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# Proper gap-labellings of unicyclic graphs

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**Abstract**

Given a simple graph $G$, an ordered pair $(\pi, c_\pi)$ is said to be a gap-$[k]$-edge-labelling (resp. gap-$[k]$-vertex-labelling) of $G$ if $\pi$ is an edge-labelling (vertex-labelling) on the set $\{1, \ldots, k\}$, and $c_\pi$ is a proper vertex-colouring such that every vertex of degree at least two has its colour induced by the largest difference among the labels of its incident edges (neighbours), with isolated and degree-one vertices treated separately. These proper labellings were introduced by M. Tahraoui et al. in 2012 [6], and by A. Dehghan et al. in 2013 [3], respectively. In the latter, the authors investigate complexity aspects of decision problems associated with these labellings. In this work, we investigate both variants of this labelling for the family of unicyclic graphs.

## 1 Introduction

Let $G$ be a simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *elements* of $G$ are its vertices and its edges. An edge $e \in E(G)$ with ends $u, v \in V(G)$ is denoted by $uv$. The degree of a vertex

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$v \in V(G)$ is denoted by $d(v)$, and the minimum degree of $G$, by $\delta(G)$. The set of edges incident with $v$ is denoted by $E(v)$ and its neighbourhood, by $N(v)$.

For a set $\mathcal{C}$ of *colours*, a *(proper vertex-)colouring* of $G$ is a mapping $c : V(G) \to \mathcal{C}$, such that $c(u) \neq c(v)$ for every pair of adjacent vertices $u, v \in V(G)$. If $|\mathcal{C}| = k$, mapping $c$ is called a *k-colouring.* The *chromatic number* of $G$, denoted by $\chi(G)$, is the least number $k$ for which $G$ admits a $k$-colouring. For a set $S$ of elements of $G$ and a set of *labels* $[k] = \{1, \ldots, k\}$, a *labelling* $\pi$ of $G$ is a mapping $\pi : S \to [k]$. Also, given a set of elements $S' \subseteq S$, we denote by $\Pi_{S'}$ the set of labels assigned to $S'$ in $\pi$.

A *gap-*$[k]$*-edge-labelling* of a graph $G$ is an ordered pair $(\pi, c_\pi)$ such that $\pi : E(G) \to [k]$ is a labelling of $G$ and $c_\pi : V(G) \to \mathcal{C}$, a colouring of $G$ such that, for every $v \in V(G)$, its colour is defined as:

$$c_\pi(v) = \begin{cases} \max\limits_{e \in E(v)} \{\pi(e)\} - \min\limits_{e \in E(v)} \{\pi(e)\}, & \text{if } d(v) \geq 2; \\ \pi(e)_{e \in E(v)}, & \text{if } d(v) = 1; \\ 1, & \text{otherwise.} \end{cases}$$

We say that colour $c_\pi(v)$ of a vertex $v$ (with $d(v) \geq 2$) is *induced* by the largest *gap* between the labels in $\Pi_{E(v)}$. The least $k$ for which $G$ admits a gap-$[k]$-edge-labelling is called the *edge-gap number* of $G$ and is denoted by $\chi^{\mathrm{g}}_{E}(G)$.

Similarly, a *gap-*$[k]$*-vertex-labelling* of $G$ is also an ordered pair $(\pi, c_\pi)$, with $\pi : V(G) \to [k]$ and the colour of a vertex $v$ with $d(v) \geq 2$ is induced by the largest gap between the labels in $\Pi_{N(v)}$; degree-one vertices receive as induced colour the label assigned to its only neighbour and isolated vertices receive colour 1. The least $k$ for which $G$ admits a gap-$[k]$-vertex-labelling is the *vertex-gap number* of $G$, denoted by $\chi^{\mathrm{g}}_{V}(G)$. An interesting remark is that all graphs without connected components isomorphic to $K_2$ admit a gap-$[k]$-edge-labelling for some $k$, while there are graphs that, for any $k$, do not admit a gap-$[k]$-vertex-labelling; such is the case of complete graphs $K_n$, $n \geq 4$.

Gap-$[k]$-edge-labellings were introduced as a generalization of gap-$k$-colour-ings in 2012 by M. Tahraoui et al. [6]. This labelling was first studied by A. Dehghan et al. [3] in 2013. The authors proved that deciding whether a given graph $G$ admits a gap-$[k]$-edge-labelling, $k \geq 3$, is NP-complete. For the particular case of $k = 2$, they showed a dichotomy regarding bipartite graphs: it is NP-complete to decide whether a bipartite graph $G$ admits a gap-[2]-edge-labelling; however, if $G$ is bipartite and planar, with $\delta(G) \geq 2$, then the problem can be solved in polynomial time. Observe that this result indicates that the existence of degree-one vertices in a bipartite planar graph seems to contribute significantly to the hardness of the problem.

In 2015, R. Scheidweiler and E. Triesch [4, 5] continued investigating gap-$[k]$-edge-labellings, providing the first formal definition of the edge-gap number of graphs[1]. They established that for any graph $G$, $\chi(G) - 1 \leq \chi^{\mathrm{g}}_{E}(G) \leq \chi(G) + 5$. These bounds were further improved in 2016, when A. Brandt et al. [1] proved that $\chi^{\mathrm{g}}_{E}(G) \in \{\chi(G), \chi(G) + 1\}$ for all graphs except stars; they also determined the edge-gap number for complete graphs, cycles and trees.

The vertex variant of gap-labellings was introduced by A. Dehghan et al. [3] in 2013, who proved that deciding whether a graph admits a gap-$[k]$-vertex-labelling, $k \geq 3$, is NP-complete. Similar to their result for the edge variant, they investigated the particular case $k = 2$ and showed that, once again, a similar dichotomy appears in bipartite graphs: it is NP-complete to decide whether a bipartite graph admits a gap-[2]-vertex-labelling, but it is polynomial-time solvable if the graph is both planar and bipartite. They showed that this problem is also NP-complete when restricted to 3-colourable graphs and that the vertex-gap number of trees and $r$-regular bipartite graphs, $r \geq 4$, is 2. A. Dehghan [2] continued his pursuit into this family in 2016, proving that it is NP-complete to decide whether a bipartite graph $G$ admits a gap-[2]-vertex-labelling such that

[1] In their article, the authors refer to this parameter as the *gap-adjacent-chromatic number*.

the induced colouring is a 2-colouring of the graph. It is important to remark that in a gap-[2]-vertex-labelling of a graph $G$ with $\delta(G) \geq 2$, the induced colouring is a 2-colouring of the graph, with the fixed colour set $\{0, 1\}$, whereas if there exists (at least) one degree-one vertex, say $v$, then it is possible to induce colour $c_\pi(v) = 2$ by assigning label 2 to its neighbour. Once again, the existence of degree-one vertices (or lack thereof) seems to play an important role in determining the boundary of tractability in gap-[2]-vertex-labellings.

In order to fully understand the true nature of a problem's hardness, it is important to study the limits of polynomial-time solvability as well as NP-hardness over the instances of the problem. In this regard, here we extend the "positive" news to a family of bipartite planar graphs with $\delta(G) \geq 1$, namely even-length unicyclic graphs. We show that every such graph admits a gap-[2]-vertex-labelling, and we also provide a polynomial-time algorithm that decides when these graphs admit gap-[2]-edge-labellings. For completeness, we show results for the case of odd-length unicyclic graphs, establishing both the vertex-gap and edge-gap numbers for this case.

## 2 Results for unicyclic graphs

A *unicyclic graph* is a connected simple graph $G = (V, E)$ with $|V| = |E|$, as exemplified in Figure 1(a). We denote the vertices of the only cycle of $G$ by $V(C_p) = \{v_0, \ldots, v_{p-1}\}$. Also, we denote by $T_i$ the tree *rooted* at $v_i$, with $E(T_i) \cap E(C_p) = \emptyset$. A *leaf* of $T_i$ is a vertex $w \in V(T_i)$ such that $d(w) = 1$ and an *internal vertex* of $T_i$ is one that is neither the root nor a leaf of $T_i$. Finally, we define $L_i^j \subset V(T_i)$ as the set of vertices of $T_i$ that are at distance $j$ from $v_i$, that is, $L_i^j = \{v \in V(T_i) :$ the path between $v_i$ and $v$ has $j$ edges$\}$. We refer to $L_i^j$ as the $j$-th *level* of tree $T_i$. Figure 1(b) exhibits a tree $T_i$ of a unicyclic graph $G$, rooted at $v_i$, highlighting its three levels (other than $L_i^0$).

Our first results are on gap-[$k$]-vertex-labellings of unicyclic graphs. In

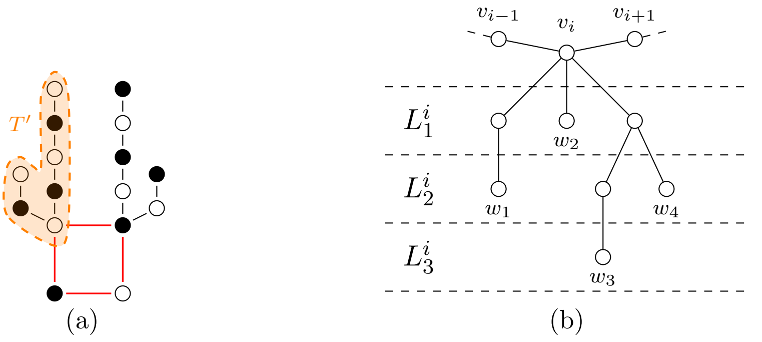


Figure 1: (a) An example of a unicyclic graph; (b) a tree $T_i$ with three levels. Level $L_0^i$ is omitted.

order to establish tightness, we use the following theorem from C. Weffort-Santos' masters thesis [7].

**Theorem 1** (cf. [7])**.** *Let $G$ be a gap-$[k]$-vertex-labelable graph that is not isomorphic to $K_{1,m}$, $m \geq 2$. Then, $\chi(G) \leq \chi_V^g(G) \leq 2^{|V(G)|}$.* □

In the following theorem, we establish the vertex-gap number for the family of unicyclic graphs. Since this result has already been established for cycles [7, 8], we consider only unicyclic graphs that have at least one nontrivial tree, i.e., there is at least one vertex $v_i \in C_p$ such that $d(v_i) \geq 3$.

**Theorem 2.** *Let $G \ncong C_n$ be a unicyclic graph. Then, $\chi_V^g(G) = \chi(G)$.*

*Sketch of the proof.* Let $G$ be as stated in the hypothesis. Since Theorem 1 establishes that $\chi_V^{\mathrm{g}}(G) \geq \chi(G)$, it suffices to show a gap-[2]-vertex-labelling for bipartite unicyclic graphs, and one that uses $k = 3$ labels for others. In both cases, we define a partial[2] labelling $\pi : S \to \{1, \ldots, \chi(G)\}$, where $S$ consists of the vertices of cycle $C_p$ together with all vertices in the first level of each tree $T_i$, i.e. $S = \bigcup_{i=0}^{p-1} L_1^i \cup V(C_p)$. This first step is done so as to induce a proper $\chi(G)$-colouring of cycle $C_p$. Furthermore, this initial colouring of root vertices $v_i \in V(C_p)$ creates three possible

[2]A *partial* labelling is one that assigns labels to some subset of elements of the graph.

combinations of pairs $(\pi(v_i), c_\pi(v_i))$ for cycles of even length, and five combinations for odd-length cycles. For each of these combinations, we assign labels to the vertices of each level of tree $T_i$ such that the induced colours of internal vertices alternate between 0 and 1 and the leaves receive induced colours 1 and 2 that do not conflict with their neighbour. This labelling technique is inspired by the labelling of trees formulated by A. Dehghan et al. [3] in 2013. □

Regarding the edge-labelling variant, we can prove that every odd-length unicyclic graph admits a gap-[3]-edge-labelling by using a similar technique. Therefore, given the bounds by Brandt et al. [1], the following result also holds.

**Theorem 3.** *Let $G$ be an odd-length unicyclic graph with at least one nontrivial tree. Then, $\chi^g_E(G) = \chi(G) = 3$.* □

Thus, it remains to consider only the gap-[$k$]-edge-labelling of even-length (and, consequently, bipartite) unicyclic graphs. Recall that A. Brandt et al. [1] established that for every graph $G$ with no connected components isomorphic to $K_2$, it holds that $\chi^{\mathrm{g}}_E(G) \in \{\chi(G), \chi(G)+1\}$. Hence, for this family, it is a matter of deciding whether $k = 2$ or $k = 3$. Although a labelling technique such as the one employed in Theorem 2 seems like a natural approach to the problem, it is a fruitless one. We explain by recalling the unicyclic graph $G$ in Figure 1(a), drawing the reader's attention to tree $T'$ highlighted in the image. For this graph, no gap-[2]-edge-labelling exists; observe that any attempt to induce colour 0 to the root of $T'$ causes the labels in edges incident with the leaves of the tree to be distinct. Therefore, the one that receives label 1 induces a conflicting colour on its respective leaf. Since tree $T'$ is isomorphic to the other nontrivial tree rooted in $C_p$ and none of them can have induced colour 0 at its root, no gap-[2]-edge-labelling exists.

Based on this observation, we argue that there exists an infinite family of bipartite unicyclic graphs which do not admit any gap-[2]-edge-labelling. Consider $G$ the graph obtained by the following construction. Let $C_p$ be any even-length cycle of order $n \geq 4$ and $v_i, v_j \in V(C_p)$ be two vertices in distinct parts of a bipartition $\{A, B\}$ of $G$. We *root* a copy of tree $T'$ in $v_i$, meaning we identify the root of $T'$ with $v_i$, and another copy in $v_j$. Next, consider the operation of rooting copies of tree $T'$ in any vertex of part $A$ in $G$. By a similar reasoning, the resulting graph also does not admit a gap-[2]-edge-labelling. Furthermore, this result also holds if we root any other nontrivial tree in other vertices of $V(C_p)$. Although we do not fully characterize bipartite unicyclic graphs which do not admit a gap-[2]-edge-labelling, we contribute to the problem providing a polynomial-time algorithm which correctly decides if such a graph admits a labelling using only two labels.

Let $T$ be a tree rooted at a vertex $r$ and let $u_1, \ldots, u_{d(r)}$ be the children of $r$. We denote the *subtree* rooted at $u_i$ by $T_i$. In order to present our result, we introduce an auxiliary recursive algorithm that, given a child vertex $u_i$, a label $l \in \{1, 2\}$ and a colour $c \in \{0, 1\}$, decides whether $T_i$ admits a gap-[2]-edge-labelling $(\pi, c_\pi)$ such that: (i) the induced colour in $u_i$ is $c$; (ii) colour $c_\pi(u_i)$ is determined by the largest gap in the entirety of $\Pi_{E(u_i)}$, i.e., considering the label assigned to edge $ru_i$; and (iii) label $l$ is assigned to $ru_i$. The base case for this algorithm is when $u_i$ is a leaf. If the input $c$ is 0, its parent must have induced colour 1, and the algorithm returns TRUE if and only if $l = 2$. Otherwise, if $c = 1$, it always returns TRUE.

For internal vertices, the idea is that, in any gap-[2]-edge-labelling of $T$, the only possible induced colours in $u_i$ are 0 and 1. Therefore, if $c = 0$, we make a recursive call for each child of $u_i$, passing label $l$ and colour $\bar{c} \in \{1, 2\} \setminus c$ as parameters. The algorithm returns TRUE if and only if all subtrees below $u_i$ admit a gap-[2]-edge-labelling (under the aforementioned assumptions). Otherwise, e.g. $c = 1$, we do as follows. For each child of $u_i$, we make two recursive calls, both with colour $\bar{c}$, and each one with a possible value for $l$. Then, we verify (in linear time) if there is an assignment of labels to the edges incident with $u_i$ which induces $c_\pi(u_i) = c$. Thus, for a tree on $n$ vertices, the algorithm takes time $T(n) = \sum_{i=1}^{d(r)-1} T(n_i) + \mathcal{O}(n)$, where $n_i$ corresponds to the number of vertices in each subtree $T_i$ rooted in each child of $r$. By induction, it is possible to prove that this algorithm executes in time bounded by $\mathcal{O}(n^2)$.

Now, for a bipartite unicyclic graph $G$, let $e = v_i v_j$ be an edge of $C_p \subset G$ with $d(v_i) \geq 3$. Let $T = G - e$. Since there are only two possible label assignments to $e$ in any gap-[2]-edge-labelling of $G$, we need only verify if there exists a gap-[2]-edge-labelling of $T$ satisfying $c_\pi(v_i) = 0$ or $c_\pi(v_i) = 1$, such that the label assigned to $e$ contributes correctly to the colouring. A small adjustment to our previous algorithm is required in order to account for the value of this fixed label in $e$. However, this modification has no impact on the execution time nor the correctness of the algorithm. Therefore, the following theorem holds.

**Theorem 4.** *Let $G$ be a bipartite unicyclic graph. Then, there exists an $\mathcal{O}(n^2)$ algorithm which decides whether $G$ admits a gap-$[2]$-edge-labelling.* □

# 3 Concluding remarks and open problems

Our results contribute to a more refined knowledge of the hardness of the decision problem regarding gap-labellings of bipartite planar graphs using two labels, for which the boundaries of tractability seem quite unclear — even more so in regard to the contribution of degree-one vertices to the hardness of the problem. Concerning the edge variant, two important questions remain. Can we extend the results from our algorithm in order to provide a full characterization of gap-[2]-edge-labelable bipartite unicyclic graphs? Moreover, what other families of planar bipartite graphs with minimum degree one admit gap-[2]-edge-labellings? We leave these open problems for future research.

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