z. Hence we have an induced rational map $\kappa : \mathbb{X} \dashrightarrow \mathbb{G}(8, \mathcal{F}_3)$. Blowing up $\mathbb{X}$ along Z, we find the closure $\widetilde{\mathbb{X}} \subset \mathbb{G}(8, \mathcal{F}_3) \times \mathbb{X}$ of the graph of κ. The fiber

$$\widetilde{\mathbb{E}}_{(p,l)} = \mathbb{P}\left(\mathcal{F}_3^l / p\mathcal{F}_2^l\right)$$

where $\mathcal{F}_d^l$ denotes the space of forms of degree d vanishing on the line l. The fiber of $\widetilde{\mathbb{E}}$ over $\mathbf{y} := (x_0, \langle x_0, x_1\rangle) \in \mathbb{Y}$ contains the disjoint subspaces

$$\mathbb{M}_{\mathbf{y}} := \mathbb{P}\big(x_1^2 \left(\mathcal{F}_1/\langle x_0\rangle\right)\big) \quad \text{and} \quad \mathbb{P}\Big(\left(\mathcal{F}_2/\mathcal{F}_2^l\right)\Big) = \widetilde{\mathbb{Y}}_{\mathbf{y}}.$$

The latter embeds into $\widetilde{\mathbb{E}}_{(p,l)}$ via multiplication by $p := x_0$ and coincides with the fiber of $\widetilde{\mathbb{Y}}$. The former is the fiber of a $\mathbb{P}^2$-bundle

$$\mathbb{M} \longrightarrow \mathbb{Y}$$

to be further described in a moment.

Now, over $\widetilde{\mathbb{X}}$ we have a subbundle of cubic forms,

$$\mathcal{B} \subset \mathcal{F}_3 \times \widetilde{\mathbb{X}} \tag{31}$$

of rank 8 obtained by pullback from the tautological subbundle over $\mathbb{G}(8, \mathcal{F}_3)$. Thus we get a map of multiplication

$$\mu_4 : \mathcal{B} \otimes \mathcal{F}_1 \to \mathcal{F}_4 \times \widetilde{\mathbb{X}}$$

with generic rank 19. The scheme of zeros of $\bigwedge^{19} \mu_4$ is equal to $\widetilde{\mathbb{Y}}$ (29). In fact, it can be verified that each fiber of $\mathcal{B}$ is a linear system of cubics such that

- either it has a base locus equal to a curve with "correct" Hilbert polynomial $P_{\mathbb{W}_{eqc}}(t) = 4t$
- or it is of the form $p \cdot \mathcal{F}_2^{**}$, meaning a linear system with fixed component a plane p, and $\mathcal{F}_2^{**}$ denoting an 8-dimensional space of quadrics cutting a subscheme of p of dimension 0 and degree 2.

The exceptional divisor $\widehat{\mathbb{E}}$ is the $\mathbb{P}^8$-bundle over $\widetilde{\mathbb{Y}}$ with fiber

$$\widehat{\mathbb{E}}_{((p,l),y_1+y_2)} = \text{system of quartic curves in the plane } p \text{ which are singular at the "doublet" } y_1+y_2.$$

Precisely, assuming the plane $p := x_0$ and the line $l := \langle x_0, x_1\rangle$, a typical doublet has homogeneous ideal of the form $\langle x_0, x_1, f(x_2, x_3)\rangle$, for some binary form f, $\deg f = 2$. Our system of plane quartics lies in the ideal $\langle x_1, f\rangle^2 = \langle x_1^2, x_1 f, f^2\rangle$. Given a non-zero quartic g in this ideal, we may form the ideal $J = \langle x_0^2, x_0x_1, x_0 f, g\rangle$, (*e.g*, $\langle x_0^2, x_0x_1, x_0x_2^2, x_2^4\rangle$). It can be checked that J contains precisely 19 independent quartics and the Hilbert polynomial is correct. Moreover, the subscheme defined by J^2 has the expected Hilbert polynomial $12t-16$. The preceding description suffices to get a hold on the fixed points on $\widehat{\mathbb{X}}$ (29) together with their tangent spaces as explained in Araújo [2] (after [9], [24]). However, as in the case of nets of quadrics, once we pass to the thickenings, one last blowup is required. The new center $\mathbb{M} \subset \widetilde{\mathbb{E}}$ is supported in the locus of $W \in \mathbb{W}_{eqc}$ where the subscheme of $\mathbb{P}^3$ defined by $(\mathcal{I}_W)^2$ has "wrong" Hilbert polynomial: flatness fails. In fact, points corresponding to an ideal like

$$\langle x_0^2, x_0x_1, C\rangle \in \widetilde{\mathbb{E}}_{(x_0,\langle x_0,x_1\rangle)}, \tag{32}$$

where C denotes a cubic form arising from $x_1^2 \cdot (\mathcal{F}_1/\langle x_0\rangle)$, are legitimate members of $\mathbb{W}_{eqc}$, whereas its square has a "bad" Hilbert polynomial (namely $13t-20$). Notation as in (30),(31), let $\nu : \mathcal{A}\otimes\mathcal{B} \to \mathcal{F}_5$ be map of vector bundles over $\widetilde{\mathbb{X}}$ defined by multiplication. The generic rank of ν is 12. Set

$$\mathbb{M} = \text{ scheme of zeros of } \bigwedge^{12} \nu.$$

In a way similar to Prop. 14, local calculations (cf. [28, Appendix I.1, p.177]) show that $\mathbb{M}$ is the indeterminacy locus of the natural rational map

$$\widetilde{\mathbb{X}} \dashrightarrow \mathbb{G}(12, \mathcal{F}_5) \tag{33}$$

induced by ν. One checks that $\mathbb{M}$ is the $\mathbb{P}^2$-bundle over $\mathbb{Y}$ which parameterizes the triples $\langle p, l, C\rangle$, where p denotes a plane, $l = \langle p, p'\rangle$ a line therein and C a class in $\mathbb{P}\left((p')^2 \cdot \mathcal{F}_1/\langle p\rangle\right)$. We have the embedding $\mathbb{M} \subset \widetilde{\mathbb{E}}$ of bundles over $\mathbb{Y}$ such that in the fiber over any $(p,l) \in Y$ the point $\langle p,l,C\rangle$ with $C = (p')^2 p''$ mod $\langle p\rangle$ is mapped to the class $\overline{(p')^2 \cdot p''} \in \mathbb{P}\left(\mathcal{F}_3^l/p\mathcal{F}_2^l\right) = \widetilde{\mathbb{E}}_{(p,l)}$. Consider the blow up diagram of $\widetilde{\mathbb{X}}$ along $\mathbb{M}$

$$\begin{tikzcd} \widetilde{\mathbb{X}}' \arrow[d] & \supset & \widetilde{\mathbb{M}} \arrow[d] \\ \widetilde{\mathbb{X}} & \supset & \mathbb{M} \end{tikzcd} \tag{34}$$

By construction $\widetilde{\mathbb{X}}'$ embeds in $\widetilde{\mathbb{X}} \times \mathbb{G}(12, \mathcal{F}_5)$ as the closure of the graph of the rational map (33). Since $\mathbb{M}$ is disjoint from the blowup center $\widetilde{\mathbb{Y}}$ (cf. diagram 29), it follows that $\widetilde{\mathbb{Y}}$ lifts isomorphically to $\widetilde{\mathbb{Y}}' \subset \widetilde{\mathbb{X}}'$ so that the blowup of $\widetilde{\mathbb{X}}$ along $\widetilde{\mathbb{Y}}$ is naturally isomorphic to the blowup of $\widetilde{\mathbb{X}}'$ along $\widetilde{\mathbb{Y}}'$ over a neighborhood of $\widetilde{\mathbb{Y}}$. In special, only the fixed points of $\widehat{\mathbb{X}}$ over $\mathbb{M}$ are replaced by those in $\widetilde{\mathbb{M}}$. It turns out that a point like (32) is replaced by 9 fixed points in $\widetilde{\mathbb{M}}$ corresponding to ideals of the form

$$\langle x_0^2, x_0x_1, C\rangle^2 + \langle m\rangle$$

$$m \in \{x_0x_2C, x_0x_3C, \frac{C^2}{x_1}, x_0^2x_1x_2^2, x_0^2x_1x_2x_3, x_0^2x_1x_3^2, x_0^2x_0x_2^2, x_0^2x_0x_2x_3, x_0^2x_0x_3^2\}.$$

The technicalities of the final computation can be found in [28, Appendix I.1, p.177]. The polynomial that gives us the degree of $\Sigma(\mathbb{W}_{eqc}, d)$ is displayed below. Note once again that the degree is equal to $(1+1) \times \dim(\mathbb{W}_{eqc}) = 2 \times 16$, cp. (2).

$$
\begin{aligned}
\deg \Sigma_{\mathbb{W}_{eqc},d} = {} & \tfrac{77991978249}{47023181004800}d^{32} - \tfrac{142130943}{922746880}d^{31} + \tfrac{8109239447979}{1175579525120}d^{30} \\
& - \tfrac{4150267051797}{20992491520}d^{29} + \tfrac{47676232841150619}{11755795251200}d^{28} - \tfrac{6615027446596551}{104962457600}d^{27} \\
& + \tfrac{128385059997089001}{167939932160}d^{26} - \tfrac{103459871906659801}{14129561600}d^{25} + \tfrac{893796960041917863271}{16277254963200}d^{24} \\
& - \tfrac{312845973151702414313}{1017328435200}d^{23} + \tfrac{4312587609200253695639}{4069313740800}d^{22} \\
& + \tfrac{6155781582234103357}{7266631680}d^{21} - \tfrac{1105621403101024328482787}{24415882444800}d^{20} \\
& + \tfrac{21346179040504773262903 37}{5410337587200}d^{19} - \tfrac{10277042907520489515373377 71}{476109707673600}d^{18} \\
& + \tfrac{1568309607110425883232529237}{223176425472000}d^{17} + \tfrac{3993143356810976602006158931 91}{57133164920832000}d^{16} \\
& - \tfrac{127911974311612787565094357769}{396758089728000}d^{15} + \tfrac{7297607552669425891347140320 19}{238054853836800}d^{14} \\
& - \tfrac{182853224866832645145663999672 49}{892705701888000}d^{13} + \tfrac{15050777906503580350914982390277}{137339338752000}d^{12} \\
& - \tfrac{8362721204990643447960751421719}{17167417344000}d^{11} + \tfrac{17856528343997993007848487280 9}{98099527680}d^{10} \\
& - \tfrac{27317871287377170497361801712 43}{476872704000}d^{9} + \tfrac{11255984459447746542885158016918 61}{74392141824000}d^{8} \\
& - \tfrac{58025484355390407710374488759691}{1743565824000}d^{7} + \tfrac{16796039461040747482814365174429}{278970531840}d^{6} \\
& - \tfrac{852135024407378395199004032465 3}{96864768000}d^{5} + \tfrac{599422208545470260381592707347}{5930496000}d^{4} \\
& - \tfrac{7963270326807152872255773702 19}{9081072000}d^{3} + \tfrac{434272227079029305979707333}{8072064}d^{2} \\
& - \tfrac{1490642041280752415948 9839}{720720}d + 37131247788800 30320.
\end{aligned} \tag{35}
$$

Acknowledgement Thanks are due to Angelo F. Lopez for clarifying the argument on generic injectiveness (cf. Lemma9).

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