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An optimal algorithm to totally color some powers of cycle graphs

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Abstract

The *total chromatic number* of a graph G, denoted by $\chi_T(G)$, is the minimum number of colors needed to totally color G. A well-known bound is $\chi_T(G) \geqslant \Delta(G)+1$, where $\Delta(G)$ represents the maximum degree of a vertex in G. The *total coloring conjecture* (TCC) was proposed independently by Behzad and Vizing and states that, for every simple graph G, $\chi_T(G) \leqslant \Delta(G)+2$. This conjecture remains open for chordal and powers of cycle graphs. If $\chi_T(G) = \Delta(G)+1$, then G is said to be Type 1. If $\chi_T(G) = \Delta(G)+2$, then G is said to be Type 2. The power of the cycle graph C_n^k has C_n as spanning subgraph and additional edges between vertices at distance at most k in C_n. Campos and de Mello (A result on the total colouring of powers of cycles, Discrete Appl. Math. (2007), 55, 585–597) proved the TCC C_n^k, when $k=3$ or when n is even. In the same work, Campos and de Mello proposed a conjecture: C_n^k is Type 2 if n is odd and $k > n/3-1$ and is Type 1 otherwise.

In the present work, we prove that the conjecture proposed by Campos and de Mello holds for a graph C_n^k if $k=3$ or $k=4$,

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by giving a collage technique using the minimum number of colors needed, which yields an optimal algorithm.

1 Introduction

The well known *total coloring conjecture* (TCC) was proposed independently by Behzad and Vizing and states that for every simple graph G, all its elements (vertices and edges) can be colored with no more than $\Delta(G) + 2$ colors, with $\Delta(G)$ being the maximum degree of a vertex in $V(G)$, in such way that adjacent elements receive distinct colors. The TCC holds for many important graph families, such as the r-partite complete graphs [5], dually chordal graphs [6], and graphs with a high maximum degree [9]. However, the TCC remains open for chordal graphs and for power of cycle graphs.

It is easy to see that no graph can be totally colored with less than $\Delta(G)+1$ colors. If a graph G can be totally colored with $\Delta(G)+1$ colors it is called Type 1 . If it can not be colored with $\Delta(G)+1$ colors but can be colored with $\Delta(G)+2$ colors it is said to be Type 2 . The minimum number of colors needed to totally color all the elements of a graph G is called *total chromatic number*. McDiarmid and Sánchez-Arroyo proved in [8] that even for r-regular bipartite graphs, with fixed r, the problem of finding the total chromatic number is $\mathcal{NP}$-hard. Another interesting dichotomy related with this problem is the fact that there are some classes of graphs, like bipartite graphs, for which the TCC holds, but the problem of finding the total chromatic number is $\mathcal{NP}$-hard. EVven if we have a polynomial-time algorithm to determine the total chromatic number of the graphs of a class, this does not mean that the TCC is proven for this class.

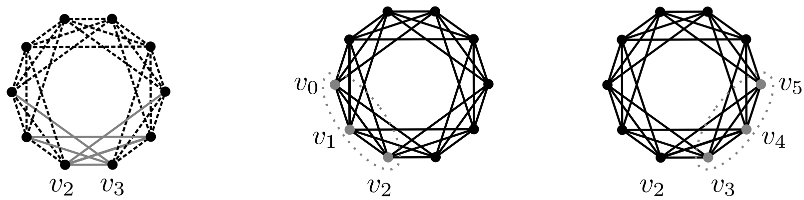
A k-th power of a graph G is a graph G' where $V(G) = V(G')$ and $E(G) = E(G) \cup \{uv \mid u, v \in V(G') \text{ and } dist(u, v) \leqslant k\}$, with $dist(u, v)$ being the distance between u and v. A power of cycle graph, denoted by C_n^k, is a graph where $V(C_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$, note that this is the order of a spanning cycle, and $E(C_n^k) = E^1 \cup \cdots \cup E^k$, where $E^i = \{v_j v_{(j+i)} \mid 0 \leqslant j \leqslant n-1\}$. In this work, when we refer to a vertex $v_i \in V(C_n^k)$ we will mean $v_{i \bmod n}$. If $k \geqslant \lfloor \frac{n}{2} \rfloor$, then C_n^k is isomorphic to K_n, the complete graph with n vertices, whose total chromatic number is established: Type 1 if n is odd and Type 2 if n is even [1]. The TCC was proved for C_n^k when n is even [3]. If $k = 1$, then the graph C_n^k is isomorphic to C_n, whose total chromatic is also known: Type 1 if n is multiple of 3 and Type 2 otherwise [9]. For a fixed value of k, the TCC was proved for C_n^3 and C_n^4, and for C_n^2, the total chromatic number is established: C_7^2 is Type 2 if $n \neq 7$ and C_n^2 is Type 1 otherwise. If n is multiple of $2k+1$ then C_n^k is Type 1 [2].

An important result about the total chromatic number problem is the result of Chetwynd and Hilton [4] that shows a sufficient condition for a graph not to be Type 1, in what is known as the *conformable* argument. Campos [2] adapted this result for the power of cycle graphs, showing that a non-complete graph C_n^k, with $k \leqslant n/3 - 1$ and n odd, is not conformable, hence it is not a Type 1 graph.

We call a total coloring a function $\pi_{C_n^k} : E(C_n^k) \cup V(C_n^k) \to [t]$, where $[t] = \{0, \ldots, t\}$, and adjacent elements must receive distinct colors. A color of a vertex v_i will be denoted by $\pi_{C_n^k}(v_i)$, similarly a color of an edge $v_i v_j$ will be denoted by $\pi_{C_n^k}(v_i v_j)$. We will denote by $\chi_T(G)$ the total chromatic number of G.

2 Preliminaries

In this section we will define some terms that will be used in Section 3. We call a semi-cut of edges of a graph C_n^k a set $S'_c(v_i) = \{v_j v_l \mid i-k < j \leqslant i,\ i < l \leqslant i+k \text{ and } l-j \leqslant k\}$. We call a semi-cut of vertices of a graph C_n^k a set of k consecutive vertices. We give name to two special semi-cuts of vertices: $S_c^-(v_i) = \{v_j \mid i-k < j \leqslant i\}$ and $S_c^+(v_i) = \{v_j \mid i < j \leqslant i+k\}$. Figure 1 shows a graph C_{10}^3 and the sets $S'_c(v_2)$, and $S_c^-(v_2), S_c^+(v_2)$. Each set has a shade highlighted in one table. The tables show total colorings,



	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	6	0	1	5	-	-	-	3	2	4
v_1	0	1	4	2	6	-	-	-	5	3
v_2	1	4	2	6	3	0	-	-	-	5
v_3	5	2	6	4	1	3	0	-	-	-
v_4	-	6	3	1	0	2	4	5	-	-
v_5	-	-	0	3	2	6	5	4	1	-
v_6	-	-	-	0	4	5	1	6	3	2
v_7	3	-	-	-	5	4	6	2	0	1
v_8	2	5	-	-	-	1	3	0	4	6
v_9	4	3	5	-	-	-	2	1	6	0

(a)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	6	0	1	5	-	-	-	3	2	4
v_1	0	1	4	2	6	-	-	-	5	3
v_2	1	4	2	6	3	0	-	-	-	5
v_3	5	2	6	4	1	3	0	-	-	-
v_4	-	6	3	1	0	2	4	5	-	-
v_5	-	-	0	3	2	6	5	4	1	-
v_6	-	-	-	0	4	5	1	6	3	2
v_7	3	-	-	-	5	4	6	2	0	1
v_8	2	5	-	-	-	1	3	0	4	6
v_9	4	3	5	-	-	-	2	1	6	0

(b)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	6	0	1	5	-	-	-	3	2	4
v_1	0	1	4	2	6	-	-	-	5	3
v_2	1	4	2	6	3	0	-	-	-	5
v_3	5	2	6	4	1	3	0	-	-	-
v_4	-	6	3	1	0	2	4	5	-	-
v_5	-	-	0	3	2	6	5	4	1	-
v_6	-	-	-	0	4	5	1	6	3	2
v_7	3	-	-	-	5	4	6	2	0	1
v_8	2	5	-	-	-	1	3	0	4	6
v_9	4	3	5	-	-	-	2	1	6	0

(c)

Figure 1: (a) The edges of the set $S'_c(v_2)$ are represented by the solid gray edges. (b) The vertices of the set $S^-_c(v_2)$ are the vertices v_0, v_1, v_2. (c) The vertices of the set $S^+_c(v_2)$ are the vertices v_3, v_4, v_5.

in such way that the color of an edge v_iv_j is given in the cell i, j of the matrix. The color of the vertex v_i is represented by the cell i, i.

Definition 2.1 (Compatibility)**.** We say that two colorings $\pi_{C^k_{n_1}}$ and $\pi_{C^k_{n_2}}$ are *compatible* if there exists a vertex $v_i \in V(C^k_{n_1})$ and a vertex $u_j \in V(C^k_{n_2})$ such that: (a) $S^-_c(v_i)$ is compatible with $S^+_c(u_j)$, meaning $\pi_{C^k_{n_1}}(v_{i-r}) \neq \pi_{C^k_{n_2}}(u_{j+s})$; (b) $S^+_c(v_i)$ is compatible with $S^-_c(u_j)$, meaning $\pi_{C^k_{n_1}}(v_{i+s}) \neq \pi_{C^k_{n_2}}(u_{j-r})$, for every $r \in \{0, \ldots, k-1\}$ and every $s \in \{1, \ldots, k-r\}$. (c) $S'_c(v_i)$ is compatible with $S'_c(u_j)$, meaning $\pi_{C^k_{n_1}}(v_{i-r}v_{i+s}) = \pi_{C^k_{n_2}}(u_{j-r}u_{j+s})$, for every $r \in \{0, \ldots, k-1\}$ and every $s \in \{1, \ldots, k\}$. In this case, we say that this total colorings are compatible through the *pivot* vertices v_i and u_j.

3 Results

The next theorem allows us to form a total coloring of a power of cycle graph by *collating* two total colorings of power of cycle graphs. Furthermore, this new total coloring will be compatible with all other total colorings that are compatible with the two *seed* total colorings through the same pivot vertex. We call *seed* total coloring a total coloring that are used to generate another total coloring.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6
v_0	0	4	1	5	2	6	3
v_1	4	1	5	2	6	3	0
v_2	1	5	2	6	3	0	4
v_3	5	2	6	3	0	4	1
v_4	2	6	3	0	4	1	5
v_5	6	3	0	4	1	5	2
v_6	3	0	4	1	5	2	6

(a)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_0	0	3	4	5	-	-	-	-	-	1	2	6
v_1	3	1	5	2	6	-	-	-	-	-	0	4
v_2	4	5	2	6	3	0	-	-	-	-	-	1
v_3	5	2	6	3	0	1	4	-	-	-	-	-
v_4	-	6	3	0	4	2	1	5	-	-	-	-
v_5	-	-	0	1	2	5	3	4	6	-	-	-
v_6	-	-	-	4	1	3	0	2	5	6	-	-
v_7	-	-	-	-	5	4	2	1	3	0	6	-
v_8	-	-	-	-	-	6	5	3	2	4	1	0
v_9	1	-	-	-	-	-	6	0	4	3	5	2
v_{10}	2	0	-	-	-	-	-	6	1	5	4	3
v_{11}	6	4	1	-	-	-	-	-	0	2	3	5

(b)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_0	0	3	1	5	-	2	6	4
v_1	3	1	4	2	6	-	5	0
v_2	1	4	2	6	3	0	-	5
v_3	5	2	6	3	1	4	0	-
v_4	-	6	3	1	0	5	4	2
v_5	2	-	0	4	5	1	3	6
v_6	6	5	-	0	4	3	2	1
v_7	4	0	5	-	2	6	1	3

(c)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	6	0	1	5	-	-	-	3	2	4
v_1	0	1	4	2	6	-	-	-	5	3
v_2	1	4	2	6	3	0	-	-	-	5
v_3	5	2	6	4	1	3	0	-	-	-
v_4	-	6	3	1	0	2	4	5	-	-
v_5	-	-	0	3	2	6	5	4	1	-
v_6	-	-	-	0	4	5	1	6	3	2
v_7	3	-	-	-	5	4	6	2	0	1
v_8	2	5	-	-	-	1	3	0	4	6
v_9	4	3	5	-	-	-	2	1	6	0

(d)

Table 1: Seed colorings which can generate colorings to any graph C_n^3, with $n \geqslant 14$. Total colorings using $\Delta + 1 = 7$ colors of the graphs: (a) C_7^3, (c) C_8^3, (d) C_{10}^3, and (b) C_{12}^3.

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_0	0	5	1	6	2	7	3	8	4
v_1	5	1	6	2	7	3	8	4	0
v_2	1	6	2	7	3	8	4	0	5
v_3	6	2	7	3	8	4	0	5	1
v_4	2	7	3	8	4	0	5	1	6
v_5	7	3	8	4	0	5	1	6	2
v_6	3	8	4	0	5	1	6	2	7
v_7	8	4	0	5	1	6	2	7	3
v_8	4	0	5	1	6	2	7	3	8

(a)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
v_0	0	4	1	6	2	-	-	-	-	-	-	-	-	5	8	3	7
v_1	4	1	0	2	7	3	-	-	-	-	-	-	-	-	5	8	6
v_2	1	0	2	7	3	8	4	-	-	-	-	-	-	-	-	6	5
v_3	6	2	7	3	8	4	0	5	-	-	-	-	-	-	-	-	1
v_4	2	7	3	8	4	1	5	0	6	-	-	-	-	-	-	-	-
v_5	-	3	8	4	1	0	2	6	5	7	-	-	-	-	-	-	-
v_6	-	-	4	0	5	2	1	3	7	6	8	-	-	-	-	-	-
v_7	-	-	-	5	0	6	3	2	1	4	7	8	-	-	-	-	-
v_8	-	-	-	-	6	5	7	1	3	0	2	4	8	-	-	-	-
v_9	-	-	-	-	-	7	6	4	0	5	1	2	3	8	-	-	-
v_{10}	-	-	-	-	-	-	8	7	2	1	0	3	5	6	4	-	-
v_{11}	-	-	-	-	-	-	-	8	4	2	3	1	6	0	7	5	-
v_{12}	-	-	-	-	-	-	-	-	8	3	5	6	2	7	0	1	4
v_{13}	5	-	-	-	-	-	-	-	-	8	6	0	7	3	1	4	2
v_{14}	8	5	-	-	-	-	-	-	-	-	4	7	0	1	6	2	3
v_{15}	3	8	6	-	-	-	-	-	-	-	-	5	1	4	2	7	0
v_{16}	7	6	5	1	-	-	-	-	-	-	-	-	4	2	3	0	8

(b)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_0	0	4	1	6	2	-	5	8	7	3
v_1	4	1	0	2	7	3	-	6	5	8
v_2	1	0	2	7	3	8	4	-	6	5
v_3	6	2	7	3	8	4	0	5	-	1
v_4	2	7	3	8	4	5	6	0	1	-
v_5	-	3	8	4	5	0	7	1	2	6
v_6	5	-	4	0	6	7	1	3	8	2
v_7	8	6	-	5	0	1	3	2	4	7
v_8	7	5	6	-	1	2	8	4	3	0
v_9	3	8	5	1	-	6	2	7	0	4

(c)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
v_0	0	4	1	6	2	-	-	-	-	-	-	-	5	7	3	8
v_1	4	1	0	2	7	3	-	-	-	-	-	-	-	6	8	5
v_2	1	0	2	7	3	8	4	-	-	-	-	-	-	-	5	6
v_3	6	2	7	3	8	4	0	5	-	-	-	-	-	-	-	1
v_4	2	7	3	8	4	0	1	6	5	-	-	-	-	-	-	-
v_5	-	3	8	4	0	5	2	1	6	7	-	-	-	-	-	-
v_6	-	-	4	0	1	2	6	3	7	5	8	-	-	-	-	-
v_7	-	-	-	5	6	1	3	7	2	0	4	8	-	-	-	-
v_8	-	-	-	-	5	6	7	2	0	4	3	1	8	-	-	-
v_9	-	-	-	-	-	7	5	0	4	1	6	2	3	8	-	-
v_{10}	-	-	-	-	-	-	8	4	3	6	2	5	7	0	1	-
v_{11}	-	-	-	-	-	-	-	8	1	2	5	3	6	4	7	0
v_{12}	5	-	-	-	-	-	-	-	8	3	7	6	4	1	0	2
v_{13}	7	6	-	-	-	-	-	-	-	8	0	4	1	5	2	3
v_{14}	3	8	5	-	-	-	-	-	-	-	1	7	0	2	6	4
v_{15}	8	5	6	1	-	-	-	-	-	-	-	0	2	3	4	7

(d)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_0	8	0	1	6	2	-	-	-	7	4	3	5
v_1	0	1	5	2	7	3	-	-	-	6	8	4
v_2	1	5	2	7	3	8	4	-	-	-	0	6
v_3	6	2	7	3	8	4	0	5	-	-	-	1
v_4	2	7	3	8	5	1	6	0	4	-	-	-
v_5	-	3	8	4	1	0	2	6	5	7	-	-
v_6	-	-	4	0	6	2	8	3	1	5	7	-
v_7	-	-	-	5	0	6	3	1	8	2	4	7
v_8	7	-	-	-	4	5	1	8	2	0	6	3
v_9	4	6	-	-	-	7	5	2	0	3	1	8
v_{10}	3	8	0	-	-	-	7	4	6	1	5	2
v_{11}	5	4	6	1	-	-	-	7	3	8	2	0

(e)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
v_0	0	4	1	6	2	-	-	-	-	-	7	5	8	3
v_1	4	1	5	2	7	3	-	-	-	-	-	6	0	8
v_2	1	5	2	7	3	8	4	-	-	-	-	-	6	0
v_3	6	2	7	3	8	4	0	5	-	-	-	-	-	1
v_4	2	7	3	8	4	0	1	6	5	-	-	-	-	-
v_5	-	3	8	4	0	5	2	1	6	7	-	-	-	-
v_6	-	-	4	0	1	2	6	3	7	5	8	-	-	-
v_7	-	-	-	5	6	1	3	0	2	8	4	7	-	-
v_8	-	-	-	-	5	6	7	2	1	3	0	8	4	-
v_9	-	-	-	-	-	7	5	8	3	2	6	0	1	4
v_{10}	7	-	-	-	-	-	8	4	0	6	3	1	2	5
v_{11}	5	6	-	-	-	-	-	7	8	0	1	4	3	2
v_{12}	8	0	6	-	-	-	-	-	4	1	2	3	5	7
v_{13}	3	8	0	1	-	-	-	-	-	4	5	2	7	6

(f)

Table 2: Seed colorings which can generate colorings to any graph C_n^4, with $n \geqslant 16$. Total colorings using $\Delta + 1 = 9$ colors of the graphs: (a) C_9^4, (c) C_{10}^4, (e) C_{12}^4, (f) C_{14}^4, and (d) C_{16}^4. Total Coloring of the graph (b) C_{17}^4 with $\Delta + 1 = 9$ colors.

Theorem 3.1. If two colorings $\pi_{C^k_{n_1}}$ and $\pi_{C^k_{n_2}}$, with $t_1, t_2 \leqslant \gamma$, are compatible, then we can build a coloring $\pi_{C^k_n}$, such that $n = n_1 + n_2$ and $t \leqslant \gamma$. Moreover, C^k_n is compatible with any coloring $\pi_{C^k_{n_w}}$ which is compatible with $\pi_{C^k_{n_1}}$ and $\pi_{C^k_{n_2}}$ through the pivot vertices $w_l \in V(C^k_{n_w})$ and the vertices v_i, u_j, respectively.

Proof. Since $\pi_{C^k_{n_1}}$ and $\pi_{C^k_{n_2}}$ are compatible, there is a vertex $v_i \in V(C^k_{n_1})$ and a vertex $u_j \in V(C^k_{n_2})$ that fulfill the restrictions of Definition 2.1. Let G_1 be a graph where $V(G_1) = V(C^k_{n_1})$ and $E(G_1) = E(C^k_{n_1}) \backslash S'(v_i)$, where $V(G_1) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-n_1}\}$ and G_2 a graph where $V(G_2) = V(C^k_{n_2})$ and $E(G_2) = E(C^k_{n_2}) \setminus S'(u_j)$, where $V(G_2) = \{u_j, v_{j+1}, \ldots, v_{j+n_2-1}\}$. Note that the graphs G_1 and G_2 are subgraphs of $C^k_{n_1}$ and $C^k_{n_2}$, respectively.

To form the coloring $\pi_{C^k_n}$ we *collate* the colorings of the graphs G_1 and G_2, to transfer the colorings of such graphs to our graph C^k_n. Figure 2a shows an example of the *collage* of the two graphs G_1 and G_2 without the final edges.

The edges that are out of G_1 and G_2 will be colored with the same color of the set $S'_c(v_i)$ or $S'_c(u_j)$.

Note that all the elements of the C^k_n receive a color. Suppose by contradiction that there is a conflict in the coloring $\pi_{C^k_n}$, but, by Definition 2.1, if there is a conflict in $\pi_{C^k_n}$ there exists a conflict in one of the colorings $\pi_{C^k_{n_1}}$ and $\pi_{C^k_{n_2}}$, which yields a contradiction.

By the construction, in $\pi_{C^k_n}$ the colors of the set $S'_c(v_i)$ and $S^-_c(v_i)$ remain the same and the colors of the set $S^+_c(v_i)$ will be exactly the same of the set $S^+_c(u_j)$. It is easy to see that the graph $\pi_{C^k_{n_w}}$ is compatible with $\pi_{C^k_n}$ through the corresponding vertex $v_i \in V(C^k_n)$ and w_l. □

In the next theorem we use Theorem 3.1 to find the chromatic number of the graphs C^3_n and C^4_n.

Theorem 3.2. A non-complete graph C^k_n, with $k = 3$ or $k = 4$, is Type 2 if n is odd and $k > n/3 - 1$ and is Type 1, otherwise.

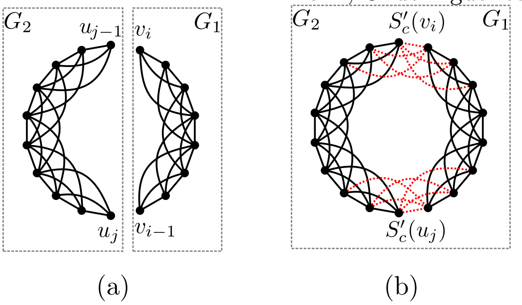


Figure 2: A graph C_{18}^3, its total coloring will be built through two colorings of the graphs C_{10}^3 and C_8^3. (a) The graphs G_1 and G_2 that will be used to form a total coloring of $\pi_{C_n^k}$. (b) The dashed edges represent the edges that are not in the graphs G_1 and G_2 but they are in C_n^k.

Proof. First, note that in her PhD thesis Campos [2] proved, in Theorem 2.29 and Theorem 2.30, that TCC works for the graphs C_n^3 and C_n^4. We can use the sets of seed colorings of Table 1 and Table 2 to form any coloring of a graph C_n^3, with $n \geqslant 14$ and for graphs C_n^4, with $n \geqslant 18$ if $k = 4$, by collate successively only graphs of the respective sets.

Note that every time that we make a collage of two seed colorings we generate another different coloring, which is compatible with all other seed colorings. The graphs C_9^3, C_{11}^3, C_{11}^4 and C_{13}^4 are known to be Type 2. So the only non-complete graphs that are not covered are the graphs C_{13}^3, C_{15}^4, and C_{17}^4, but the graphs C_{13}^3 and C_{15}^4 have Type 1 total colorings, which were shown by Campos in [2]. We show a total coloring of the graph C_{17}^4 in the Table 2b. □

4 Conclusion

This work is an alternative proof for a presentation made by Sheila de Almeida, JÃ´natas Belotti, Mayara Omai and Juliana Brim in the *VI Latin American Workshop on Cliques in Graphs*, 2014. Therefore, this work uses a more general technique that allows us to apply the same idea to other graphs, like the C_n^5. In the Table 3a we see the coloring of the graph C_{11}^5 and in the Table 3b C_{12}^5 we see the coloring of the graph C_{12}^5. As these colorings are compatible, we can generate a $(\Delta + 1)$-coloring for

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_0	0	6	1	7	2	8	3	9	4	10	5
v_1	6	1	7	2	8	3	9	4	10	5	0
v_2	1	7	2	8	3	9	4	10	5	0	6
v_3	7	2	8	3	9	4	10	5	0	6	1
v_4	2	8	3	9	4	10	5	0	6	1	7
v_5	8	3	9	4	10	5	0	6	1	7	2
v_6	3	9	4	10	5	0	6	1	7	2	8
v_7	9	4	10	5	0	6	1	7	2	8	3
v_8	4	10	5	0	6	1	7	2	8	3	9
v_9	10	5	0	6	1	7	2	8	3	9	4
v_{10}	5	0	6	1	7	2	8	3	9	4	10

(a)

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_0	0	2	1	6	3	8	-	9	5	10	7	4
v_1	2	1	0	7	8	3	9	-	4	5	10	6
v_2	1	0	2	8	7	9	4	10	-	6	5	3
v_3	6	7	8	3	2	4	10	5	0	-	9	1
v_4	3	8	7	2	4	10	5	0	6	1	-	9
v_5	8	3	9	4	10	5	1	2	7	0	6	-
v_6	-	9	4	10	5	1	0	6	3	7	2	8
v_7	9	-	10	5	0	2	6	1	8	4	3	7
v_8	5	4	-	0	6	7	3	8	2	9	1	10
v_9	10	5	6	-	1	0	7	4	9	3	8	2
v_{10}	7	10	5	9	-	6	2	3	1	8	4	0
v_{11}	4	6	3	1	9	-	8	7	10	2	0	5

(b)

Table 3: Seed colorings which can generate colorings to any graph C_n^5, with $n \geqslant 110$. Total colorings using $\Delta + 1 = 11$ colors of the graphs: (a) C_{11}^5; (b) C_{12}^5.

all but 20 graphs C_n^5. Moreover, all graphs C_n^5, with $n > 110$ have such coloring. Currently, we are working to extend those results for all graphs C_n^k.

Our work also represents a more general technique then the one used by Campos [2] to established the total coloring of the graphs C_n^2. Since Campos collage is used only for $k = 2$, uses a different construction of the seed colorings and different collage between two graphs as well. The other techniques in [2] are not closely related with the technique presented in this paper.

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